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Hochschild cohomology for algebras, by Sarah Witherspoon, Graduate Studies in Mathematics, Vol. 204, American Mathematical Society, Providence, RI, 2019, xi+250 pp., ISBN 978-1-4704-4931-5

Homological techniques appeared in algebra in the 1940s, when Eilenberg and Mac Lane introduced homology and cohomology of groups, and Hochschild introduced homology and cohomology of algebras. In the 1960s Gerstenhaber showed that Hochschild cohomology has also a nonassociative Lie bracket. Subsequently, a number of higher structures have been discovered, and a huge amount of related research has developed in many different fields.

Hochschild cohomology of algebras is indispensable to understand homological properties of algebras and their representations. Usually associative algebras occurring naturally, over fields, are not semisimple, and to understand their properties, homological methods are essential. Such algebras can be very different depending on the context, but Hochschild cohomology can be constructed in complete generality, in terms of basic linear algebra. In spite of its simplicity, it is a unifying concept for different areas. It has important invariance properties, for example it is invariant under derived, stable, and Morita equivalences.

Let A be an algebra over k, which in the most general setting could be any commutative ring. In this book, from Chapter 2 onwards k is a field. The enveloping algebra A^e of A is defined to be the algebra $A \otimes_k A^{\mathrm{op}}$, then an A-bimodule M is the same as a (left) A^e-module (by definining $(a \otimes b)m = amb$, for $m \in M$ and $a, b \in A$). The algebra A itself is an A-bimodule, more generally so is $A^{\otimes n} = A \otimes \cdots \otimes A$. The basic construction starts with a projective resolution of A as an A^e-module,

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \to A \to 0. \tag{1}$$

Applying the functor $\mathrm{Hom}_{A^e}(-, M)$ to (1) for an A^e-module M produces a complex, with differentials d_n^* given by composing with d_n. The homology of this complex is the Hochschild cohomology $\mathrm{HH}^*(A, M)$ of A with coefficients in M, the nth term is $\mathrm{Ker}(d_{n+1}^*)/\mathrm{Im}(d_n^*)$, with $\mathrm{Ker}(d_{n+1}^*)$ the Hochschild n-cocycles and $\mathrm{Im}(d_n^*)$ the Hochschild n-coboundaries. When $M = A$, this is the Hochschild cohomology $\mathrm{HH}^*(A)$ of A. For k a field, $\mathrm{HH}^*(A, M)$ is isomorphic to $\mathrm{Ext}^*_{A^e}(A, M)$. If one applies the functor $M \otimes_{A^e} (-)$ to (1), one obtains Hochschild homology, $\mathrm{HH}_*(A, M)$. When k is a field this is isomorphic to $\mathrm{Tor}_*^{A^e}(M, A)$. For $M = A$, this is the Hochschild homology, $\mathrm{HH}_*(A)$.

In his original paper, [11], Hochschild takes the bar complex in (1), an explicit free resolution of A as an A^e-module, with nth term $A^{\otimes(n+2)} = A \otimes \cdots \otimes A$. He defines the cup product, which makes $\mathrm{HH}^*(A)$ a graded commutative associative algebra. Subsequently, Gerstenhaber [8] showed that it also has the structure of a graded Lie algebra, so that the Lie bracket is a graded derivation with respect to the cup product. Many properties of Hochschild cohomology rings that are essential in today's applications can be seen in the classical definitions of Hochschild and Gerstenhaber. Accordingly, Chapter 1 starts with these historical definitions, and amongst others, it also shows how the classical approach leads to specific information encoded by $\mathrm{HH}^n(A)$ and $\mathrm{HH}_n(A)$ when n is small.

Chapter 2 works with more general resolutions or with generalized extensions. It constructs various products and also shows how Hochschild cohomology acts on Ext spaces of modules. To deal with general resolutions, one needs to construct chain maps from cocycles. With this, one can define a second product on Hochschild cohomology, the Yoneda product. Given a projective bimodule resolution of A as in (1), let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles. Extend g to a chain map where $g_m : P_{n+m} \to P_m$. With this, the map $f \cup g \in \mathrm{Hom}_{A^e}(P_{m+n}, A)$ is defined to be the composition

$$f \cup g := f g_m. \tag{2}$$

One can show that it gives rise to a well-defined product on cohomology. As well, it does not depend on the choice of the resolution. Hence taking for $P_\bullet$ the bar resolution, this new product is the same as Hochschild's original cup product. Another definition of an associative product on $\mathrm{HH}^*(A)$ which also is equivalent to the original cup product is a convolution product which arises from a tensor product of complexes and a diagonal map, also introduced in Chapter 2.

There are two ways to define a product on $\mathrm{HH}^*(A)$ in terms of generalized extensions. The first, known as Yoneda composition or Yoneda splice, uses that $\mathrm{HH}^n(A)$ is isomorphic to $\mathrm{Ext}^n_{A^e}(A, A)$, the equivalence classes of n-extensions. Taking an m-extension and an n-extension, representing elements in $\mathrm{HH}^m(A)$ and $\mathrm{HH}^n(A)$, they may be combined into a new exact sequence which is an $(m+n)$-extension of A by A, and gives an element in $\mathrm{HH}^{m+n}(A)$. It corresponds up to sign to the Yoneda product defined before. Alternatively, one can take the tensor product, over A, of an m-extension with an n-extension, to obtain an $(m+n)$-extension from the total complex, again this is equivalent to Yoneda composition.

For two left A-modules M and N, the Hochschild cohomology ring $\mathrm{HH}^*(A)$ acts on the graded vector space $\mathrm{Ext}^*_A(M, N)$ in such a way that it becomes a graded $\mathrm{HH}^*(A)$-module. Similarly, if B is an A-bimodule, $\mathrm{HH}^*(A)$ acts on $\mathrm{HH}^*(A, B)$. Let $f \in \mathrm{Hom}_{A^e}(P_i, A)$ represent an element in $\mathrm{HH}^i(A)$. Then we have

$$\phi_M(f) := f \otimes 1_M \tag{3}$$

in $\mathrm{Hom}_A(P_i \otimes_A M, M)$ representing an element of $\mathrm{Ext}^i_A(M, M)$. Extending $\phi_M(f)$ to a chain map and composing its jth term with any function $g : P_j \otimes_A M \to N$ induces a well-defined map

$$\mathrm{Ext}^j_A(M, N) \otimes \mathrm{HH}^i(A) \to \mathrm{Ext}^{i+j}_A(M, N).$$

Similarly, there is a left action of $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_A(M, N)$, and we have

$$\text{If } \alpha \in \mathrm{HH}^i(A) \text{ and } \beta \in \mathrm{Ext}^j_A(M, N), \text{ then } \ \alpha \cdot \beta = (-1)^{ij} \beta \cdot \alpha.$$

Suppose $M = N$, the graded vector space $\mathrm{Ext}^*_A(M, M)$ is itself an associative algebra, via the Yoneda product. The above identity shows that

$$\text{The image of } \phi_M \text{ is contained in the graded center, } Z_{\mathrm{gr}}(\mathrm{Ext}^*_A(M, M)).$$

(For a graded algebra B, the graded center $Z_{\mathrm{gr}}(B)$ is the subalgebra generated by all homogeneous elements $\alpha \in B$ such that $\alpha \cdot \beta = (-1)^{|\alpha||\beta|} \beta \cdot \alpha$.)

Chapter 3 investigates resolutions designed for particular types of algebras, and it gives a rich supply of examples, which are used throughout the book. It starts with

the tensor product of algebras, and the graded tensor product when the underlying algebras are graded. Taking the algebra structure on Hochschild cohomology by cup product, one gets

> Let A and B be k-algebras for which there exists a free A^e-resolution $P_\bullet$ of A and a free B^e resolution $Q_\bullet$ of B consisting of finitely generated bimodules. Then
>
> $$\mathrm{HH}^*(A \otimes B) \cong \mathrm{HH}^*(A) \otimes \mathrm{HH}^*(B)$$
>
> as algebras (where the right side is a graded tensor product algebra).

The text moves on to twisted tensor products $A_1 \otimes^t A_2$ of graded algebras, with twist via a bicharacter t on the grading groups. The total complex of the tensor product of graded projective resolutions for the factors gives rise to a projective bimodule resolution ([2]). However, there is no general description of the Hochschild cohomology ring in terms of the factors, as the following example illustrates.

Take $A_1 = k[x_1]/(x_1^2)$ and $A_2 = k[x_2]/(x_2)^2$, and let q be a nonzero scalar. Then there is a bicharacter t so that the twisted tensor product $A = A_1 \otimes^t A_2$ is isomorphic to the four-dimensional algebra

$$A = k\langle x_1, x_2\rangle/(x_i^2, x_1x_2 - qx_2x_1). \tag{4}$$

In [5] it is shown that its Hochschild cohomology is finite-dimensional when q is not a root of unity. This was a surprise, showing that an algebra of infinite global dimension can have finite-dimensional Hochschild cohomology, answering a long-standing question of Happel [10]. On the other hand when q is a root of unity, $\mathrm{HH}^*(A)$ is infinite dimensional. Furthermore, there can be gaps, it can be 0 in infinitely many degrees, also shown in [5].

The next part defines the Koszul complex associated to regular sequences of central elements in an algebra. It presents the Hochschild–Kostant–Rosenberg theorem, which describes completely the Hochschild cohomology and homology rings of smooth commutative algebras.

The text gives an introduction to Koszul algebras. Let V be a finite-dimensional vector space, and let $T(V)$ be the tensor algebra $\bigoplus_{n\geq 0} T^n(V)$ with $T^0(V) = k$, $T^1(V) = V$, and $T^n(V) = V \otimes \cdots \otimes V$ (n tensor factors). This is a graded algebra with $|v| = 1$ for all $v \in V$. Let R be a subspace of $T^2(V)$, and let $A = T(V)/(R)$, where (R) is the ideal generated by R in $T(V)$. This is a quadratic algebra, that is, a graded algebra generated by elements in degree 1 with relations in degree 2. There is a natural construction of a quadratic dual, or Koszul dual, $A^! = T(V^*)/(R^\perp)$. One can view A as an augmented algebra, and then A is defined to be a Koszul algebra if the A-module k has a linear minimal graded free resolution. Equivalent definitions of Koszul algebras are given, for example, by

> Let A be a finitely generated graded connected quadratic algebra. Then the following are equivalent:
> (i) A is a Koszul algebra;
> (ii) $\mathrm{Ext}^*_A(k,k) \cong A^!$ as graded algebra;
> (iii) $\mathrm{Ext}^*_A(k,k)$ is generated by $\mathrm{Ext}^1_A(k,k)$ as an algebra.

Suppose A is a Koszul algebra, then there is a close relationship between Hochschild cohomology and the Ext algebra $\mathrm{Ext}^*_A(k,k)$. Let ϕ_k be the map (3) giving

the action of $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_A(k,k)$, then we have

> Let A be a Koszul algebra. Then the image of the map ϕ_k is precisely the graded center, $Z_{\mathrm{gr}}(\mathrm{Ext}^*_A(k,k))$.

Let G be a finite group, acting by automorphisms on an algebra A, and let $A \rtimes G$ be the skew group algebra. When the order of G is not divisible by the characteristic of k, the Hochschild cohomology and homology for $A \rtimes G$ are described in terms of invariants and coinvariants of Hochschild cohomology and homology of A with coefficients in the bimodule $A \rtimes G$. Namely,

> There are actions of G on Hochschild cohomology or homology with coefficients in $A \rtimes G$, such that there are isomorphisms as graded algebras and graded vector spaces, respectively,
>
> $$\mathrm{HH}^*(A \rtimes G) \cong \mathrm{HH}^*(A, A \rtimes G)^G \quad \text{and} \quad \mathrm{HH}_*(A \rtimes G) \cong \mathrm{HH}_*(A, A \rtimes G)_G.$$

The last part of Chapter 3 introduces monomial algebras and presents a bimodule resolution of a monomial algebra due to Bardzell [1]. Generalizations of this have been found, and some details are given in the text. These techniques are more general than may appear since a finite-dimensional algebra over an algebraically closed field is Morita equivalent to an algebra defined by a quiver and relations, and Hochschild cohomology is invariant under Morita equivalence.

Some classical geometric notions such as smoothness may be viewed as essential homological properties of commutative function algebras, allowing interpretations of them in noncommutative settings via Hochschild cohomology. This is presented in Chapter 4.

The Hochschild dimension of A is defined as $\dim A = \mathrm{pdim}_{A^e}(A)$. The algebra A is smooth if its Hochschild dimension is finite and it has a finite projective resolution as an A^e-module consisting of finitely generated projective modules.

The text introduces a noncommutative version of Kähler differentials, following work of Cuntz and Quillen [6]. Square-zero extensions of algebras already occur in [11], Hochschild proved that for an A-bimodule M, there is a bijection between $\mathrm{HH}^2(M)$ and equivalence classes of square-zero extensions of A by M. The text shows that square-zero extensions, noncommutative differential forms, and quasi-free algebras (that is, $\dim(A) \leq 1$) are related.

For smooth algebras there is a duality between Hochschild homology and cohomology, an analogue of Poincaré duality in geometry, proved by van den Bergh [17]. A bimodule U is invertible if there is an A-bimodule V such that $U \otimes_A V \cong A$ and $V \otimes_A U \cong A$ as A-bimodules.

> Let A be a smooth algebra. Assume that there is a positive integer d for which $\mathrm{HH}^i(A, A^e) = 0$ for all $i \neq d$ and that $U = \mathrm{HH}^d(A, A^e)$ is an invertible A-bimodule. Then there is an isomorphism of vector spaces
>
> $$\mathrm{HH}^n(A, M) \cong \mathrm{HH}_{d-n}(A, U \otimes_A M)$$
>
> for all A-bimodules M and $0 \leq n \leq d$, and $\mathrm{HH}^n(A, M) = 0$ for $n > d$.

In this case the algebra A satisfies van den Bergh duality. A smooth algebra A is Calabi–Yau if it has van den Bergh duality with $U \cong A$. Let G be a finite group, let k be a field of characteristic not dividing $|G|$, and let V be a kG-module of finite vector space dimension d. This section shows that the skew group algebra $A = S(V) \rtimes G$ has van den Bergh duality and determines conditions under which

it is Calabi–Yau, namely this holds when G acts on V via linear transformations of determinant 1.

The final part of Chapter 4 introduces the Connes differential on Hochschild cohomology, which arises in cyclic homology. For Calabi–Yau algebras, this differential in combination with van den Bergh duality is used to define a new operation on Hochschild cohomology, called a Batalin–Vilkovisky operator.

Understanding how some algebras may be viewed as deformations of others involves Hochschild cohomology, as discussed in Chapter 5. This introduces formal deformations of an algebra A over the ring $k[[t]]$ (or $k[t]$, or $k[t]/(t^n)$ for some $n \geq 2$), and it explains the role of $\mathrm{HH}^3(A)$ to obtain new associative products on $A[[t]]$. This also illustrates that Gerstenhaber brackets naturally appear. It defines infinitesimal deformations, leading to associative algebra structures on $A[t]/(t^2)$.

There is a natural notion of equivalence and triviality of formal deformations. If $\mathrm{HH}^2(A) = 0$, then A has no nontrivial formal deformations, that is, A is rigid. Furthermore, there is a short section on the Maurer–Cartan equation and on deformation quantizations of Poisson algebras.

The last part of this chapter is an account on the theory of Braverman and Gaitsgory [3] for graded deformations of Koszul algebras. As an application of this, one obtains a proof of the classical Poincare–Birkhoff–Witt theorem for Lie algebras.

In algebraic deformation theory, the Lie structure on Hochschild cohomology arises naturally, and Chapter 6 spends some time studying this in detail. Part of this is based on the bar resolution, but it also presents results using other resolutions and exact sequences. In particular it presents the notion of homotopy liftings that allow Gerstenhaber brackets to be expressed on an arbitrary resolution as essentially graded commutators; this follows [18]. Towards a topological approach, the brackets are constructed as loops in the classifying space of an extension category (see [13]).

Chapter 7 gives a brief introduction to infinity structures and applications to Hochschild cohomology. It discusses A_∞-algebras; any differential graded algebra and hence any associative algebra can be viewed as A_∞-algebra, but in general there may be many possible A_∞-structures. An infinitesimal n-deformation of an algebra B is defined to be a $k[x]/(x^2)$-multilinear A_∞-algebra structure on A that lifts the multiplcation of B. With this,

> Let $n \geq 2$. The Hochschild n-cocycles on B are in one-to-one correspondence with the infinitesimal n-deformations of B.

The text defines minimal models and formality of A_∞-algebras, and it presents a characterization of Koszul algebras in this setting ([12]). Using the notion of an A_∞-center of an A_∞-algebra as introduced in [4] gives a generalization of a previous result:

> Let B be an augmented algebra. The image of $\mathrm{HH}^*(B)$ in the Ext algebra $\mathrm{Ext}^*_B(k,k)$ is precisely its A_∞-center.

Furthermore, the text defines L_∞-algebras and formality of associative algebras, making a connection with deformations of algebras and Deligne's conjecture.

In representation theory, one may sometimes use Hochschild cohomology to define support varieties, which are geometric objects associated to modules and which encode representation theoretic information. Chapter 8 discusses support varieties defined via Hochschild cohomology for finite-dimensional algebras, as introduced in [15].

Assume A is a finite-dimensional algebra over a field k. Recall that $\mathrm{HH}^*(A)$ acts on $\mathrm{Ext}^*_A(M,M)$ for finite-dimensional modules. We take $M = A/\mathfrak{r}$ with $\mathfrak{r}$ is the Jacobson radical of A, so that every simple A-module occurs as a direct summand of M. Then we say that algebra A satisfies *condition* (fg) if

(1) $\mathrm{HH}^*(A)$ is a noetherian ring and
(2) $\mathrm{Ext}^*_A(A/\mathfrak{r}, A/\mathfrak{r})$ is a finitely generated $\mathrm{HH}^*(A)$-module.

One may replace $\mathrm{HH}^*(A)$ by any graded subalgebra H which is finitely generated commutative and has $H^0 = \mathrm{HH}^0(A)$ (see [16]). Many algebras are known to satisfy this but not all. For example, let A be four-dimensional algebra in (4) with q not a root of unity. Then $\mathrm{HH}^*(A)$ is finite dimensional [5], but the algebra $\mathrm{Ext}^*_A(k,k)$ is infinite dimensional and hence is not finitely generated over $\mathrm{HH}^*(A)$.

Assume A satisfies (fg), and let H be a subalgebra of $\mathrm{HH}^*(A)$ as above. For finite-dimensional A-modules M, N let $I_H(M,N)$ be the annihilator ideal in H of $\mathrm{Ext}^*_A(M,N)$, that is

$$I_H(M,N) = \{\alpha \in H \mid \alpha \cdot \beta = 0 \text{ for all } \ \beta \in \mathrm{Ext}^*_A(M,N)\}.$$

Then the support variety $V_H(M,N)$ is the variety associated to the ideal $I_H(M,N)$, that is, the maximal ideals of H containing $I_H(M,N)$. The support variety of M is $V_H(M) = V_H(M,M)$. When A is self-injective, this has properties similar to that of support varieties for finite group representations defined via group cohomology (see [7]).

There are strong connections between Hochschild cohomology and group cohomology; this is analyzed in the more general context of Hopf algebras in Chapter 9. Hopf algebras are those algebras whose module categories are tensor categories, and they include many examples of interest, such as group algebras, universal enveloping algebras of Lie algebras, and quantum groups.

Suppose A is a Hopf algebra, that is, A is an algebra over k together with algebra homomorphisms $\Delta : A \to A \otimes A$, the comultiplication, and $\varepsilon : A \to k$, the counit, and an algebra antihomomorphism $S : A \to A$, the antipode, satisfying certain axioms. Then also A^e is a Hopf algebra. Tensor products of A-modules and k-linear maps between A-modules are again A-modules, and k is an A-module. There is a cup product,

$$\cup : \mathrm{Ext}^m_A(M,M') \times \mathrm{Ext}^n_A(N,N') \longrightarrow \mathrm{Ext}^{m+n}_A(M \otimes N, M' \otimes N')$$

constructed from tensor products of functions. Furthermore,

> If $\alpha \in \mathrm{Ext}^m_A(M,M')$ and $\beta \in \mathrm{Ext}^n_A(N,N')$, then the cup product $\alpha \cup \beta$ is equivalent to the Yoneda composite of $\alpha \otimes 1_{N'}$ and $1_M \otimes \beta$ in $\mathrm{Ext}^m_A(M \otimes N', M' \otimes N')$ and $\mathrm{Ext}^n_A(M \otimes N, M \otimes N')$, respectively.

Applying this with $M = M' = N = N' = k$, one gets the algebra $\mathrm{H}^*(A,k) := \mathrm{Ext}^*_A(k,k)$, the Hopf algebra cohomology. Moreover, the cup product leads naturally to an action of $\mathrm{H}^*(A,k)$ on $\mathrm{H}^*(A,M) := \mathrm{Ext}^*_A(k,M)$ for any A-module M. As well, if we apply the above with $M = M'$ and $N = N'$, and compose with $k \otimes N \cong N$, we get an action of $\mathrm{H}^*(A,k)$ on $\mathrm{Ext}^*_A(N,N)$. This action corresponds to that on $\mathrm{Ext}^*_A(k, N \otimes N^*)$ under the isomorphism $\mathrm{Ext}^*_A(k, N \otimes N^*) \cong \mathrm{Ext}^*_A(N,N)$.

There is an injective algebra homomorphism $\delta : A \to A^e$ which allows one to relate the Hopf algebras A and A^e and their modules. Furthermore, there is an isomorphism of A^e-modules, $A \cong A^e \otimes_A k$ where $A^e \otimes_A k$ is the A^e-module induced from the trivial A-module k via the embedding given by δ.

We see how to relate Hochschild cohomology to Hopf algebra cohomology. The following was first proved in [9].

> Let A be a Hopf algebra with S bijective. Then there is an isomorphism of algebras
>
> $$\mathrm{HH}^*(A) \cong \mathrm{H}^*(A, A^{\mathrm{ad}}).$$

Here M^{ad} is the A-module structure on an A-module M by twisting with S. When $A = kG$, the group algebras of a finite group, the adjoint action is conjugation by group elements, and by the above, one can rewrite Hochschild cohomology as

$$\mathrm{HH}^*(kG) \cong \bigoplus_i \mathrm{H}^*(C(g_i), k),$$

an isomorphism of graded vector space, where the g_i are representatives of G-conjugacy classes, and $C(g_i)$ is the centralizer of g_i in G. A product formula in terms of group data was discovered in [14].

For an A-module algebra R, we have the smash product $R\#A$. There is a spectral sequence relating its Hochschild cohomology with that of R, and of Hopf algebra cohomology of A:

> Let M be an $(R\#A)^e$-module. There is a right action of A on $\mathrm{HH}^q(R, M)$ and a spectral sequence
>
> $$E_2^{p,q} = \mathrm{H}^p(A, \mathrm{HH}^q(R, M)) \;\Rightarrow\; \mathrm{HH}^{p+q}(R\#A, M).$$

This book is a superb introduction to the basic theory of Hochschild cohomology for algebras and some of its current uses in algebra and representation theory. It takes a concrete approach, with many examples, written for graduate students and working mathematicians. I expect it will also be appreciated by those who are experts in part of the area, learning more of the wider context. It gives a complete and carefully written account of core material, and it serves as a reference for many results that are currently only found in research papers and as a a bridge to some more advanced topics. There is no comparable text available at present, and it is likely that this book will become a standard reference.

References

[1] M. J. Bardzell, *The alternating syzygy behavior of monomial algebras*, J. Algebra **188** (1997), no. 1, 69–89, DOI 10.1006/jabr.1996.6813. MR1432347

[2] P. A. Bergh and S. Oppermann, *Cohomology of twisted tensor products*, J. Algebra **320** (2008), no. 8, 3327–3338, DOI 10.1016/j.jalgebra.2008.08.005. MR2450729

[3] A. Braverman and D. Gaitsgory, *Poincaré-Birkhoff-Witt theorem for quadratic algebras of Koszul type*, J. Algebra **181** (1996), no. 2, 315–328, DOI 10.1006/jabr.1996.0122. MR1383469

[4] B. Briggs and V. Gélinas, *The A_∞ centre of the Yoneda algebra and the characteristic action of Hochschild cohomology on the derived category*, (2017), `arXiv:1702.00721`.

[5] R.-O. Buchweitz, E. L. Green, D. Madsen, and Ø. Solberg, *Finite Hochschild cohomology without finite global dimension*, Math. Res. Lett. **12** (2005), no. 5-6, 805–816, DOI 10.4310/MRL.2005.v12.n6.a2. MR2189240

[6] J. Cuntz and D. Quillen, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc. **8** (1995), no. 2, 251–289, DOI 10.2307/2152819. MR1303029

[7] K. Erdmann, M. Holloway, R. Taillefer, N. Snashall, and Ø. Solberg, *Support varieties for selfinjective algebras*, K-Theory **33** (2004), no. 1, 67–87, DOI 10.1007/s10977-004-0838-7. MR2199789

[8] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) **78** (1963), 267–288, DOI 10.2307/1970343. MR161898

[9] V. Ginzburg and S. Kumar, *Cohomology of quantum groups at roots of unity*, Duke Math. J. **69** (1993), no. 1, 179–198, DOI 10.1215/S0012-7094-93-06909-8. MR1201697

[10] D. Happel, *Hochschild cohomology of finite-dimensional algebras*, Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988), Lecture Notes in Math., vol. 1404, Springer, Berlin, 1989, pp. 108–126, DOI 10.1007/BFb0084073. MR1035222

[11] G. Hochschild, *On the cohomology groups of an associative algebra*, Ann. of Math. (2) **46** (1945), 58–67, DOI 10.2307/1969145. MR11076

[12] B. Keller, *A-infinity algebras in representation theory*, Representations of algebra. Vol. I, II, Beijing Norm. Univ. Press, Beijing, 2002, pp. 74–86. MR2067371

[13] S. Schwede, *An exact sequence interpretation of the Lie bracket in Hochschild cohomology*, J. Reine Angew. Math. **498** (1998), 153–172, DOI 10.1515/crll.1998.048. MR1629858

[14] S. F. Siegel and S. J. Witherspoon, *The Hochschild cohomology ring of a group algebra*, Proc. London Math. Soc. (3) **79** (1999), no. 1, 131–157, DOI 10.1112/S0024611599011958. MR1687539

[15] N. Snashall and Ø. Solberg, *Support varieties and Hochschild cohomology rings*, Proc. London Math. Soc. (3) **88** (2004), no. 3, 705–732, DOI 10.1112/S002461150301459X. MR2044054

[16] Ø. Solberg, *Support varieties for modules and complexes*, Trends in representation theory of algebras and related topics, Contemp. Math., vol. 406, Amer. Math. Soc., Providence, RI, 2006, pp. 239–270, DOI 10.1090/conm/406/07659. MR2258047

[17] M. van den Bergh, *A relation between Hochschild homology and cohomology for Gorenstein rings*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1345–1348, DOI 10.1090/S0002-9939-98-04210-5. MR1443171

[18] Y. Volkov, *Gerstenhaber bracket on the Hochschild cohomology via an arbitrary resolution*, Proc. Edinb. Math. Soc. (2) **62** (2019), no. 3, 817–836, DOI 10.1017/s0013091518000901. MR3974969

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