BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 58, Number 3, July 2021, Pages 457–460
https://doi.org/10.1090/bull/1735
Article electronically published on April 6, 2021

Sobolev and viscosity solutions for fully nonlinear elliptic and parabolic equations, by N. V. Krylov, Mathematical Surveys and Monographs Volume, Vol. 233, American Mathematical Society, Providence, RI, 2018, xiv+441 pp., ISBN 978-1-4704-4740-3

A huge literature deals with fully nonlinear partial differential equations of the types

$$H(u(x), Du(x), D^2u(x), x) = 0 \quad \text{(elliptic case)} \tag{1}$$

or

$$\partial_t u(t,x) + H(u(t,x), Du(t,x), D^2u(t,x), t, x) = 0 \quad \text{(parabolic case)}. \tag{2}$$

Here, Du denotes the gradient of u and D^2u denotes the Hessian matrix of u. The above classes of problems include Monge-Ampère equations, Hamilton-Jacobi and Hamilton-Jacobi-Bellman equations in control theory, Isaacs equations in differential game theory, obstacle problems, etc. All these equations lead to various families of Hamiltonians H and various sets of assumptions.

In 1982, L.C. Evans and A.N. Krylov independently succeeded in exhibiting smooth solutions to equations (1) and (2), respectively, provided that $H(\cdot,\cdot,X,\cdot)$ (resp., $H(\cdot,\cdot,X,\cdot,\cdot)$) is strictly convex w.r.t. the matrix X and satisfies regularity conditions w.r.t. the other variables.

However, in many real applications where the above problems appear, the corresponding Hamiltonian may not be convex w.r.t. X or smooth w.r.t. the other variables. Consequently, equations (1) and (2) cannot have smooth solutions and the problems need to be posed in a weak sense.

The notion of viscosity solutions introduced by M.G. Crandall and P-L. Lions in 1980 rapidly appeared as the good notion of weak solutions to get well-posedness and to prove that the solutions are limits of natural numerical methods (see, e.g., Crandall et al. [2]). For example, the viscosity solution theory applies to Hamilton-Jacobi-Bellman equations in stochastic control:

$$\partial_t u(t,x) + \sup_{\alpha\in\mathcal{A}} (L^\alpha u(t,x) + \ell(t,x,\alpha)) = 0. \tag{3}$$

Here $\mathcal{A}$ is the set of the possible values for the control and, for any $\alpha \in \mathcal{A}$, the elliptic operator L^α is defined as

$$L^\alpha := \sum_{j=1}^{d} b_j(t,x,\alpha)\ \partial_j + \sum_{i,j=1}^{d} a_{ij}(t,x,\alpha)\ \partial_{ij},$$

for some vector-valued function $b(t,x,\alpha)$ and some symmetric semidefinite matrix-valued function $a(t,x,\alpha)$. The function ℓ is the instantaneous cost of the control problem. Existence and uniqueness of viscosity solutions can be obtained without supposing that the matrix $a(t,x,\alpha)$ is uniformly strongly elliptic. In addition, the coefficients b, a do not need to be smooth.

Which kind of Sobolev regularity can one expect for solutions to Hamilton-Jacobi-Bellman (HJB) equations? That question was already present in N.V. Krylov's seminal book [3] where he combined probabilistic and analytical methods to connect HJB equations and value functions for controlled diffusion stochastic processes. However, regularity estimates were not the core of the book.

Actually, systematic treatments of the subject for general nonlinear PDEs (1) or (2) and their viscosity solutions were originated around ten years later by Caffarelli in his pioneering paper [1].

In the book under review, the point of view is somehow different.

Namely, the objective consists in exhibiting **weak** conditions on the unspecified Hamiltonian H which allow one to get solutions in suitable Sobolev spaces without necessarily using the viscosity solution machinery. In particular, the methodologies presented within the book allow us to consider Hamiltonians which are not convex or concave w.r.t. X and may be discontinuous w.r.t. time and space variables.

In counterpart, structure conditions need to be imposed to get the desired regularity estimates. In most of the cases, the Hamiltonian is supposed to be Lipschitz w.r.t. X and close to a Hamiltonian satisfying a uniform strong ellipticity condition. The structure conditions, which vary from chapter to chapter, are presented in an abstract way. A few examples illustrate the generality of potential applications, including Isaacs' equations and Monge-Ampère equations. However, some of the examples may seem artificial. More examples coming from real applications would have been useful to the reader.

Under the structure conditions, Krylov succeeds in getting a wide family of regularity results, either in Sobolov or Hölder spaces, mostly by using standard tools in real analysis.

One of the main ingredients in Krylov's approach is presented in Chapter 1. Consider the elliptic version of the HJB equation (3) and suppose that the coefficients b_j and a_{ij} depend on α only. Approximate the operator L^α by finite difference operators with discretization steps (h_n) tending to 0. Solve the corresponding finite difference problems. The sequence of solutions (u_n) converge uniformly on any ball to a limit u which enjoys the following properties:

(1) the first-order Sobolev derivatives $\partial_i u$ exist and are bounded;
(2) the generalized second-order derivatives $\partial_{ij} u$ are locally finite measures;
(3) a second-order Taylor formula holds true for u almost everywhere (which extends the celebrated Aleksandrov-Busemann-Feller theorem).

To treat the equation (1) in full generality, the author combines the preceding approximation procedure with the cut-off procedure consisting in introducing a positive convex function P such that, for any positive constant K (which will further go to infinity) there exists a unique solution in $C(\overline{\Omega}) \cap C^{1,1}_{\mathrm{loc}}(\Omega)$ to

$$\max(H(u(x), Du(x), D^2u(x), x), P(D^2u(x)) - K) = 0 \quad \text{a.e. in } \Omega. \tag{4}$$

In addition, the solution has locally bounded second-order derivatives in Ω. Here, Ω is a bounded domain in $\mathbb{R}^d$. It is understood that (1) and (4) are posed in Ω with the same Dirichlet boundary condition.

For the parabolic problem (2), the solution to the cut-off equation is sought in the space $C([0,T] \times \overline{\Omega}) \cap W^{1,2}_{\infty,\mathrm{loc}}([0,T) \times \Omega)$.

The cut-off procedure leads to technical difficulties to get estimates on the solutions which are uniform in K. However, it is a poweful tool to exhibit solutions in Sobolev spaces which, by construction, solve (1) or (2) almost everywhere, and to get accurate estimates on their Sobolev norms.

Striking complementary results concern the local behaviour of the solutions close to the boundary, under suitable conditions which ensure that these solutions belong to $W^{1,2}_p$. The global modulus of continuity of the solutions is also examined in detail.

Furthermore, the author studies nonlinear elliptic equations in Hölder spaces $C^{2+\alpha}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ and extends Evans-Krylov estimates. In particular, regularity assumptions on H are relaxed.

Similarly, he studies the Hölder regularity of continuous or L^p-viscosity solutions to Isaacs parabolic problems with coefficients in the class of functions of almost vanishing mean oscillation.

To conclude, the truly remarkable monograph by V.N. Krylov gathers important results obtained by the author and a few coauthors from the early 1970s on fully nonlinear elliptic and parabolic PDEs. The author has revisited and rewritten some parts of the book [4] as well as a series of published papers, making them easier to read and to study.

In spite of the many demanding technicalities involved in the proofs, the quality and clarity of the redaction is impressive. Most of the chapters start with a short summary, the list of assumptions to be held along the whole chapter and the main theorems to be proved. The statements of the main theorems are commented: useful indications are provided to the reader on the motivations, the existing bibliography, and the strategy followed in the proofs.

The book will be useful to mathematicians interested in regularity properties of solutions to fully nonlinear PDEs. It will also be useful to any reader interested in finding elegant proofs or extensions to celebrated results such as Fefferman-Stein's theorem, Harnack inequalities, and Krylov-Safonov estimates.

A.N. Krylov is an outstanding expert in stochastic analysis. However, as the book under review concerns fairly general problems which may not have natural probabilistic interpretations, it does not contain any tool or notion from probability theory.

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