BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 58, Number 2, April 2021, Pages 289–294
https://doi.org/10.1090/bull/1720
Article electronically published on January 15, 2021

Nonlinear Dirac equations: Spectral stability of solitary waves, by N. Boussaïd and A. Comech, Mathematical Surveys and Monographs, Vol. 244, American Mathematical Society, Providence, RI, 2019, vi+297 pp., ISBN 978-1-4704-4395-5

1. History of the Dirac equation

The Dirac equation was written down by Paul Dirac in December 1927 as the relativistically invariant equation for electrons. Its discovery was preceeded by the following developments in quantum physics.

In 1923, following discussions of Max Planck's and Albert Einstein's research on wave-particle duality in the context of photons, Louis de Broglie formulated an idea that the electrons could be described as waves. Attracted by this idea, Erwin Schrödinger wrote down in 1925 a relativistically invariant equation for the wave function of the electron, which is now known as the Klein–Gordon equation,

$$\frac{\partial^2\psi}{\partial t^2} + m^2c^4\psi - c^2\nabla^2\psi = 0, \tag{1}$$

where $\psi(t,x) : \mathbb{R}\times\mathbb{R}^3 \mapsto \mathbb{C}$, m is electron's mass, and c is the speed of light.

The electron energy levels (which were known from experiments) were recovered from the new equation; however, the relativistic corrections to the energy levels of the hydrogen atom were inconsistent with the experimental measurements. As a result, Schrödinger shelved his relativistic equation and retreated to the non-relativistic limit, where he derived what is now known as the Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2m}\nabla^2\psi = 0. \tag{2}$$

The main reason why an electron's wave function cannot be described by the Klein–Gordon equation (1) is that under spatial rotations, an electron's wave function transforms according to the four-dimensional spinorial representation of $\mathbf{SO}(3)$, whereas the Klein–Gordon equation is written for the scalar wave function. The relativistically invariant equation of Paul Dirac was intended to cure this problem. In particular, it yielded an accurate explanation of the relativistic corrections to the energy levels of the hydrogen atom. It also allowed Dirac to predict the existence of the positron a year before it was triumphally discovered by Carl Anderson in 1932.

As was noted by Viktor Weisskopf and widely cited since then, "a great deal more was hidden in the Dirac equation than the author had expected when he wrote it down. Dirac himself remarked in one of his talks that his equation was more intelligent than its author."

2. Background on stability of solitary waves

Since the time of developments in quantum mechanics, nonlinear versions of the Klein–Gordon, Schrödinger, and Dirac equations have been proposed in many modeling problems, e.g., in atomic physics, nonlinear optics, molecular dynamics, and general relativity. Supported by the balance between nonlinear and linear terms, solitary waves appear to be spatially localized solutions which either travel or oscillate in time. If such solutions are stable with respect to the time evolution, they are important in practical applications of such models.

For the nonlinear Schrödinger (NLS) equation of the form

$$i\partial_t\psi + \nabla^2\psi + |\psi|^{2\kappa}\psi = 0, \tag{3}$$

where $\psi(t,x) : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{C}$ and $\kappa > 0$, solitary waves are solutions of the form $\psi(t,x) = \phi_\omega(x)e^{i\omega t}$, where $\omega > 0$ is a free parameter and $\phi(x) \to 0$ as $|x| \to \infty$. Their stability is defined by using the concept of orbital stability, which is needed to eliminate transformations of solitary waves along symmetries of the NLS equation.

Definition 1. The solitary wave $\psi(t,x) = \phi_\omega(x)e^{i\omega t}$ of the NLS equation (3) is called orbitally stable in the Banach space X with the norm $\|\cdot\|_X$ if for every $\epsilon > 0$ there is $\delta > 0$ such that for every initial data $\psi_0 \in X$ such that $\|\psi_0 - \phi\|_X < \delta$, there exists a unique global solution $\psi \in C(\mathbb{R}, X)$ to the NLS equation (3) satisfying the bound

$$\inf_{\theta\in\mathbb{R},\xi\in\mathbb{R}^n} \|\psi(t,\cdot) - e^{i\theta}\phi_\omega(x-\xi)\|_X < \epsilon, \quad t \in \mathbb{R}.$$

It was realized in the 1980s that the orbital stability can be proven by exploring the Hamiltonian formulation of the NLS equation (3) in the energy space, e.g., in $X = H^1(\mathbb{R}^n)$. Moreover, the proof of the orbital stability is developed based on the spectral information about linearization of the NLS equation (3) at the solitary wave $\psi(t,x) = \phi_\omega(x)e^{i\omega x}$. In the case of real-valued ϕ_ω, this linearization is achieved by using the decomposition

$$\psi(t,x) = e^{i\omega t}\left[\phi_\omega(x) + u(t,x) + iv(t,x)\right],$$

where u and v are real. The linearized equations of motion are

$$u_t = L_- v, \quad v_t = -L_+ u, \tag{4}$$

where

$$\begin{aligned} L_- &= -\nabla^2 + \omega - \phi_\omega^{2\kappa}, \\ L_+ &= -\nabla^2 + \omega - (2\kappa+1)\phi_\omega^{2\kappa}. \end{aligned}$$

The self-adjoint operators $L_\pm : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \mapsto L^2(\mathbb{R}^n)$ are bounded from below. Since $\omega > 0$ and $\phi_\omega(x) \to 0$ exponentially as $|x| \to \infty$, the absolutely continuous spectrum of $L_\pm$ is strictly positive and is bounded from below by ω (by Weyl's theorem on the essential spectrum). Hence, there exist finitely many negative eigenvalues of finite multiplicity, the total number of which is denoted by $n(L_\pm)$, and a zero eigenvalue of finite multiplicty, denoted by $z(L_\pm)$. The zero eigenvalue exists due to $\mathbf{U}(1)$-invariance and translational symmetries of the NLS equation (3).

Related to the linearized system (4), one can define a weaker stability concept for the solitary waves of the NLS equation (3).

Definition 2. The solitary wave $\psi(t,x) = \phi_\omega(x)e^{i\omega t}$ of the NLS equation (3) is called spectrally stable if $\lambda \in i\mathbb{R}$ for every solution $(u,v) \in H^2(\mathbb{R}^n)$ of the non-self-adjoint spectral problem

$$L_- v = \lambda u, \quad -L_+ u = \lambda v. \tag{5}$$

The following theorem was proven by M. Grillakis, J. Shatah, and W. Strauss in 1987 [3] as a generalization of the earlier studies of J. Shatah and W. Strauss [8], and M. Weinstein [10]. As it happens often, the same result was actually proven as early as 1973 in the works of N. G. Vakhitov and A. A. Kolokolov [9] and has been widely cited in physics literature as the Vakhitov–Kolokolov stability criterion.

Theorem 1 (Weinstein, 1986; Grillakis, Shatah, and Strauss, 1987)**.** *The solitary wave* $\psi(t,x) = \phi_\omega(x)e^{i\omega t}$ *of the NLS equation* (3) *is orbitally stable in the sense of Definition* 1 *if* $n(L_+) = 1$, $n(L_-) = 0$, $z(L_+) = n$, $z(L_-) = 1$, *and the map* $\omega \mapsto \|\phi_\omega\|^2_{L^2}$ *is monotonically increasing.*

If the spectral information on $n(L_\pm)$ and $z(L_\pm)$ does not allow us to prove the orbital stability theorem, one can often retreat to analysis of spectral stability of solitary waves, from which further analysis can be developed to understand nonlinear dynamics of perturbations to the solitary waves. The spectral stability results appeared in the papers of M. Grillakis [2] and C. Jones [4], but a complete development on the subject was only done 15 years later in the works of T. Kapitula, P. Kevrekidis, and B. Sandstede [5] and in the works of the author of this review [7] (see also [1, 6] for generalizations).

Theorem 2 (Kapitula, Kevrekidis, and Sandstede, 2004; Pelinovsky, 2005)**.** *The solitary wave* $\psi(t,x) = \phi_\omega(x)e^{i\omega t}$ *of the NLS equation* (3) *is spectrally stable in the sense of Definition* 2 *if* $n(L_+) = n(L_-) + 1$, $z(L_+) = n$, $z(L_-) = 1$, *the map* $\omega \mapsto \|\phi_\omega\|^2_{L^2}$ *is monotonically increasing, and there exist* $n(L_-)$ *pairs of purely imaginary eigenvalues of the spectral problem* (5) *counting their algebraic multiplicities with the eigenvectors* $(u,v) \in H^2(\mathbb{R})$ *satisfying* $\langle L_+u,u\rangle + \langle L_-v,v\rangle < 0$.

Given the success in analysis of orbital and spectral stability of solitary waves in the NLS equation in Theorems 1 and 2, one can wonder if the same problem can be easily solved in the nonlinear generalization of the relativistically invariant Dirac equation. The answer to this question is known to be negative, because the linearized operators for the nonlinear Dirac equations are unbounded both from above and from below so that indices $n(L_\pm)$ are not defined. This property is related to sign-indefiniteness of the energy of the Dirac equations. This is why each result on stability of solitary waves in the nonlinear Dirac equations is so valuable. The monograph under review covers some results obtained after the ten-year work of its two authors (and several other researchers) on the subject.

3. The monograph

As the title suggests, this monograph is devoted to the spectral stability of solitary waves in the nonlinear Dirac equations. In some sense, the entire monograph of 297 pages is devoted to just one nonlinear partial differential equation written in the form

$$i\partial_t\psi = D_m\psi - f(\psi^*\beta\psi)\beta\psi,$$

where $\psi(t,x) : \mathbb{R}\times\mathbb{R}^n \mapsto \mathbb{C}^N$ and the Dirac operator D_m with mass $m > 0$ is given by

$$D_m = -i\alpha\cdot\nabla + \beta m : \quad L^2(\mathbb{R}^n,\mathbb{C}^N) \mapsto L^2(\mathbb{R}^n,\mathbb{C}^N),$$

with α^i, $1 \le i \le n$, and β being the Dirac matrices. The nonlinearity f is represented by $f \in C^1(\mathbb{R}\backslash\{0\}) \cap C(\mathbb{R})$ satisfying $f(z) = |z|^\kappa + \mathcal{O}(|z|^K)$ with $0 < \kappa < K$.

The main result of this monograph is the proof of spectral stability of solitary waves in the weakly relativistic limit for weakly subcritical and critical nonlinearities $\kappa \lesssim 2/n$ and $\kappa = 2/n$, where the criticality is defined for the limiting nonlinear Schrödinger equation (3).

In particular, the spectral stability of solitary waves is proven for the quintic nonlinear Dirac equations in one spatial dimension with $n = 1$ and $\kappa = 2$. Note that the nonlinear Schrödinger equation (3) is L^2-critical for $\kappa = 2/n$ and that its solitary waves are unstable in H^1 norm in the L^2-critical case.

Solitary waves of the nonlinear Dirac equation enjoy better stability properties than their nonrelativistic counterparts in the nonlinear Schrödinger equation. This is rather surprising because the kinetic energy of the Dirac equation is sign-indefinite which could potentially lead to developing instabilities. Even more surprisingly, the nonlinear Dirac equation does not seem to have blow-up instability, which makes it particularly attractive from the modeling point of view. This stability of solitary waves in the nonlinear Dirac equation does not have a consistent explanation yet, being hidden under the wealth of features such as its internal symmetries. Further interpretations of these facts are expected to be achieved in the future from both physical and mathematical viewpoints.

As the authors admit on the first page, their argument only applies for subcritical values of $\kappa \in (0, 2/n)$ sufficiently close to $2/n$ since the point spectrum of the limiting nonlinear Schrödinger equation linearized at the solitary wave becomes rich for small κ.

The main result described above is based on the authors' research published in the sequence of papers:

- J. Funct. Anal. **271** (2016), 1462–1524;
- SIAM J. Math. Anal. **49** (2017), 2527–2572;
- Commun. Pure Appl. Anal. **17** (2018), 1331–1347;
- J. Funct. Anal. **277** (2019), 108289.

The proof of the main result requires many specialized analytical tools to be mastered, such as the limiting absorption principle and the Carleman–Berthier–Georgescu estimates. The authors describe all these technical tools in great detail before they focus on the particular studies of the nonlinear Dirac equations; this is why the proof of the main result has been expanded as the full-scale monograph.

The overview of specialized analytical tools follows a remarkable introduction to functional-analytic methods for partial differential equations, comprehensive and self-contained spectral theory for non-self-adjoint linear operators, where, in particular, Weyl's theorem on the essential spectrum is proven in the general framework of Banach spaces, and mathematical properties of the nonlinear Dirac and Schrödinger equations. These introductory chapters feature many examples and problems suitable for graduate students and useful for many researchers. Thus, the book starts as a graduate text, evolves like a specialized monograph, and ends with the state-of-the-art in the study of the nonlinear Dirac equations and the spectral stability of solitary waves.

Next, I will describe the structure of this book. The first five chapters cover the introductory material suitable for graduate students and general audience:

(1) the history of the electron theory and the nonlinear Dirac equations;
(2) the distribution theory, Sobolev spaces (with a detailed derivation of the Gagliardo–Nirenberg–Sobolev inequalities), the Pólya–Szegő inequality, and the Paley–Wiener theorem;

(3) the spectral theory of non-self-adjoint linear operators in Banach spaces, with the Gohberg–Krein theory of normal eigenvalues (the discrete spectrum), the Weyl theorem on the essential spectrum, the Schur complement theory, and the Keldysh theory of characteristic roots;
(4) the linear stability of solitary waves in the nonlinear Schrödinger equation;
(5) the existence theory for solitary waves in the nonlinear Schrödinger equation.

The next three chapters give the tools needed for specialized analysis of the Dirac operators:

(6) the limiting absorption principle;
(7) the Carleman–Berthier–Georgescu estimates;
(8) the Dirac–Pauli theorem on the choice of the Dirac matrices.

The final five chapters collect together the main technical results on the spectral stability of weakly relativistic solitary waves in the nonlinear Dirac equation with weakly subcritical and critical nonlinearities. The presentation covers

(9) properties of the Soler model;
(10) the bifrequency solitary waves (this is a new class of solutions discovered by the authors and is important for their arguments);
(11) bifurcations of eigenvalues from the essential spectrum;
(12) nonrelativistic asymptotics of solitary waves;
(13) spectral stability in the nonrelativistic limit.

The spectral stability results obtained in this monograph open the way to the proofs of asymptotic stability of solitary waves, with only very few results obtained presently in the context of the nonlinear Dirac equations. Thus, this monograph can be considered an introduction to an active area of research with much more work to be done in the future.

In summary, the monograph is clearly written and includes a historical perspective, the results are well-motivated and ample background is provided, numerous well-chosen references are given, and the subject of studies is properly focused. The book is suitable as a textbook on spectral theory and as an introduction into nonlinear wave equations.

References

[1] M. Chugunova and D. Pelinovsky, *Count of eigenvalues in the generalized eigenvalue problem*, J. Math. Phys. **51** (2010), no. 5, 052901, 19, DOI 10.1063/1.3406252. MR2666968

[2] M. Grillakis, *Analysis of the linearization around a critical point of an infinite-dimensional Hamiltonian system*, Comm. Pure Appl. Math. **43** (1990), no. 3, 299–333, DOI 10.1002/cpa.3160430302. MR1040143

[3] M. Grillakis, J. Shatah, and W. Strauss, *Stability theory of solitary waves in the presence of symmetry. I*, J. Funct. Anal. **74** (1987), no. 1, 160–197, DOI 10.1016/0022-1236(87)90044-9. MR901236

[4] C. K. R. T. Jones, *An instability mechanism for radially symmetric standing waves of a nonlinear Schrödinger equation*, J. Differential Equations **71** (1988), no. 1, 34–62, DOI 10.1016/0022-0396(88)90037-X. MR922197

[5] T. Kapitula, P. G. Kevrekidis, and B. Sandstede, *Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems*, Phys. D **195** (2004), no. 3-4, 263–282, DOI 10.1016/j.physd.2004.03.018. MR2089513

[6] T. Kapitula and K. Promislow, *Stability indices for constrained self-adjoint operators*, Proc. Amer. Math. Soc. **140** (2012), no. 3, 865–880, DOI 10.1090/S0002-9939-2011-10943-2. MR2869071

[7] D. E. Pelinovsky, *Inertia law for spectral stability of solitary waves in coupled nonlinear Schrödinger equations*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **461** (2005), no. 2055, 783–812, DOI 10.1098/rspa.2004.1345. MR2121936

[8] J. Shatah and W. Strauss, *Instability of nonlinear bound states*, Comm. Math. Phys. **100** (1985), no. 2, 173–190. MR804458

[9] N. G. Vakhitov and A. A. Kolokolov, *Stationary solutions of the wave equation in a medium with nonlinearity saturation*, Radiophys. Quantum Electron **16** (1973), 783–789.

[10] M. I. Weinstein, *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math. **39** (1986), no. 1, 51–67, DOI 10.1002/cpa.3160390103. MR820338

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