BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 58, Number 2, April 2021, Pages 283–288
https://doi.org/10.1090/bull/1709
Article electronically published on January 7, 2021

*Introduction to approximate groups*, by Matthew C. H. Tointon, London Mathematical Society Student Texts, Vol. 94, Cambridge University Press, Cambridge, 2020, xiii+205 pp., ISBN 978-1-108-45644-9; 978-1-108-47073-5

**What is an approximate group?** At the time that this review is being written, the term "approximate group" is roughly twelve years old, and has probably not yet entered into mathematicians' common parlance. It was coined by Terry Tao in 2008 in his seminal paper [9], where it was used as a convenient language to discuss product sets in noncommutative groups. The term has since then been grounded in the additive combinatorics community.

Roughly speaking, an *approximate group* is a subset $\Lambda$ of a group $G$, assumed to be symmetric and to contain the identity element, which is "almost closed under multiplication". The latter condition, which is the key condition, can be spelled out in a myriad of different ways. My personal favourite is to assume that there is a finite set $F_\Lambda$ in the group $G$ such that the product set $\Lambda^2$ is contained in $\Lambda F_\Lambda$, that is to say, the product set is contained in a finite union of right translates of $\Lambda$. The smallest cardinality $D_\Lambda$ for a set $F_\Lambda$ as above is called the *doubling constant* of $\Lambda$, and we often refer to $\Lambda$ as a $D_\Lambda$-approximate group.

Since $\Lambda$ is assumed symmetric, $\Lambda^2$ is also contained in $F_\Lambda^{-1}\Lambda$, so it does not matter if you insist on left translates, rather than right translates. Note that if $D_\Lambda = 1$, then $\Lambda$ is really closed under multiplication, whence it is a subgroup of $G$, since $\Lambda$ is symmetric. What about approximate groups with $D_\Lambda = 2$? Well, this is a much trickier question.

**First examples.** For simplicity, let us confine our attention to $G = (\mathbb{Z}, +)$, the additive group of integers. Then a family of simple examples of 2-approximate groups in $G$ can be constructed as follows: for $N \geq 1$, consider the (symmetric) arithmetic progression

$$\Lambda_N = \{-N, \ldots, 0, \ldots, N\}. \tag{1}$$

We see that

$$\Lambda_N + \Lambda_N = \{-2N, \ldots, 2N\} = \Lambda_N + \{-N, N\},$$

and thus $\Lambda_N$ is a (finite) 2-approximate group in $G$.

To give a more complicated example, let

$$b = 3\sqrt{2} - 4 \approx 0,2426\cdots \quad \text{and} \quad I = [-b, b] \subset \mathbb{R}/\mathbb{Z},$$

and define the so-called *Bohr set*

$$\Lambda = \{n \in \mathbb{Z} \mid n\sqrt{2} \bmod 1 \in I\}. \tag{2}$$

It turns out that $\Lambda$ is a proper (infinite) approximate subgroup of $\mathbb{Z}$ with doubling constant equal to 2. To see this, note that $D_\Lambda > 1$ (since $\Lambda$ is not a subgroup), and note that $I + I = [-2b, 2b] = I + \{-b, b\}$ (inside $\mathbb{R}/\mathbb{Z}$, since $0 < b < 1/2$). Thus,

$$\begin{aligned}\Lambda + \Lambda & \\ &\subset \{n \in \mathbb{Z} \mid n\sqrt{2} \bmod 1 \in [-2b, 2b]\} = \{n \in \mathbb{Z} \mid n\sqrt{2} \bmod 1 \in [-b, b] + \{-b, b\}\} \\ &= \{n \in \mathbb{Z} \mid (n+3)\sqrt{2} \bmod 1 \in [-b, b]\} \cup \{n \in \mathbb{Z} \mid (n-3)\sqrt{2} \bmod 1 \in [-b, b]\} \\ &= (\Lambda - 3) \cup (\Lambda + 3) = \Lambda + \{-3, 3\},\end{aligned}$$

and hence $D_\Lambda = 2$. As far as I am aware, a complete classification of 2-approximate groups in $\mathbb{Z}$ (as well as in other groups), let alone of $D_\Lambda$-approximate groups for a fixed $D_\Lambda \geq 2$, is still lacking.

**Two general constructions.** Let $G$ and $H$ be groups, and let $\phi : G \to H$ be a homomorphism. Suppose that $W \subset H$ is an approximate group in $H$ with the property that there is a finite set $F$ in $G$ such that $W^2 \subset W\phi(F)$. Then $\Lambda = \phi^{-1}(W)$ is an approximate group in $G$. Indeed,

$$\Lambda^2 \subset \phi^{-1}(W^2) \subset \phi^{-1}(W\phi(F)) = \Lambda F.$$

In other words, preimages under homomorphisms of approximate groups are approximate groups, at least if the image of $\phi$ is "large enough" in $H$ to ensure that there is a finite set $F$ in $G$ such that $W^2 \subset W\phi(F)$.

The latter condition holds for example if $H$ is a topological group and $W$ is a compact symmetric identity neighbourhood in $H$. Indeed, in this case, one can always cover the compact set $W^2$ with right translates of the nonempty (open) interior $W^o$ by elements from the dense subgroup $\phi(G)$. The second example above (the Bohr set) is of this form with

$$G = \mathbb{Z} \quad \text{and} \quad H = \mathbb{R}/\mathbb{Z} \quad \text{and} \quad \phi(n) = n\sqrt{2} \bmod 1 \quad \text{and} \quad W = [-b, b],$$

since $\phi(\mathbb{Z})$ is a dense subgroup of $\mathbb{R}/\mathbb{Z}$.

Once you have an approximate subgroup $\Lambda$ of a given group $G$ at your disposal, you can construct tons of new approximate subgroups in $G$ by passing to large (symmetric) subsets of $\Lambda$. This works as follows. Suppose that $\Xi$ is a symmetric subset of $\Lambda$, containing the identity, which is large enough in $\Lambda$ in the sense that there is a finite set $M \subset G$ such that $\Lambda \subset \Xi M$. Then $\Xi$ is also an approximate subgroup of $G$. Indeed, since $\Lambda$ is an approximate group, there is a finite set $F_\Lambda$ such that $\Lambda^2 \subset \Lambda F_\Lambda$, and thus

$$\Xi^2 \subset \Lambda^2 \subset \Lambda F_\Lambda \subset \Xi M F_\Lambda.$$

In particular, $D_\Xi \leq |M| D_\Lambda$, where $D_\Xi$ and $D_\Lambda$ denote the doubling constants of $\Xi$ and $\Lambda$, respectively.

**A generalization.** It might be prudent here to stress that it is not necessary in the first construction above to assume that $\phi$ is a homomorphism, and that with some minor modifications, the same argument goes through if $\phi$ is only assumed to be a quasi-morphism between $G$ and $H$. Let us spell out these modifications. Recall that a map $\phi : G \to H$, where $G$ is an abstract group and $H$ is a topological group, is a (*symmetric*) *quasi-morphism* if $\phi(e) = e$ and $\phi(g)^{-1} = \phi(g)^{-1}$ for all $g \in G$, and the set

$$Q_\phi = \{\phi(y)^{-1}\phi(x)^{-1}\phi(xy) \mid x, y \in G\}$$

is precompact in $H$. Note that every homomorphism is a quasi-morphism, as is every function with precompact image.

If $H$ is abelian, the product of a homomorphism and a function with precompact images is a quasi-morphism. Such quasi-morphisms are called *trivial*. If $G$ is discrete and amenable and $H$ is abelian, then only trivial quasi-morphisms exist. However, nontrivial quasi-morphisms (into abelian groups) exist in abundance for many hyperbolic groups.

Suppose that $\phi$ is a quasi-morphism with dense image in $H$. Let $W \subset H$ be a compact and symmetric set in $H$ which contains the identity, which is large enough in the sense that the intersection $W' := \bigcap_{q \in Q_\phi} Wq^{-1}$ has nonempty interior in $H$. We claim that the preimage $\Lambda = \phi^{-1}(W)$ is an approximate group in $G$. First note that our two first conditions on $\phi$, together with the symmetricity and identity containment for $W$, implies that $\Lambda$ is a symmetric set in $G$ which contains the identity element. Furthermore,

$$\begin{aligned}\phi(\Lambda^2) &= \big\{\phi(xy) \mid \phi(x), \phi(y) \in W\big\} \\ &= \big\{\phi(x)\phi(y)\,(\phi(y)^{-1}\phi(x)^{-1}\phi(xy)) \mid \phi(x), \phi(y) \in W\big\} \subset W^2 Q_\phi.\end{aligned}$$

Since $Q_\phi$ is precompact and $W$ is compact, $W^2 Q_\phi$ is precompact. Since $W'$ has nonempty interior and $\phi(G)$ is dense in $H$, we have $W'\phi(G) = H$. In particular, $W'\phi(G)$ is an open covering of the precompact set $W^2 Q_\phi$, and thus there is a finite subset $L \subset G$ such that $W^2 Q_\phi \subset W'\phi(L)$. We conclude that

$$\Lambda^2 \subset \phi^{-1}(W'\phi(L)) = \bigcup_{l \in L} \phi^{-1}(W'\phi(l)).$$

It thus suffices to show that $\phi^{-1}(W'\phi(l)) \subset \Lambda l$ for all $l \in L$. This follows from the inclusions,

$$\begin{aligned}\phi^{-1}(W'\phi(l)) &= \big\{x \in G \mid \phi(x) \in W'\phi(l)\big\} = \big\{x \in G \mid \phi(x)\phi(l^{-1}) \in W'\big\} \\ &= \big\{x \in G \mid \phi(xl^{-1})\,(\phi(l^{-1})^{-1}\phi(x)^{-1}\phi(xl^{-1}))^{-1} \in W'\big\} \\ &\subset \big\{x \in G \mid \phi(xl^{-1}) \in W'Q_\phi\big\} \subset \big\{x \in G \mid \phi(x) \in W\big\} l = \Lambda l,\end{aligned}$$

where in the last inclusion we have used that $W'Q_\phi \subset W$. In particular, $\Lambda^2 \subset \Lambda L$, and thus $\Lambda$ is an approximate group in $G$.

**Classification results.** One of the fundamental questions in approximate group theory (which is still a very young, but active, subject) is whether all approximate groups really just stem from some combination of the two constructions outlined above. More precisely, are all approximate groups in the world simply *large* subsets of preimages of *nicer* approximate groups (say arithmetic progressions or compact sets with nonempty interiors) under quasi-morphisms (perhaps modulo some minor technicalities)? This, admittedly vague, question has been answered in the affirmative for two special, but very important, classes of approximate groups.

*Finite approximate groups.* In the early 1960s Freiman initiated the study of *finite* approximate groups in (torsion-free) abelian groups, and he proved they are (in a quantitative manner) large subsets of (higher-dimensional) arithmetic progressions. His work was later extended by Green and Ruzsa to cover all abelian groups. More recently, Breuillard, Green, and Tao [2] established a far-reaching generalization, which states that (modulo finite subgroups, in a technical sense), every finite approximate subgroup of an abstract group is essentially nilpotent, and contained in a nilpotent version of a Bohr set.

*Euclidean quasi-crystals* (*approximate lattices*). In the late 1960s Meyer, motivated by some problems in Diophantine properties of Pisot numbers, initiated the study of infinite, but uniformly discrete and relatively dense, approximate subgroups of $\mathbb{R}^d$. These sets are nowadays often referred to as (Euclidean) quasi-crystals or approximate lattices. Meyer showed that every approximate lattice in $\mathbb{R}^d$ is a *large* (relatively dense) subset of a so-called cut-and-project set. These sets are preimages of compact sets under homomorphisms from a subgroup of $\mathbb{R}^d$ into locally compact abelian groups, and thus they fit the scheme described above. For general nonabelian groups, no such classification result is currently known (see however the recent preprints [6, 7] by Machado for an extension of Meyer's result to solvable Lie groups).

**Where do approximate groups appear?** Approximate groups have appeared in many different areas of mathematics long before the name "approximate group" was known. The simultaneous appearance of approximate groups in additive combinatorics and the theory of mathematical quasi-crystals has already been mentioned.

For a very different example, let $\Gamma$ be a finitely generated group, and let $d$ be a word metric with respect to some finite symmetric generating set of $\Gamma$. Gromov has proved that if $\Gamma$ has polynomial growth with respect to $d$, then it is virtually nilpotent. The assumption of polynomial growth is, however, equivalent to the existence of an an infinite sequence of $d$-balls in $\Gamma$ which are $k$-approximate groups for some fixed integer $k$. Shalom and Tao [8], and later Breuillard, Green, and Tao [2], have used this connection to give a new proof of Gromov's theorem based on approximate groups.

Approximate groups have also appeared in analytic group theory. For example, Cornulier [3, Example 1.12] has constructed examples of discrete "non-a-(T)-menable" (non-Haagrup) groups which do not have Property (T) relative to any infinite subgroup, but which do have Property (T) relative to infinite *approximate* subgroups.

The construction of (approximate) groups as preimages of compact identity neighbourhoods in locally compact groups is very familiar to number theorists in the case where the target group is totally disconnected and the identity neighbourhood is actually a compact open subgroup. This construction is then often referred to as a special instance of a Hecke correspondence. For example, let $p$ be a prime, and let $\phi$ denote the identity map on $\mathrm{SL}_n(\mathbb{Z}[1/p])$, where the domain is viewed as a (dense) subgroup of $\mathrm{SL}_n(\mathbb{R})$ and the target is viewed as a (dense) subgroup of $\mathrm{SL}_n(\mathbb{Q}_p)$. Let $W := \mathrm{SL}_n(\mathbb{Z}_p)$, and note that $\Lambda := \phi^{-1}(W) = \mathrm{SL}_n(\mathbb{Z})$, which is a discrete subgroup of $\mathrm{SL}_n(\mathbb{R})$, produced by *killing* the non-Archimedean place. In the larger world of approximate groups, we are also allowed to *kill Archimedean places.*

**What are approximate groups good for?** Here, I am not looking for the answer, They are good for proving theorems in additive combinatorics. Rather, I am asking about applications in other areas, not a priori related to additive combinatorics. Perhaps the most convincing applications so far have appeared in sieving theory and in the construction of expander graphs (see the excellent surveys by Breuillard [1] and Green [4] for more detailed expositions). However, it is not always easy to properly illuminate the (key) role played by approximate groups in these applications. Indeed, in nature, away from the zoo that is additive combinatorics, approximate groups are exceedingly stealthy animals and thrive best deep inside technical lemmas, where they lurk in the shadows until they are being chased out into the daylight, in the seventeenth and final subcase to be dealt with before the proof is over. There are perhaps less evasive breeds of approximate groups, but to the best of my knowledge, they have not yet been discovered.

**About the book.** This is a book about *finite* approximate groups, and it concentrates for the most part on various special cases of the Breuillard, Green, and Tao (BGT) theorem. There are eleven chapters in total. In Chapter I, approximate groups are defined and put in an historical context. In Chapter II, some rudiments of approximate group theory are outlined. In particular, the behaviour of approximate groups under iterations, intersections and (Freiman) homomorphisms are discussed in detail. In Chapters III and IV Bohr sets and coset progressions are motivated from the viewpoint of geometry of numbers, and the powerful inverse theorem of Freiman, Green, and Ruzsa for abelian groups is proved. In Chapter V and VI the analysis is extended to nilpotent groups. Chapters III–VI cover almost 100 pages and form the technical core of the book.

In Chapter VII the general version of the BGT theorem is briefly surveyed. The proof is too difficult to get into as it requires heavy tools from a diverse field of different subjects. The author outlines a self-contained proof in the special setting of *residually* nilpotent groups in Chapter VIII.

In the subsequent chapters, the book changes its perspective. The focus is now rather on what the *absence* of interesting approximate subgroups in nonnilpotent groups says about general product sets. The key phenomena here are summarized under the umbrella of *sum-product theorems.* In Chapter IX and X, Tointon discusses such results in the setting of linear groups over the complex numbers. In the eleventh and final chapter, the author surveys a few applications to growth in groups, most notably to Gromov's theorem on groups of polynomial growth (which is presented as a fairly straightforward consequence of the BGT theorem) and to the construction of expander graphs.

As the author points out in the preface, the theory of approximate groups is still being worked out, and "the book should therefore be thought of as giving a snapshot of the current state of an active research topic". That said, an aspiring student who wants to enter the world of approximate groups will surely find the first chapters of the book, which cover the fundamentals, invaluable. Moreover, anyone willing to climb the mountain that is the BGT theorem should be grateful for the webbing ladders laid out in Chapters IV–VI. Less ambitious readers might still enjoy the small gems, scattered throughout the text, like Solymosi's sum-product theorem in Chapter IX or the Sanders–Croot–Sisask power set argument in Chapter X, both of which are a delight to read.

Although there is some overlap between the material presented in Chapters I–IV with Chapters 2–5 in Tao and Vu's book [10] as well as with the preliminary chapters in the more recent book [5] by Grynkiewicz, this is perhaps the first book that provides a systematic treatment of approximate groups as a mathematical subject. It is very likely to become one of standard texts in this rapidly developing field.

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