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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
BUCKMASTER AND VICOL

MR0065993 (16,515e) 53.0X

Nash, John

C^1 isometric imbeddings.

Annals of Mathematics. Second Series **60** (1954), 383–396.

This paper contains some surprising results on the C^1-isometric imbedding into an Euclidean space of a Riemannian manifold with a positive definite C^0-metric. The theorems are: 1) Any closed Riemannian n-manifold has a C^1-isometric imbedding in E^{2n} (the Euclidean space of dimension $2n$). 2) Any Riemannian n-manifold has a C^1-isometric immersion in E^{2n} and an isometric imbedding in E^{2n+1}. 3) If a closed Riemannian n-manifold has C^1-immersion or imbedding in E^k with $k \geq n+2$, it also has respectively an isometric immersion or imbedding in E^k. The basic idea is a perturbation process defined in a neighborhood and relative to two normal vector fields. The imbedded or immersed manifold is of course generally quite pathological.

S. Chern

From MathSciNet, November 2020

MR1428905 (98e:76002) 76-02; 35Q30, 76D05, 76Fxx, 76M35

Frisch, Uriel

Turbulence. (English)

Cambridge University Press, Cambridge,, 1995, xiv+296 pp., \$80.00,
ISBN 0-521-45103-5

In 1941 A. N. Kolmogorov published three short papers on turbulence [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30** (1941), 301–305; MR0004146 (reprinted in Proc. Roy. Soc. London Ser. A **434** (1991), no. 1890, 9–13; MR1124922); C. R. (Doklady) Acad. Sci. URSS (N. S.) **31** (1941), 538–540; MR0004568; C. R. (Doklady) Acad. Sci. URSS (N.S.) **32** (1941), 16–18; MR0005851 (reprinted in Proc. Roy. Soc. London Ser. A **434** (1991), no. 1890, 15–17; MR1124923)]. Making "physical" assumptions about homogeneity, isotropy, the zero viscosity limit of the Navier-Stokes equations and the long-time asymptotic behavior of its solutions, he defined the "Kolmogorov dissipation scale", and showed that the "structure functions" (averages of pth powers of velocity differences at points separated by distance l) could be written as $S_p(l) = C_p\epsilon^{p/3}l^{p/3}$, where ϵ is the average energy dissipation per unit of time and mass. From the case $p = 2$ the famous "5/3 law" can be deduced: that the power spectrum falls off with wave number k as $E(k) = C\epsilon^{2/3}k^{-5/3}$ in the inertial range, intermediate between energy production and dissipation scales. Kolmogorov also considered turbulent dissipation and the decay of unforced isotropic turbulence. His papers are widely quoted but probably little read, being very concise, indeed, almost gnomic. Nonetheless, they are sometimes described as "the only results on turbulence which have been derived from the Navier-Stokes equations";

in any event they have had a significant influence in the physics and engineering communities, and they are of considerable interest to applied mathematicians. They form the starting point and heart of this unusual book.

The book derives from a graduate course on "Turbulence and dynamical systems" that Uriel Frisch taught at the University of Nice (Sophia-Antipolis). On page 1 he immediately won this reviewer's heart by pointing out that although the Navier-Stokes equation "probably contains all of turbulence, it would be foolish to try to guess what its consequences are without looking at experimental facts". The reason for this is that, global existence difficulties aside, we cannot in any real sense solve this equation. Frisch's approach is very much a physicist's, concerned neither with mathematical rigor nor with flows of current engineering interest involving complex geometry or chemistry, for example, although it does focus on "open" (unbounded) flows, rather than idealized situations such as the Taylor-Couette or Rayleigh-Bénard systems. It is primarily concerned with energy transport in the inertial (intermediate wavenumber) range and it focusses on intermittency and corrections to the "Kolmogorov laws" noted above. Its strongest points are that it rapidly develops the necessary background in probability theory, provides an accessible and systematic account of Kolmogorov's work as well as more recent developments, and includes much relevant experimental data. It contains interesting historical remarks (for example, I learned that Heisenberg and von Weizsäcker, while detained at Farm Hall in 1945, worked on a turbulence closure theory (as well as wondering how the Allies had managed to build a bomb)), an extensive bibliography, and brief discussions of a broad range of experimental, numerical and theoretical work.

Chapter One introduces the basic idea of loss of symmetry of individual solutions with increasing Reynolds number, followed by a gain in "statistical symmetry" in fully developed turbulence (averaging over ensembles of solutions). Here Frisch makes good use of flow visualization pictures. Chapter Two turns to the Navier-Stokes (NS) equations, discussing symmetries in greater precision and describing conservation laws and the energy production/transport/dissipation budget in terms of Fourier-filtered wavenumber scales.

Chapter Three motivates the statistical theory of turbulence by discussing chaotic dynamics of (one-dimensional) iterated maps. However, although some of the spirit of dynamical systems permeates the book, little use is made of it, and this section is merely to persuade the reader that a statistical analysis of the deterministic NS equation is appropriate. Chapter Four introduces elements of probability theory, including a version of Birkhoff's ergodic theorem, correlation functions and spectra.

Equipped with the basics after only 56 pages, in Chapter Five Frisch describes two key empirical laws: the mean square velocity differences between points separated by l scale as $l^{2/3}$, and the energy dissipation ϵ defined above has a finite limit as viscosity tends to zero. These provide an experimental basis for his treatment of the Kolmogorov theory in Chapter Six, where it is developed in a systematic and fairly complete way, with hypotheses clearly stated. However, the derivation differs from Kolmogorov's, and Frisch adds his hypotheses one by one, so that one has to skip back and forth to get the whole picture. Not all steps are included, and there is at least one irritating error (a factor of 6π in equations (6.51–6.52)). (For the experts, the sequence is: derivation of the "4/5 law" for the third-order structure function, followed by the scaling exponent $h = \frac{1}{3}$, and the spectrum

$E(k) = C\epsilon^{1/3}k^{-5/3}$, where ϵ denotes dissipation and k wavenumber.) He then discusses the effect of finite viscosity, deriving the Kolmogorov dissipation scale, and Landau's objections to the supposed universality of the constant C above.

In contrast to the rational (albeit non-rigorous) presentation of Chapter Six, Chapter Seven presents a "phenomenological" (physical) derivation of scaling and related results on the energy cascade and the decay of unforced turbulence. Probability densities of velocity gradients, coherent structures, and (numerical, inconclusive) existence for blow-up of solutions of the inviscid equations are also briefly discussed. These three chapters conclude the "classical" material and together with Chapter Eight form the heart, and well over half, of the book.

Chapter Eight covers recent work on intermittency and "corrections" to the Kolmogorov scaling theory, including fractal and multifractal cascade models, fractal dissipation fluctuation models and Fourier mode "shell" models. The models and their analysis become quite baroque. The chapter ends with a discussion of numerical and experimental evidence for the formation of concentrated vortex filaments. Much of this work is tentative and a considerable amount is due to the author and his colleagues.

Chapter Nine outlines other approaches, including rigorous existence and blow-up results, closure, eddy viscosity, homogenization, functional (Hopf), diagrammatic (Kraichnan), and renormalization methods, dynamical systems and two-dimensional turbulence. It contains a useful literature survey.

For those who already know the elements of fluid mechanics and have a reasonable (first-year graduate) applied mathematical or theoretical physics education, and are not repelled by order of magnitude estimates and a mix of hypotheses and formal deduction, Frisch's book provides an attractive account of Kolmogorov's work and current attempts to extend it. Suitably supplemented by standard material and exercises, it would be a reasonable graduate course text.

Philip J. Holmes

From MathSciNet, November 2020

MR1983780 (2005i:35028) 35D10; 35J45, 35J50, 49J10, 49N60

Müller, S.; Šverák, V

Convex integration for Lipschitz mappings and counterexamples to regularity.

Annals of Mathematics. Second Series **157** (2003), *no.* 3, 715–742.

In this fundamental paper, examples are given of nowhere differentiable Lipschitz solutions to the Euler-Lagrange equation $\operatorname{div} DF(\nabla u) = 0$ corresponding to the functional $I(u) = \int_\Omega F(\nabla u(x))\,dx$, where $\Omega \subset \mathbf{R}^2$ is a disk, $u\colon \Omega \to \mathbf{R}^2$, and F is a smooth function on the set $M^{2\times 2}$ of real 2×2 matrices that is strongly quasiconvex with uniformly bounded second derivatives D^2F. By definition F is strongly quasiconvex if there exists $\gamma > 0$ such that $\int_\Omega (f(A + \nabla\varphi) - f(A))\,dx \geq \gamma \int_\Omega |\nabla\varphi|^2 dx$ for each $A \in M^{2\times 2}$ and each smooth, compactly supported $\varphi\colon \Omega \to \mathbf{R}^2$. Since by the result of L. C. Evans [Arch. Rational Mech. Anal. **95** (1986), no. 3, 227–252; MR0853966] absolute minimizers of I are smooth outside a closed subset of Ω of measure zero (this is even true for local minimizers according to a result of J. Kristensen and A. Taheri [Arch. Ration. Mech. Anal. **170** (2003), no. 1, 63–89; MR2012647]), the examples demonstrate a striking difference between the regularity of weak solutions and that of minimizers.

The construction of the examples has various ingredients. First, the problem is reduced to that of solving a differential inclusion $\nabla w \in K$, where $w\colon \Omega \to \mathbf{R}^4$ and K is a suitable subset of 4×2 matrices. This enables the authors to use a suitable modification of the theory of convex integration of M. L. Gromov [*Partial differential relations*, Springer, Berlin, 1986; MR0864505]. The integrand F is defined in terms of a special quasiconvex function f_0 on symmetric 2×2 matrices defined by $f_0(X) = \det X$ if X is positive definite, $f_0(X) = 0$ otherwise, previously discovered by Šverák [Arch. Rational Mech. Anal. **119** (1992), no. 4, 293–300; MR1179688], and uses also a "T_4-configuration" of 2×2 matrices $A_1, \dots, A_4$ that was first considered by V. Scheffer in his 1974 Princeton thesis ["Regularity and irregularity of solutions to nonlinear second-order elliptic systems of partial differential equations and inequalities", Princeton Univ., Princeton, NJ], and later used by R. J. Aumann and S. Hart [Israel J. Math. **54** (1986), no. 2, 159–180; MR0852476] and L. C. Tartar [in *Microstructure and phase transition*, 191–204, Springer, New York, 1993; MR1320538]. In fact Scheffer used T_4-configurations to prove a version of the counterexample in this paper with F rank-one convex; the extension to quasiconvex F is very significant on account of the counterexample of Šverák [Proc. Roy. Soc. Edinburgh Sect. A **120** (1992), no. 1-2, 185–189; MR1149994] that rank-one convexity does not imply quasiconvexity, and the central role played by quasiconvexity in the multi-dimensional calculus of variations, for example as an essentially necessary and sufficient condition for weak lower semicontinuity.

John M. Ball

From MathSciNet, November 2020

MR2214822 (2007g:76108) 76F02; 01A60, 76-03, 76F55, 82-03

Eyink, Gregory L.; Sreenivasan, Katepalli R.

Onsager and the theory of hydrodynamic turbulence.

Reviews of Modern Physics **78** (2006), *no.* 1, 87–135.

This is an excellent review of the contribution of Lars Onsager to the theory of turbulence. It is written by two outstanding experts in the fields of experimental and theoretical turbulence.

Onsager, a giant of twentieth-century science and the 1968 Nobel Laureate in Chemistry, made deep contributions to several areas of physics and chemistry. Perhaps less well known is his groundbreaking work and lifelong interest in the subject of hydrodynamic turbulence. He wrote two papers on the subject in the 1940's, one of them just a short abstract. Unbeknownst to Onsager, one of his major results was derived a few years earlier by A. N. Kolmogorov, but Onsager's work contains many gems and shows characteristic originality and deep understanding. His only full-length article on the subject, in 1949, introduced two novel ideas—negative-temperature equilibria for two-dimensional ideal fluids and an energy-dissipation anomaly for singular Euler solutions—that stimulated much later work. However, a study of Onsager's letters to his peers around that time, as well as his private papers of that period and the early 1970's, shows that he had much more to say about the problem than he published. Remarkably, his private notes of the 1940's contain the essential elements of at least four major results that appeared decades later in the literature: (1) a mean field Poisson-Boltzmann equation and other thermodynamic relations for point vortices; (2) a relation similar to Kolmogorov's 4/5 law connecting singularities and dissipation; (3) the modern physical picture of spatial intermittency of velocity increments, explaining anomalous scaling of the spectrum; and (4) a spectral turbulence closure quite similar to the modern eddy-damped quasinormal Markovian equations. This paper is the summary of Onsager's published and unpublished contributions to hydrodynamic turbulence and an account of their place in the field as the subject has evolved over the years. A discussion is also given of the historical context of the work, especially of Onsager's interactions with his contemporaries who were acknowledged experts in the subject at the time. Finally, a brief speculation is offered as to why Onsager may have chosen not to publish several of his significant results.

The review is lucidly written and is a pleasure to read. The study of Onsager's unpublished work on turbulence uses documents from trusted sources such as the Caltech Archive and the Lars Onsager Archive in Trondheim. Modern research in turbulence which either stems from Onsager's work or is rediscovering his ideas is duly reviewed and referenced. This review will be useful to theoretical physicists, engineers, mathematicians and historians of science interested both in the fascinating subject of turbulence and in Lars Onsager's contribution to its understanding.

Oleg V. Zaboronsky

From MathSciNet, November 2020

MR2422377 (2009g:76008) 76B03; 76F02

Cheskidov, A.; Constantin, P.; Friedlander, S.; Shvydkoy, R.

Energy conservation and Onsager's conjecture for the Euler equations.

Nonlinearity **21** (2008), *no.* 6, 1233–1252.

The paper deals with certain conserved quantities for the Euler equations representing ideal incompressible fluid flows such as the kinetic energy in $\mathbb{R}^3$, the helicity in $\mathbb{R}^3$, and the enstrophy in two dimensions. The kinetic energy conservation, in particular, of a smooth (weak) solution of order greater than 'only' one third fractional derivatives is called Onsager's conjecture. A seemingly close connection to turbulent flows in the inviscid limit explains the importance of the conjecture.

This paper gives a satisfactory answer to Onsager's conjecture in the Besov spaces $B^{1/3}_{3,p}(\mathbb{R}^3)$. In fact, it is shown in the paper that the kinetic energy is preserved for velocities in the Besov space $B^{1/3}_{3,c(\mathbb{N})}(\mathbb{R}^3)$, where $c(\mathbb{N})$ is the space of sequences convergent to zero. It also presents an example of a divergence-free vector field in $B^{1/3}_{3,\infty}$ with energy flux of a positive lower bound in order to explain a possible break-down of the conservation in that space. The crucial observation is that the Littlewood-Paley energy flux of a divergence-free vector field $u \in L^2$ satisfies the following estimate:

$$\left| \int_{\mathbb{R}^3} Tr\,[S_q(u \otimes u) \cdot \nabla S_q u d]\, dx \right| \leq C(K * d^2)^{3/2}(q),$$

where the localization kernel K is a suitable (geometric) sequence and $C > 0$ is a constant independent of $d^2 := \{2^{\frac{2}{3}q} \|\Delta_q u\|^2_{-3}\}_{q \geq -1}$.

By the same analysis, it is shown that the helicity is conserved for every weak solution of the Euler equations that belongs up to $B^{2/3}_{3,c(\mathbb{N})}(\mathbb{R}^3)$, and the conservation may be violated by a divergence-free vector field in $B^{2/3}_{3,\infty}(\mathbb{R}^3)$. The authors also point out, by constructing a concrete example, that in two dimensions, the locality in the enstrophy cascade is strong only in the ultraviolet range. An (optimal) estimate for the trilinear map $(u, v, w) \mapsto \int_{\mathbb{R}^3} \mathbb{P}(u \cdot \nabla v) \cdot w \colon dx$ is presented in the final section.

The paper is well presented, so that the whole picture can be easily seen, and it provides a good guide to some important conserved quantities of the ideal incompressible fluid flows.

Hee Chul Pak
From MathSciNet, November 2020

MR2917063 76F02; 35D30, 35Q31, 35Q35

De Lellis, Camillo; Székelyhidi, László, Jr.

The h-principle and the equations of fluid dynamics.

American Mathematical Society. Bulletin. New Series **49** (2012), *no.* 3, 347–375.

This paper presents a survey of well-known results about Euler equations in fluid dynamics. As a by-product, it seeks to show how the h-principle can shed new light on the nonuniqueness of weak solutions for the incompressible Euler equations. Weak solutions and related problems of nonuniqueness are introduced in the first part. Theorems of existence are stated for compactly supported bounded weak solutions of the incompressible Euler equations in any space dimension. Being an heuristic candidate to explain nonuniqueness, oscillations are used in iteration schemes to generate solutions. This was theorized by the introduction of so-called subsolutions which are characterized by their behaviour with respect to some averaged energy and associated space of coarse-grained—that is, macroscopically averaged—solutions. Existence of global weak solutions is gained by the construction of a subsolution with bounded energy.

An alternative to the approximation of coarse-grained vector fields by sequences of weak solutions lies in the construction of Young measures which are parametrized probability measures accounting for weakly convergent oscillating sequences. High-frequency oscillations and concentrations in Euler flows are handled by measure-valued solutions derived from Young measures which give sense to weak convergence with respect to not necessarily bounded functions and present themselves as the sum of an oscillation measure, a concentration measure and a concentration-angle measure, therefore providing a featured description of the above-mentioned phenomena.

Discrimination between the infinitely numerous measure-valued solutions comes from the kinetic energy density which is nonincreasing for the classical solution, when it exists. That being the case, the associated Young measure coincides with the Dirac measure of the classical solution. In that respect, Lions introduced the class of dissipative weak solutions to which the natural energy constraint applies to gain back weak-strong uniqueness. These results come in contrast with the fact that classical criteria of energy boundedness inspired by the theory of hyperbolic equations do not restore uniqueness of weak solutions which originate in a very large space of wild initial data including the usual shear flow, namely a dense set in the space of L^2 solenoidal vector fields. Unlike the theory of hyperbolic conservation laws, although criteria of energy boundedness were formulated in the framework of measure-valued solutions, they do not yield uniqueness of the weak solution.

Many nondissipative systems of evolutionary partial differential equations fall in the scope of these methods after some additional considerations have been taken into account. A digression is made to introduce the so-called Baire-category method based on an auxiliary system with compact set-valued solutions. Interestingly, the set of solutions contains the above-mentioned subsolutions as long as they satisfy some compatibility condition involving the Λ-convex hull of the compact image and some wave cone. A few important examples are listed, starting with nonuniqueness theorems for admissible solutions of the system of isentropic gas dynamics in Eulerian coordinates—that is, bounded weak solutions satisfying an inequality of conservation type. The second example is that of active scalar equations, namely systems involving active unknowns, e.g. a scalar function and the velocity, coupled by an integral operator, as may occur in systems of partial differential equations of fluid dynamics. Under the hypothesis that the integral operator is translation invariant and after an adequate rewriting of the problem where the existence of a large set of plane solutions eventually comes to the fore, the additional considerations above are used to draw conclusions about nonuniqueness in geometrically involved problems such as the surface quasi-geostrophic and the incompressible porous medium equations. The arguments are based on specific tools developed in the theory of laminates and differential inclusions sharing common features with gradient vector fields. The article yields a detailed account of technical issues such as the computation of a Λ-convex hull and the analysis of the coarse-grained flow by Otto. It is noticed that the latter yields a good example of the ideas at work in these developments, namely the alternative to nonuniqueness of weak solutions lies in the observation that the associated coarse-grained density is a good candidate to discriminate between all the subsolutions.

The existence of dissipative weak solutions to the Euler equations was conjectured in a formalized statement known as Onsager's Conjecture in the framework of Hölder functions and then extended to L^2 and L^∞ functions. An alternative is presented on the basis of the above-mentioned Baire-category method, resulting in a strongly convergent iteration scheme. The method is illustrated by a scalar toy example.

An analogy is drawn between Euler's existence theorems and the Nash-Kuiper Theorem dealing with the isometric embedding problem where short embeddings are used to construct an essentially C^0-dense set of solutions thanks to the Gromov h-principle. More precisely, it is noticed that in the same way as the Reynolds stress measures the defect to being a solution to the Euler equations and is in general a nonnegative symmetric tensor, short embeddings measure the defect for an embedding to being isometric and are associated with a nonnegative symmetric tensor, therefore leading to the approximation that short maps may be seen as subsolutions to the isometric embedding problem. Noticing that the analogy extends to problems displaying sharp regularity threshold and overdetermination, the authors present a new proof of the h-principle in the framework of Hölder Riemannian metrics based on an iteration scheme in the spirit of the Nash-Kuiper Theorem. The Weyl problem about rigidity linked to overdetermination in the isometric embedding problem is also revisited.

To conclude, several open problems are listed. The analogy between Riemann problems and Euler equations may be fruitful as regards the latter since the former class is very large, even if some restriction is made because of the specific behavior of the pressure.

Isabelle Gruais
From MathSciNet, November 2020

MR3374958 35Q31; 35B65

Buckmaster, Tristan; De Lellis, Camillo; Isett, Philip; Székelyhidi, László, Jr.

Anomalous dissipation for 1/5-Hölder Euler flows.

Annals of Mathematics. Second Series **182** (2015), *no.* 1, 127–172.

A famous conjecture by L. Onsager asserts that, for any $\theta < 1/3$, there exist weak solutions to the 3-D incompressible Euler equations which are θ-Hölder continuous in space and which dissipate the energy during the evolution.

C. De Lellis and L. Székelyhidi have recently introduced a new viewpoint on the subject, based on linking Euler flows with differential inclusions and Gromov's h-principle. This new approach has allowed them to construct Hölder continuous weak solutions with prescribed (compactly supported in time) energy; they have been able to reach the Hölder exponent 1/10.

It is worth mentioning here that this new approach has been shown to be fruitful in many other models of fluid dynamics (e.g. compressible Euler equations, active scalar equations, porous-medium equations).

Coming back to the problem of dissipative Euler flows, P. Isett has improved the Hölder exponent to 1/5 by constructing solutions with compact support in space and time; on the other hand, he has lost any information on the exact shape of the energy.

The main goal of the present paper is to give a new and simpler proof of this result, based on the method of De Lellis and Székelyhidi; besides, this allows the possibility of prescribing the total kinetic energy.

The proof consists in implementing an iterative scheme. More precisely, at any step $n \in \mathbb{N}$ a solution (v_n, p_n, R_n) to the Euler-Reynolds system is constructed (where v_n is the velocity field, p_n the pressure and R_n the Reynolds stress), so that the difference between two consecutive solutions is kept small (in C^0 and C^1 norms). The smallness is measured in terms of a small amplitude parameter δ_n and a large frequency parameter λ_n; the precise link between these two quantities will determine the Hölder regularity of the limit solution. The perturbation for passing from step n to step $n+1$ is essentially a finite sum of modulated Beltrami modes, highly oscillating in space (with frequency λ_n) and with fixed direction; this ansatz ensures a special structure on the oscillatory part of the error terms appearing when solving the Euler-Reynolds systems.

The main points introduced by Isett, and exploited in the present paper (with some simplifications), are the following:

- to use non-linear phase functions in the ansatz (this is simplified in this work);
- to use space-time dependent directions of the vector fields in the ansatz;
- to define the phase functions of the Beltrami fields using the flow map related to the velocity fields v_n;

- to include additional (better) a priori estimates for the advective derivative of R_n, to be proved in the iteration process.

The implementation of all these ideas requires new technical ingredients when performing the proof; these novelties play a key role in the present paper.

Francesco Fanelli

From MathSciNet, November 2020

MR3866888 35Q31; 35A02, 35D30, 76B03, 76F02, 76F05

Isett, Philip

A proof of Onsager's conjecture.

Annals of Mathematics. Second Series **188** (2018), *no.* 3, 871–963.

The author proves the "negative" part of Onsager's famous conjecture for 3D incompressible Euler equations, that is, part (b) of the following conjecture.

Conjecture [L. Onsager, Nuovo Cimento (9) **6** (1949), Supplemento, no. 2 (Convegno Internazionale di Meccanica Statistica), 279–287; MR0036116]. Consider periodic 3-dimensional weak solutions of the incompressible Euler equations

$$\begin{cases} \partial_t v + (v\cdot\nabla)v + \nabla p = 0, \\ \operatorname{div} v = 0, \end{cases}$$

where the velocity v satisfies the uniform Hölder condition

$$|v(x,t) - v(x',t)| \le C|x-x'|^\alpha$$

for constants C and α independent of x, x' and t. (a) If $\alpha > \frac{1}{3}$, then the total kinetic energy $E(t) = \frac{1}{2}\int |v(t,x)|^2 dx$ is constant. (b) For any $\alpha < \frac{1}{3}$, there are v for which it is not constant.

Indeed, it is proved in Theorem 1 that for any $\alpha < \frac{1}{3}$ there is a nonzero weak solution to the incompressible Euler equations in the class

$$v \in C^\alpha_{t,x}, \quad p \in C^{2\alpha}_{t,x},$$

such that v is identically 0 outside a finite time interval. In particular, the solution v fails to conserve the energy.

Therefore, this paper gives the complete solution to part (b) of the conjecture. Partial results have previously been given in a series of papers by C. De Lellis and L. Székelyhidi Jr. [Invent. Math. **193** (2013), no. 2, 377–407; MR3090182; J. Eur. Math. Soc. (JEMS) **16** (2014), no. 7, 1467–1505; MR3254331] (where the failure of energy conservation is possible for solutions $v \in L^\infty_t C^\alpha_x$ for $\alpha < \frac{1}{10}$), by the present author himself in his thesis [*Holder continuous Euler flows with compact support in time*, Ph.D. thesis, Princeton Univ., 2013; MR3153420] (for the range $\alpha < \frac{1}{5}$, and a simpler proof is from [T. Buckmaster et al., Ann. of Math. (2) **182** (2015), no. 1, 127–172; MR3374958]), and by Buckmaster, De Lellis and Székelyhidi Jr. [Comm. Pure Appl. Math. **69** (2016), no. 9, 1613–1670; MR3530360] (for continuous solutions in the class $L^1_t C_x^{\frac{1}{3}-\epsilon}$).

The technique used in the paper under review is partially based on these previous results but improves some part of them, e.g., by making use of Mikado flows (used by S. Daneri and Székelyhidi Jr. [Arch. Ration. Mech. Anal. **224** (2017), no. 2, 471–514; MR3614753]) instead of the Beltrami ones.

The paper is very well written. There are four parts with 18 sections. The bibliography is very rich and the interested reader can follow all the details of the proofs. Some open questions are present in the introduction.

Benedetta Ferrario
From MathSciNet, November 2020

MR3896021 76B03; 35Q31

Buckmaster, Tristan; de Lellis, Camillo; Székelyhidi, László, Jr.; Vicol, Vlad

Onsager's conjecture for admissible weak solutions.

Communications on Pure and Applied Mathematics **72** (2019), *no.* 2, 229–274.

In this paper the authors study the statement of the Onsager conjecture concerning the behavior of solutions of the incompressible Euler equations in the case C^β with $\beta < 1/3$. In particular, P. Isett in [Ann. of Math. (2) **188** (2018), no. 3, 871–963; MR3866888] showed the existence of a non-conservative solution for which the kinetic energy fails to be monotone in any interval of time. The question of whether or not it is possible to construct solutions that in addition dissipate the kinetic energy remained open.

In this paper the same approach is modified to prove that if $e\,[0,t] \to \mathbf{R}$ is a strictly positive smooth function, then for any $0 < \beta < 1/3$ there exists a weak solution v of the Euler equations such that $v \in C^\beta(\mathbf{T}^3 \times [0,T])$, where $\mathbf{T}^3$ is the three-dimensional torus, and

$$\int_{\mathbf{T}^3} |v(x,t)|^2\,dx = e(t).$$

Two proofs of this result are given. The first one is rather short and self-contained and the second one follows as a corollary of a more general result on subsolutions.

The most important technical improvement is a precise estimate of the regions where the perturbation is added in the intermediate steps made with the Euler-Reynolds system. This fact, used in addition to the "gluing steps" and the Mikado flows, allows the authors also to prove an h-principle in the sense of [S. Daneri and L. Székelyhidi Jr., Arch. Ration. Mech. Anal. **224** (2017), no. 2, 471–514; MR3614753]. Namely, the second result is that if $(\overline{v},\overline{p},\overline{R})$ is a smooth subsolution of the Euler equations, then there exists a sequence (v_k,p_k) of weak solutions of the Euler equations such that $v_k \in C^\beta(\mathbf{T}^3 \times [0,T])$,

$$v_k \overset{*}{\rightharpoonup} \overline{v} \quad \text{and} \quad v_k \otimes v_k \overset{*}{\rightharpoonup} \overline{v} \otimes \overline{v} + \overline{R} \quad \text{in } L^\infty,$$

uniformly in time, and furthermore

$$\int_{\mathbf{T}^3} |v_k|^2\,dx = \int_{\mathbf{T}^3} (|\overline{v}|^2 + \operatorname{tr}\overline{R})\,dx.$$

Luigi Carlo Berselli
From MathSciNet, November 2020