SELECTED MATHEMATICAL REVIEWS
related to the papers in the previous section on the work of
JEAN BOURGAIN

Lindenstrauss, Joram

Weakly compact sets—their topological properties and the Banach spaces they generate.

The paper under review is a survey of the finer structure of weakly compact sets in Banach spaces. It was written in 1967 when the understanding of this structure was still quite incomplete. The paper contains a list of 16 open problems whose study has been the motivation behind much of the most beautiful abstract functional analysis of the past decade. Lindenstrauss’ survey still bears careful reading and the review below will concentrate on the progress that has been made regarding the problems set forth.

Problem 1: Let $X$ be a Banach space which is generated by a $w$-compact set. Is every closed subspace of $X$ generated by a $w$-compact set? No. H. P. Rosenthal [see MR0417762 below] has shown that there exists a probability measure $\mu$ and a subspace of $L_1(\mu)$ which is not WCG. On the other hand, K. John and V. Zizler [J. Functional Analysis 15 (1974), 1–11; MR0417759 above] have shown that if $X$ is a subspace of WCG space and $X$ has an equivalent Fréchet differentiable norm or if $X$ and $X^*$ are both subspaces of WCG spaces then $X$ is WCG. It seems plausible at this time that the following is true: if $X$ is a subspace of a WCG and $X^*$ has the Radon-Nikodým property then $X$ is WCG. Another case of good heredity is due to Y. Benyamini, M. E. Rudin and M. Wage [“Continuous images of weakly compact subsets of Banach spaces”, preprint]: $C(K)$ subspaces of WCG spaces are WCG.

Problem 2: Is it true that a Banach space is generated by a $w$-compact set if and only if it is Lindelöf in its $w$-topology? M. Talagrand [Bull. Sci. Math. 99 (1975), 211–212] has shown that every WCG Banach space enjoys the Lindelöf property in its weak topology; however, the Rosenthal example cited above is an example of a non-WCG space whose weak topology is Lindelöf. Open is whether or not having a Lindelöf weak topology characterizes subspaces of WCG spaces.

Problem 3: Let $X$ be a Banach space such that $X^*$ is generated by a $w$-compact set. Is $X$ generated by a $w$-compact set? R. D. James [“On a conjecture about $l_1$ subspaces”, preprint] has constructed a space JT, now known as James’ tree space with the startling properties that all the even duals of JT are WCG while none of the odd duals are. Independently, W. B. Johnson and the author [see MR0417760 above] constructed a space $X$ with Fréchet differentiable norm where dual is WCG though $X$ is not WCG; in the same paper Johnson and Lindenstrauss present several stability results for WCG spaces, in particular, if $X$ and $X^*$ are WCG then every subspace of $X$ is WCG. This result in turn has been extended by K. John and V. Zizler to read: if both $X$ and $X^*$ are subspaces of some WCG then every subspace of $X$ is WCG. It is noteworthy that the Johnson-Lindenstrauss
and James examples are quite different each with other revealing peculiarities than those cited above; a good source for the pathology inherent in JT is the paper of the author and C. Stegall [Studia Math. 54 (1975), no. 1, 81–105; MR0390720].

Problem 4: Let $K$ be an Eberlein compact and let $H$ be a continuous image of $K$. Is $H$ an Eberlein compact? Yes. This is the main result of the aforementioned paper of Benyamini, Rudin and Wage. Their solution, by the way, is dependent on the elegant topological classification of Eberlein compacts due to Rosenthal [see MR0417762 below].

Problem 5: Let $X$ be a Banach space such that the unit cell $S^*$ of $X^*$ is an Eberlein compact in its $w^*$-topology. Is $X$ generated by a weakly compact set? Benyamini, Rudin and Wage showed that $S^*$ is Eberlein compact if and only if $X$ is a subspace of a WCG space.

Problem 6: Let $K$ be a compact Hausdorff space. Is $C(K)$ $w$-Lindelöf if and only if $K$ is an Eberlein compact? Talagrand’s result cited above answers half this question affirmatively while the question of whether $K$ is Eberlein compact whenever $C(K)$ is weakly Lindelöf seems to be still open.

Problem 6′: Let $K$ be a compact Hausdorff space. Is $C(K)$ Lindelöf in the topology of pointwise convergence if and only if $K$ is an Eberlein compact? This one seems to be relatively untested as of this date.

Problem 7: Let $K$ be a $w$-compact convex subset of a Banach space in which every point is a $G_δ$ point. Is $K$ metrizable? In a note added in proof, the author notes a negative response. Recently, Benyamini, Rudin and Wage have shown that every Eberlein compact contains a dense $G_δ$ metrizable set of $G_δ$ points.

Problem 8: Let $X$ be a strictly convex Banach space. Does there exist a one-to-one linear operator from $X$ into $c_0(\Gamma)$ for some $\Gamma$? Not necessarily. F. K. Dashiell and the author [Israel J. Math. 16 (1973), 329–342; MR0348466] provided several natural counter-examples.

Regarding Problem 9 (are smooth spaces embeddable in WCG spaces), Problem 10 (are smooth spaces strictly convexifiable) and Problem 11 (characterize those compact Hausdorff spaces $K$ such that $C(K)$ is strictly convexifiable or smoothable) the reviewer is unaware of any substantial progress.

Problem 12: Let $X$ be a Banach space generated by a weakly compact set. Does $X$ have an equivalent locally uniformly convex norm? In particular, does every reflexive space have an equivalent locally uniformly convex norm? This was perhaps the first of Lindenstrauss’ problems that was answered and the positive response has some very spectacular consequences. The respondent was S. L. Trojanski [Studia Math. 37 (1970/71), 173–180; MR0306873]. As a consequence of Trojanski’s result and the ingenious averaging method discovered by E. Asplund [Israel J. Math. 5 (1967), 227–233; MR0222610], every reflexive Banach space can be renormed in such a way that the space and its dual are locally uniformly convex and have a Fréchet differentiable norm. This answers the second part of Problem 13: Let $X$ be a Banach space such that $X^*$ is generated by a weakly compact set. Does $X^*$ have an equivalent Fréchet differentiable norm (at every $x \neq 0$)? In particular, does every reflexive space have an equivalent Fréchet differentiable norm? The first part of the above problem is still open. Thus far all the evidence is towards a positive solution. John and Zizler [ibid. 12 (1972), 331–336; MR0344853] have shown that if $X$ and $X^*$ are WCG then $X$ has an equivalent norm whose dual norm is locally uniformly convex, smooth and whose second dual norm is strictly convex.
Actually, there has been some very spectacular progress on a closely related problem. Recall that a Banach space is called an Asplund space [cf. I. Namioka and R. R. Phelps, Duke Math. J. 42 (1975), no. 4, 735–750; MR0390721] if each continuous real-valued convex function on an open convex domain in \( X \) has a dense domain of Fréchet differentiability. Namioka and Phelps noted that if \( X \) is an Asplund space then \( X^* \) has the Radon-Nikodým property (similarly if \( X \) has a Fréchet differentiable norm). I. Ekelund and G. Lebourg [“Generis Fréchet differentiability and perturbed optimization problems in Banach spaces”, preprint] were able to show that a space with Fréchet differentiable norm is an Asplund space. More recently, C. Stegall (private communication) was able to show that if \( X^* \) has the Radon-Nikodým property then \( X \) is an Asplund space thereby giving extremely strong evidence that Problem 13’s response is affirmative.

Problem 14: Is it true that \( B(X, Y) = B_K(X, Y) \) if and only if \( K \) is weakly compact and \( TK \) is norm compact for every \( T \in B(X, Y) \)? (Notation: \( B_K(X, Y) \) denotes the operators \( T: X \to Y \) for which \( \|Tx\| \) attains a max on \( K \).) This problem seems to not yet have received much attention. Related to this is a recent result of J. Bourgain (“On dentability and the Bishop-Phelps property”, to appear in Israel J. Math.) showing that the question of existence of many norm attaining operators is equivalent to the Radon-Nikodým property.

Problem 15: Is every weakly compact convex set in a Banach space the closed convex hull of its strongly exposed points? Yes. Trojanski’s affirmative response to Problem 12 implies an affirmative response to this problem. Recently, Bourgain [Proc. Amer. Math. Soc. 58 (1976), 197–200; MR0415272] has given a more elementary proof of this result. Also recent is the surprising relationship between the existence of many strongly exposed points in arbitrary closed bounded convex sets and the Radon-Nikodým property established by Phelps [J. Functional Analysis 7 (1974), 78–90; MR0352941]. Finally, Bourgain (see the reference cited above in Israel J. Math.) has shown that a closed bounded convex set in a Banach space is subset dentable if and only if each of its closed convex subsets is the closed convex hull of its strongly exposed points.

Problem 16: Let \( X \) be a conjugate space which is also generated by a \( w \)-compact set. Is every closed convex and bounded subset of \( X \) the closed convex hull of its extreme (perhaps even exposed or strongly exposed) points? Yes. This result was obtained by Namioka [Pacific J. Math. 51 (1974), 515–531; MR0370466]. However, Namioka’s solution only touches the tip of a geometric bonanza. The complete characterization of dual spaces for which the above strong form of the Kreĭn-Mil’man theorem holds was provided by R. E. Huff and P. D. Morris [Proc. Amer. Math. Soc. 49 (1975), 104–108; MR0361775], who, building on some deep results of Stegall [Trans. Amer. Math. Soc. 206 (1975), 213–223; MR0374381], showed that for \( X^* \) to have the property that each closed bounded convex set be the closed convex hull of its extreme points it is necessary that \( X^* \) have the Radon-Nikodým property. Earlier, J. Lindenstrauss had shown that the Radon-Nikodým property always implies this strong version of the Kreĭn-Mil’man theorem whether in a dual space or not. It remains open whether or not this Kreĭn-Mil’man property implies the Radon-Nikodým property in general.

{For the entire collection see MR0346790.}

J. Diestel
From MathSciNet, February 2021
Diestel, Joseph

Sequences and series in Banach spaces. (English)
Graduate Texts in Mathematics, 92.

This work is a very useful introduction to the linear topological, geometrical and structural theory of Banach spaces. Several topics, especially $w^*$-compactness and unconditional convergence, are worked out in detail and many very recent discoveries are presented. It should be pointed out that the author by no means aims to give a total description or picture of the field, which in this limited space would make the book much less readable.

In infinite-dimensional Banach space theory, a central role is played by the sequence spaces $c_0$ and $l^1$. Nowadays, we know of many characterizations of their presence in a given space. In the book, several of them are described in a detailed way. This topic intersects also the geometrical theory, centered around the extremal structure of convex sets. In this context, the author covers the structure of uniformly convex spaces, the Krein-Milman-Choquet theory and the various later developments, emphasizing the role of convexity and extreme points for the global structure of the space. A different direction is the beautiful characterization due to H. P. Rosenthal of the existence of $l^1$-subsequences belonging to descriptive set theory and motivating the author to include an introduction to Ramsey theory in his book.

From the point of view of connections and applications to classical analysis, the most penetrating part of modern Banach space theory lies in the so-called local theory. The starting point is the $p$-summing operators and Grothendieck’s fundamental theorem, to which the book provides a brief introduction and surveys related literature. Again, a treatment of this subject in more depth and in a self-contained style would make the book much longer.

The author presents his selection of results in fourteen sections. The four initial ones deal with various notions of compactness and convergence, centered around the Eberlein-Shmulyan and Orlicz-Pettis theorems. Also, here an illustration of the concepts by concrete examples is indeed the best way to motivate them. Section 5 presents generalities on basic sequences, for instance the Bessaga-Pelczynski construction and its application to $c_0$ and $l^1$. The examples of the Schauder and Haar system, important in classical approximation, are discussed. In Section 6, the reader is initiated into the theory of $p$-summing operators, the classical Grothendieck-Pietsch factorization principle and the Dvoretzky-Rogers theorem. Section 7 is a summary of some main properties of the classical Banach spaces $c_0$, $l^1$, $l^\infty$, $(l^\infty)^*$ and $L^p[0, 1]$ ($1 \leq p < \infty$). The theory of finite measures (including Rosenthal’s and Phillips’ lemma, etc.) and a detailed proof of Khinchin’s inequality are included. Sections 8 and 9 belong to the geometrical theory. Section 8 is focussed on the theorems of Kakutani, Kadets and Gurarii on the weak and unconditional convergence in uniformly convex spaces. In a different spirit, Section 9 describes various aspects of extreme point theory. The well-known theorem of M. Krein and Milman, and Choquet’s integral representation, are the starting point. This section includes some highlights in the purely geometrical theory of infinite-dimensional convex bodies and extreme point characterizations of $c_0$. Section 10 contains a proof of Grothendieck’s theorem, an introduction to $L^p$ theory and uniqueness of
unconditional structure. Further reading on those subjects is indicated through a detailed discussion. The next chapter contains an initiation to Ramsey’s theorem, mainly for the needs of the proof of H. Rosenthal’s fundamental result on \( l^1 \)-embedding, presented in Section 11. The special theme of Sections 12 and 13 is the \( w^* \)-convergence of sequences in dual spaces and the relation with the presence of \( l^1 \)-subspaces. The first Baire class plays a central role in those questions. Several tools developed earlier in the book are employed to obtain the main result of the last section, stating that each infinite-dimensional Banach space contains a \((1 + \varepsilon)\)-separated sequence.

The book seems suited for students with some knowledge of functional analysis (especially in view of the elementary level of several parts and the many exercises). For the analyst in general, it gives some idea of the research done over recent years in the theory of infinite-dimensional Banach spaces. An extensive discussion of the relevant literature broadens the actual content of the book considerably and makes it a valuable source for references.

J. Bourgain

From MathSciNet, February 2021

MR1678031 (2000a:05183) 05C80; 42C10

Friedgut, Ehud

Sharp thresholds of graph properties, and the \( k \)-sat problem. (English)

A graph property \( \varphi \) is a predicate on finite, undirected graphs that is invariant under graph automorphism. A graph property \( \varphi \) is said to be monotone if \( \varphi(H) \Rightarrow \varphi(G) \) whenever \( H \) is a spanning subgraph of \( G \). “Containing a triangle” and “being connected” are examples of monotone graph properties. An important phenomenon in the theory of the Erdős-Rényi random graph model \( G_{np} \) is the existence of thresholds for monotone graph properties. Thus, for any \( \varepsilon > 0 \), a random graph \( G \in G_{np} \) with \( n \) vertices and edge-probability \( p \) is almost surely not connected when \( p = (1 - \varepsilon)n^{-1} \ln n \) and almost surely connected when \( p = (1 + \varepsilon)n^{-1} \ln n \). (Here, “almost surely” means “with probability 1 as \( n \to \infty \).”) Observe that there is a well-defined critical probability \( p_c = n^{-1} \ln n \), and that the change from almost sure non-connectedness to almost sure connectedness occurs over an interval that is small in relation to \( p_c \). Connectivity is thus an example of a graph property with a sharp threshold.

In contrast, as \( p \) is raised from 0.86 \( n^{-1} \) to 2.40 \( n^{-1} \), the probability that a random graph \( G \in G_{np} \) contains a triangle rises from about 0.1 to about 0.9. “Containing a triangle” is an example of a graph property with a coarse threshold, since the onset of the property is not precisely located; one can say no more than that it occurs at edge-probability \( p \) of order \( n^{-1} \). There is a clear distinction between the two graph properties \( \varphi \) we have considered: in one case, there is a finite collection of graphs (in fact the singleton \( \{K_3\} \)) such that \( \varphi(G) \) is equivalent to \( G \) containing at least one graph in the collection as a subgraph; in the other case there is no such collection. Another way of expressing this distinction is to say that the minimal graphs (with respect to subgraph order) in the first case form a finite set, and in the second an infinite one.

Now consider the property \( \varphi(G) \) described by the statement “\( G \) contains a triangle or is connected”. The property \( \varphi \) is, in an almost sure sense, equivalent to the
simple property of containing a triangle, and therefore exhibits a coarse threshold; yet the set of all minimal graphs satisfying \( \varphi \) is infinite. Nevertheless, \( \varphi \) is closely approximated by another property—containing a triangle—whose minimal graphs do form a finite (indeed singleton) set. The striking main result of this article is that in fact every monotone graph property exhibiting a coarse threshold can be closely approximated by another whose minimal graphs are all “small”. To be a little more precise, it is necessary to realise that what I have been calling a monotone graph property \( \varphi \) is more properly a family \((\varphi_n: n \in \mathbb{N})\) of monotone properties of \( n \)-vertex graphs. (Monotonicity constrains \( \varphi \) only on pairs of graphs with the same number of vertices.) The main result is then that there exists a function \( k(\varepsilon, c) \) satisfying the following: for any \( n \), any monotone graph property \( \varphi_n \) with a coarse threshold, and any \( \varepsilon > 0 \), there exists a monotone graph property \( \psi_n \) such that (a) \( \Pr(\varphi_n(G) \oplus \psi_n(G)) \leq \varepsilon \) and (b) the minimal graphs satisfying property \( \psi_n \) have size (i.e., number of edges) at most \( k(\varepsilon, c) \). Here, \( G \) is assumed to be chosen according to the random graph model \( G_{np} \), and \( c \) is a parameter, depending on \( \varphi_n \), that measures the coarseness of the threshold.

The result rests on a simple, elegant idea, but turning this idea into a proof presents a major challenge. Very briefly, one regards a graph property as a real function \( f \) on \( Z_2^n \)—where \( m = \binom{n}{2} \), and 0 and 1 encode “false” and “true”—and considers the Fourier-Walsh transform of \( f \). In the case of a coarse threshold, the coefficients of this transform are concentrated on a small (up to isomorphism) collection of well-behaved subgraphs. Retaining just these large coefficients and applying the inverse transform one obtains a function \( \tilde{f} \) that approximates \( f \) and that is largely determined by small subgraph counts. Finally it is necessary to approximate \( \tilde{f} \) by a Boolean function.

The author applies his techniques to a question concerning random CNF Boolean formulas, which had been open for some time. Let \( x_1, \ldots, x_n \) be Boolean variables, and consider a random CNF formula obtained by conjoining \( M \) clauses selected uniformly at random from the set of all \( 2^k \binom{n}{k} \) possible (nontrivial) clauses with exactly \( k \) literals from the set \( \{ x_1, \neg x_1, \ldots, x_n, \neg x_n \} \). It had been conjectured, partly on the basis of computer simulation, that satisfiability of such random formulas exhibited a sharp threshold: specifically, that there is a constant \( c_k \) such that the formula is almost surely satisfiable if \( M \leq (c_k - \varepsilon)n \) and almost surely unsatisfiable if \( M \geq (c_k + \varepsilon)n \). This conjecture is settled positively here, with the proviso that \( c_k \) may possibly be a function of \( n \) rather than a universal constant.

The proofs in the paper rely crucially on graph properties being invariant under graph isomorphism. The author speculates on whether symmetry is necessary for the results. This question is partly answered in an appendix written by Jean Bourgain. The exact relationship between the results in the main article and the appendix are too complicated to describe here; suffice it to say, (a) the main result of the appendix does not assume symmetry but is weaker in some other respect and does not imply the main result of the article proper, and (b) the methods of the appendix can be used to shorten some proofs in the article. In many applications of interest, either result can be applied.

Mark R. Jerrum

From MathSciNet, February 2021
A sharp bilinear cone restriction estimate.

The main result of this groundbreaking paper is a bilinear estimate for the adjoint of the Fourier restriction operator in \( \mathbb{R}^d \), where \( d \geq 3 \). Consider two disjoint conical subsets \( \Gamma_1, \Gamma_2 \),

\[ \Gamma_i = \{ x = (x', x_d) : x_d = |x'|, x'/|x_d| \in \Omega_i \} \]

where \( \Omega_1 \) and \( \Omega_2 \) are separated closed subsets of the unit sphere. Then the estimate

\[ \| \hat{f} \hat{g} \|_{L^p(\mathbb{R}^d)} \lesssim \| f \|_2 \| g \|_2 \]

holds for two functions \( f \) and \( g \) supported in \( \Gamma_1 \) and \( \Gamma_2 \), respectively, provided that \( p > 1 + 2/d \). In three dimensions this estimate provides an affirmative answer to a conjecture by S. Klainerman and M. Machedon, but Wolff actually proves a sharp bound (modulo endpoints) in all dimensions. The relevant range is \( p \in (1 + 2/d, 1 + 2/(d - 2)) \) as for \( p \geq 1 + 2/(d - 2) \) the bound follows from the Strichartz estimate. Only very partial results in three and four dimensions were previously known; see articles by J. Bourgain [in Geometric aspects of functional analysis (Israel, 1992–1994), 41–60, Birkhäuser, Basel, 1995; MR1353448] and by T. C. Tao and A. M. Vargas [Geom. Funct. Anal. 10 (2000), no. 1, 185–215; MR1748920]. A corollary of Wolff’s theorem is a sharp \( L^p \) result for the Fourier restriction operator on the cone in \( \mathbb{R}^4 \) (here \( p < 3/2 \)).

In a very interesting appendix the author establishes new \( L^p \rightarrow L^q \) results for families of X-ray transforms associated to cones, as well as improved mixed norm estimates. The bounds are essentially best possible in three or four dimensions, while in dimensions \( d \geq 5 \) partial but sharp results are given in the range \( p \leq (d + 1)/2 \).

This article, and [Geom. Funct. Anal. 10 (2000), no. 5, 1237–1288; MR1800068], appear to be the last two journal publications by Thomas Wolff, who tragically died in July 2000. Both contributions are highly original masterpieces and should have a long lasting impact in analysis.

Andreas Seeger

From MathSciNet, February 2021
The heart of the proof is §4, which the authors say is based on the recent solution to the Erdős ring problem by G. A. Edgar and C. Miller [Proc. Amer. Math. Soc. 131 (2003), no. 4, 1121–1129 (electronic); MR1948103]. The argument is elegant.

The rest of the paper is devoted to applications. We state simple versions of the three main results.

Theorem 6.2 (Szemerédi-Trotter type theorem). Let $0 < \alpha < 2$, and let $P$ be a set of $p^\alpha$ points in $\mathbb{F}_p^2$, and let $L$ be a set of $p^\alpha$ lines. Then the number of pairs $(p, l) \in P \times L$ such that $p \in l$ is less than $C(\alpha)p^{3\alpha/2-\varepsilon}$, where $\varepsilon = \varepsilon(\alpha) > 0$.

Theorem 7.1 (Distance sets). Let $0 < \alpha < 2$, and suppose that $P \subseteq \mathbb{F}_p^2$ has cardinality $p^\alpha$. Then the number of “distances”, that is to say elements of the form $(x_1 - x_2)^2 + (y_1 - y_2)^2$ with $(x_1, y_1), (x_2, y_2) \in P$, is at least $C(\alpha)p^{\alpha/2+\varepsilon}$ for some $\varepsilon = \varepsilon(\alpha) > 0$.

Theorem 8.1 (Kakeya problem). Let $A \subseteq \mathbb{F}_p^3$ be a set which contains a line in each of the $p^2 + p + 1$ different directions. Then $|A| > Cp^{5/2+\varepsilon}$ for some $\varepsilon > 0$.

In large part the interest in these issues is in their analogues in Euclidean spaces. Full answers to these questions would have wide-ranging implications in analysis and combinatorics. The finite field models discussed in this paper allow one to strip away much of the complexity of the Euclidean situation (in particular, there is no notion of “scale”) and as such provide a pleasant framework for thinking about these important problems.

The main result of this paper is already finding important applications. For example S. V. Konyagin and the first author have used it in their spectacular new estimates for exponential sums over subgroups of $(\mathbb{Z}/p\mathbb{Z})^*$ [C. R. Math. Acad. Sci. Paris 337 (2003), no. 2, 75–80; MR1998834].

REVISED (February, 2005)

Ben Joseph Green
From MathSciNet, February 2021
the one that gives the concept its name: there is some constant \( c > 0 \), independent of \( p \), such that for any set \( X \) consisting of at most one half the vertices of \( \mathcal{G}_p \) the 1-neighbourhood \( N_1(X) = X \cup \{ y : xy \in E(\mathcal{G}_p) \} \) has size at least \((1 + c)|X|\). The second property, which is nontrivially equivalent to the first, is that the lim sup as \( p \to \infty \) of the second largest eigenvalue \( \lambda_1(\mathcal{G}_p) \) is strictly less than \( d \). The expansion property, which has been extensively written about in many places, should be thought of as asserting that the family \( \mathcal{G}_p \) is in a sense a family of pseudorandom graphs. A great deal more on expanders and their importance may be found in the article [S. Hoory, N. Linial and A. Wigderson, Bull. Amer. Math. Soc. (N.S.) \textbf{43} (2006), no. 4, 439–561; MR2247919].

There is another interesting result in this paper, namely that by taking a random set of \( 2k \) generators \( \{g_1^{\pm 1}, \ldots, g_k^{\pm 1}\} \) for a Cayley graph on \( \text{SL}_2(\mathbb{F}_p) \), for each \( p \), we almost surely get a family of graphs whose second largest eigenvalues are bounded away from \( 2k \) as \( p \to \infty \).

We focus on the ideas behind the proof of the first theorem. Fix a prime \( p \) and consider the probability measure \( \mu_S(x) = |S|^{-1} \sum_{g \in S} \delta_g(x) \) which places equal mass on each point of \( S \). The first main idea is to use the trace formula to conclude that

\[
\frac{1}{N} \sum_{j=0}^{N-1} \lambda_j^2 = (2k)^{2m} \mu_S(2m)(1),
\]

where the eigenvalues of \( \mathcal{G}_p \) are listed as \( 2k = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_{N-1} \geq -2k \) and \( \mu_S^{(j)} \) denotes the \( j \)th convolution power of \( \mu_S \).

Thus the main business of the paper is to examine these convolution powers \( \mu_S^{(j)} \). The first step (Proposition 4) is merely stated; the proof may be found in [A. Gamburd, Israel J. Math. \textbf{127} (2002), 157–200; MR1900698]. The proposition claims that the graphs \( \mathcal{G}_p \) have girth at least \( c \log p \) for an appropriate constant \( c > 0 \). This means that if \( l_0 < \frac{1}{2}c \log p \) then the measure \( \mu_S^{(l_0)} \) (and hence all measures \( \mu_S^{(l)} \) with \( l \geq l_0 \)) is fairly “spread out”, meaning that it does not resemble a \( \delta \) peak too closely. More specifically we have a bound \( \|\mu_S^{(l)}\|_\infty < p^7 \), which may be thought of as saying, roughly, that the support of \( \mu_S^{(l)} \) behaves like a subset \( \text{SL}_2(\mathbb{F}_p) \) of size at least \( p^7 \).

The next stage of the argument consists in bootstrapping this information rather considerably by looking at repeated convolution squares, that is to say by looking at the relationship between \( \nu = \mu_S^{(j)} \) and \( \nu \ast \nu = \mu_S^{(2j)} \). For each \( j = l_0, 2l_0, 4l_0, \ldots \) one of two possibilities eventually occurs: either (i) \( \nu \ast \nu \) is “not much more spread out” than \( \nu \), meaning that \( \|\nu \ast \nu\|_2 > p^{-\epsilon}\|\nu\|_2 \), or (ii) \( \nu \) is almost uniform in the sense that \( \|\nu\|_2 < p^{-3/2+\epsilon} \) (note that \( |\text{SL}_2(\mathbb{F}_p)| \sim p^3 \)).

Suppose for the moment that (ii) holds, in which case we have a convolution power \( \mu_S^{(l_1)} \), \( l_1 \sim C_{\epsilon,k} \log p \), which is almost uniformly distributed in the sense that \( \|\mu_S^{(l_1)}\|_2 < p^{-3/2+\epsilon} \). A little representation theory, specifically the fact that \( \text{SL}_2(\mathbb{F}_p) \) has no nontrivial representations of degree less than \((p-1)/2\), then allows one to conclude that still somewhat higher convolution powers \( \mu_S^{(l_2)} \) are extremely uniform and hence to obtain a bound on an appropriate \( \mu_S^{(2m)}(1) \) to use in the trace formula mentioned at the start of the review. This fact, that in groups with no small-dimensional representations convolution smoothes things out very dramatically, seems to have first been observed in a related context by P. C. Sarnak and X. Xue [Duke Math. J. \textbf{64} (1991), no. 1, 207–227; MR1131400]. More recently it has been elaborated upon and placed in a more general context by

It remains to rule out the possibility that (i) holds, and this is done using the techniques of additive combinatorics together with some combinatorial group theory. Here is a very brief summary. Supposing that \(|\|\nu\|_2 > p^{-\epsilon}||\nu||_2|\), a rather tedious but essentially straightforward decomposition of \(\nu\) into level sets produces as a set \(A \subseteq \text{SL}_2(F_p)\) with \(|A| < p^{1-\gamma}\) with large “additive energy”, that is to say with many solutions to \(xy = zw\). By T. C. Tao’s noncommutative version of the Balog-Szemerédi-Gowers theorem [Combinatorica 28 (2008), no. 5, 547–594; MR2501249] one may locate an “approximate group” \(H\) related to \(A\). By the work of H. A. Helfgott [Ann. of Math. (2) 167 (2008), no. 2, 601–623; MR2415382] any such approximate subgroup \(H\) must fail to generate all of \(\text{SL}_2(F_p)\). By classifying the proper subgroups of \(\text{SL}_2(F_p)\), one sees that \(H\) must in fact be contained in a 2-step solvable group \(G_0\). Working backwards, it follows that the measure \(\nu\) concentrates near a coset of this solvable group \(G_0\), and it is this possibility which must be ruled out in order to complete the argument. A result of Kesten concerning walks in the free group is applied to conclude that if this were the case then many different words of length \(l_0\) in the generating set \(S\) would lie in \(G_0\). A combinatorial group theory argument is then applied to contradict this, essentially because the solvability forces too much commutation between the aforementioned words.

This last paragraph, in particular, has been a very brief sketch. However, the reviewer hopes that it has adequately conveyed the amazingly rich array of ideas which have been brought to bear on this problem.

Ben Joseph Green
From MathSciNet, February 2021

MR2892611 11F72; 11N36
Bourgain, Jean; Gamburd, Alex; Sarnak, Peter
Generalization of Selberg’s \(3/16\) theorem and affine sieve.

This impressive paper is concerned with non-elementary (i.e., the commutator of some pair of elements of infinite order does not have trace 2) finitely generated subgroups \(\Lambda\) of \(\text{SL}_2(\mathbb{Z})\) which may be quite “thin”. In particular the authors are concerned with finding asymptotics for the number of elements \(\gamma \in \Lambda\) such that \(||\gamma|| \leq T\) and \(\gamma \equiv g \mod q\), where \(g\) is some fixed matrix and \(q\) can be as large as \(T^{\kappa_\Lambda}\) for some \(\kappa_\Lambda > 0\). Here, \(||\gamma||\) is the Archimedean norm, defined to be the \(\ell^2\)-norm of the matrix entries of \(\gamma\).

Such a bound can be used to deduce results about matrices in \(\Lambda\) having prime or almost-prime entries, or more generally for which some polynomial \(f(a, b, c, d)\) in the matrix entries has prime or almost-prime entries. This further develops a theme first announced in [J. Bourgain, A. Gamburd and P. C. Sarnak, C. R. Math. Acad. Sci. Paris 343 (2006), no. 3, 155–159; MR2246331] and developed in subsequent papers of the authors. The result (Theorem 1.6 of the paper) is that if \(\Lambda\) is Zariski-dense then one has an upper bound within a constant factor of the expected truth for primes, and a matching lower bound for almost primes, that is to say for numbers with \(O(1)\) prime factors. This is the situation one would expect from a basic application of the combinatorial sieve, and indeed improvements to the sieve itself are not the aim of this paper.
The main result of the paper (Theorems 1.3 and 1.4) is the asymptotic formula
\[ |\{ \gamma \in \Lambda : \| \gamma \| \leq T, \gamma \equiv g \pmod{q} \}| \sim c_{\Lambda} T^{2\delta(\Lambda)} |SL_2(q)|,\]
uniformly for \( g \in SL_2(q) \) and \( q \leq T^{\kappa_{\Lambda}} \), provided that \( q \) is squarefree and coprime to some modulus \( q_0 \) “provided by the strong approximation theorem”, that is to say for which the image of \( \Lambda \) under projection \( (\pmod q) \) is surjective. Here, \( \delta(\Lambda) \) is the Hausdorff dimension of the limit set of \( \Lambda \), which is taken to be positive. The content of this result is that as \( \gamma \) ranges over the ball
\[ \{ \gamma \in \Lambda : \| \gamma \| \leq T \}, \]
the projections of \( \gamma \pmod{q} \) equidistribute in \( SL_2(q) \), even for somewhat large values of \( q \).

The proof is different according as \( \delta(\Lambda) > \frac{1}{2} \) or \( \delta(\Lambda) < \frac{1}{2} \), the first case being easier (and leading to better error terms in the above asymptotic formula). It is beyond this reviewer to say anything meaningful about the proofs, save to remark that for \( \delta(\Lambda) > \frac{1}{2} \) the \( L^2 \)-methods of Lax and Phillips are applied, whereas for \( \delta(\Lambda) < \frac{1}{2} \) techniques of Lalley coming from symbolic dynamics are applied.

Indeed, a detailed reading of the paper would severely test any experienced mathematician. To give one example, this reviewer recalls being mystified, as a Ph.D. student, about how to construct a bump function such as the one whose existence is stated in (9.4). (The details of this particular construction may be found in [Y. Katznelson, An introduction to harmonic analysis, second corrected edition, Dover, New York, 1976; MR0422992 (Chapter V, Lemma 2.7)].)

Ben Joseph Green
From MathSciNet, February 2021

MR3374964 42B37; 11E76, 46E30, 53C40

Bourgain, Jean; Demeter, Ciprian

The proof of the \( l^2 \) decoupling conjecture. (English)


\[ \| f \| \leq C(\delta) \left( \sum_{\vartheta \in P_\delta} \| f_{\vartheta} \|_p^2 \right)^{1/2} \]
for a function \( f \) with the Fourier transform \( \hat{f} \) having support in the \( \delta \)-neighborhood \( N_\delta \) of a \( C^2 \) compact hyper-surface \( S \) in \( \mathbb{R}^n \) possessing a positive definite second quadratic form. By \( P_\delta \) one denotes a finite multiplicity covering of \( N_\delta \) by rectangular boxes \( \vartheta \) having size of the order \( \delta^{1/2} \times \cdots \times \delta^{1/2} \times \delta \), the last dimension directed ‘almost’ along the normal to \( S \), and \( f_{\vartheta} \) denotes the Fourier restriction of \( f \) to \( \vartheta \). The main results consist of proving the inequality in question for the largest possible interval of values of \( p \), with ‘almost’ sharp order of dependence of the constant \( C(\delta) \) on \( \delta \). The classical result by Wolff has a number of striking applications, including the ‘local smoothing for the wave equation’, the regularity for convolutions with arclength measure on helices and the boundedness of the Bergman projection in
tube domains over full light cones. Applications of the new theorem improve the above ones; additionally they provide the full range of expected $L^p_{x,t}$ Strichartz estimates for both the rational and irrational tori. Also, sharp estimates are derived for additive energies of various sets. The most unexpected range of applications involve almost sharp estimates for the number of solutions of various Diophantine inequalities.

G. V. Rozenblum
From MathSciNet, February 2021

MR3548534 11P05; 11N25
Bourgain, Jean; Demeter, Ciprian; Guth, Larry

This remarkable paper is a landmark contribution to that part of analytic number theory connected with Waring’s problem and the estimation of Weyl sums $\sum_{x=1}^N e(p(x))$, $p$ a polynomial. (Here $e(t) = e^{2\pi it}$. The authors succeed in completely resolving, up to factors of $N^\epsilon$, the so-called “Vinogradov mean-value conjecture”.

This is an estimate for the quantity $J_{s,n}(N)$, the number of solutions to the system of $n$ equations

$$X_1^i + \cdots + X_s^i = Y_1^i + \cdots + Y_s^i, \quad i = 1, \ldots, n,$$

with $X_i, Y_i$ integers less than or equal to $N$, which may also be written as the mean value

$$\int_{\deg p=n} \left| \sum_{x=1}^N e(p(x)) \right|^{2s},$$

where the integral is over all polynomials $p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$, the uniform measure being placed on coefficient sequences $(\alpha_0, \ldots, \alpha_n) \in (\mathbb{R}/\mathbb{Z})^{n+1}$.

The estimate obtained by the authors is

$$J_{s,n}(N) \ll N^{s+\epsilon} + N^{2s-n\frac{(n+1)}{2}+\epsilon}.$$

Up to the factors of $N^\epsilon$, this is best possible: the first term counts the trivial solutions in which $\{X_1, \ldots, X_s\} = \{Y_1, \ldots, Y_s\}$, and the second term counts solutions which are forced to exist on density grounds (the tuple $(\sum X_1, \sum X_1^2, \ldots, \sum X_1^n)$ takes values in a set of size $\sim N^{\frac{1}{2}n(n+1)}$).

Previously, this estimate was only completely known for $s \leq 3$ due to deep work of T. D. Wooley [Adv. Math. 294 (2016), 532–561; MR3479572]. Other work of Wooley starting with [Ann. of Math. (2) 175 (2012), no. 3, 1575–1627; MR2912712] provided approximations to the conjecture in certain ranges when $s \geq 4$.

The proof of the conjecture is long and very involved. Perhaps the most important thing to be said about it is that it is a method of Euclidean harmonic analysis, whereas the previous work of Wooley involved consideration of congruences modulo a suitable prime $p$. One remarkable consequence of this is that the argument works just as well if the $X_i$ (and the $Y_i$) are merely assumed to be 1-separated, rather than integers.
It is beyond the capability of the reviewer to give anything like a meaningful description of the argument here, save to repeat the authors’ comments that the three major ingredients are estimates for the multilinear Kakeya problem due to J. M. Bennett, A. Carbery and T. C. Tao [Acta Math. 196 (2006), no. 2, 261–302; MR2275834], an induction on scales argument and an elaborate iteration scheme. Anyone seriously interested in reading the paper would be well advised to start with the overview found in [J. Bourgain, “On the Vinogradov mean value”, preprint, arXiv:1601.08173] or the blog post of T. Tao [“Decoupling and the Bourgain-Demeter-Guth proof of the Vinogradov main conjecture”, posted December 10, 2015].

It is natural to ask what implications the new results have for Weyl sums and for Waring’s problem. According to Bourgain’s preprint cited above, we now know that the asymptotic formula in Waring’s problem for sums of $s$ $k$th powers holds when $G(k) \geq k^2 - \omega(k)$, where $\omega(k) \geq 0$ is a certain quantity (the previous best bound was $\approx \frac{1}{54}k^2$, due to Wooley). Writing $G(k)$ for the smallest $s$ such that every sufficiently large integer is a sum of $s$ $k$th powers, Wooley has noted that, for example, the new methods may be used to obtain $G(7) \leq 31$ whereas previously one only knew $G(7) \leq 33$.

Regarding Weyl sums, Bourgain remarks that the new methods lead to the estimate

$$\left| \sum_{x \leq N} e(\alpha x^n + \ldots) \right| \ll N^{1+\epsilon}(q^{-1} + N^{-1} + qN^{-k})^{1/k(k-1)}$$

under the assumption that $|\alpha - \frac{a}{q}| \leq \frac{1}{q^2}, (a, q) = 1$. This is an improvement over existing results for $k \geq 7$.

Ben Joseph Green

From MathSciNet, February 2021

MR3678500 37C30; 11B30, 37F30

Bourgain, Jean; Dyatlov, Semyon

Fourier dimension and spectral gaps for hyperbolic surfaces. (English)


Let $\Gamma \subset \text{PSL}_2(\mathbb{R})$ be a finitely generated torsion-free Fuchsian group, let $M = \Gamma \backslash \mathbb{H}^2$ be the associated hyperbolic surface and let $\delta \in [0, 1]$ be the exponent of convergence of the Poincaré series associated to $\Gamma$. Recall that $\delta = 0$, if $\Gamma$ is elementary; $\delta = 1$, if $\Gamma$ is a Fuchsian group of the first kind; and $\delta \in (0, 1)$ if $\Gamma$ is a Fuchsian group of the second kind. By work of S. J. Patterson [Acta Math. 136 (1976), no. 3-4, 241–273; MR0450547] and D. P. Sullivan [Inst. Hautes Études Sci. Publ. Math. No. 50 (1979), 171–202; MR0556586], $\delta$ is equal to the Hausdorff dimension of the limit set $\Lambda_\Gamma \subset \partial \mathbb{H}^2$ of $\Gamma$.

The Selberg zeta function is defined by

$$Z_M(s) = \prod_{\ell \in \mathcal{L}_M} \prod_{k=0}^{\infty} (1 - e^{-(s+k)\ell}), \quad \text{Re}(s) > \delta,$$

where $\mathcal{L}_M$ is the primitive length spectrum of $M$. If $\Gamma$ is of the first kind, as a consequence of the Selberg trace formula, $Z_M(s)$ extends meromorphically to $s \in \mathbb{C}$. For a general $\Gamma$, the same result was proved by L. Guillopé [in Zeta functions in geometry (Tokyo, 1990), 33–70, Adv. Stud. Pure Math., 21, Kinokuniya, Tokyo,
Moreover, $Z_M(s)$ has a simple zero at $s = \delta$, and has no other zeroes in $\{\text{Re}(s) \geq \delta\}$.

In the paper under review the authors consider the cases where $M$ is noncompact and convex co-compact. The main result (Theorem 1) is the following: there is an $\epsilon > 0$ depending only on $\delta \in (0, 1)$ such that $Z_M(s)$ has only finitely many zeroes in $\{\text{Re}(s) \geq \delta - \epsilon\}$. This generalizes F. Naud’s result [Ann. Sci. École Norm. Sup. (4) 38 (2005), no. 1, 116–153; MR2136484] where $\epsilon$ depends on $M$.

In the proof, the authors used S. Dyatlov and J. Zahl’s result [Geom. Funct. Anal. 26 (2016), no. 4, 1011–1094; MR3558305], which states that the existence of the essential spectral gap $\epsilon$ is a consequence of a fractal uncertainty principle for the limit set $\Lambda \Gamma$. The latter is a consequence of Theorem 2, which establishes a Fourier decay bound for the Patterson-Sullivan measure on $\Lambda \Gamma$.

Shu Shen

From MathSciNet, February 2021