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SELECTED MATHEMATICAL REVIEWS

related to the work of
JOHN HORTON CONWAY

MR0827219 (88g:20025) 20D05; 20-02

Conway, J. H.; Curtis, R. T.; Norton, S. P.; Parker, R. A.; Wilson, R. A.

𝔸𝕋𝕃𝔸𝕊 of finite groups. (English)

Oxford University Press, *Eynsham*, 1985, xxxiv+252 pp., $45.00

At last, an official collection of character tables and related information about many finite simple groups has appeared in book form. This information is important to specialists in finite group theory and the volume contains neatly presented instructional material which the nonspecialists can appreciate. For years, the authors have used the material at a very high level. It has been reworded and refined by experience. At the month-long 1979 Santa Cruz conference on finite groups, Simon Norton carried a shopping bag of tattered printouts and character tables to deal with urgent questions about simple groups. Now, we can all have the power of such rapid access, but in a classier format!

The "classic" character table of a finite group G is by definition a $k \times k$ matrix of complex numbers, whose rows are indexed by the k irreducible characters and whose columns are indexed by the k conjugacy classes; of course, it is not unique because there is no generally accepted way to order the index sets, though the principal character (corresponding to the trivial homomorphism $G \to \mathrm{GL}(1, \mathbf{C})$) is always listed first. The (i, j) entry is $\chi_i(g_j)$, the value of the ith irreducible character on a representative of the jth conjugacy class, and this algebraic number is always a sum of d $|g_j|$th roots of unity, where $d = \chi_i(1)$ is the degree of χ_i.

The efforts of the last 25 years to classify finite simple groups created a greater need to have numerical and combinatorial information about the known groups. The occasional tables produced by R. Brauer or J. S. Frame or J. Todd years ago were followed by a flood of tables in the 1960s and 1970s. Generally, these were distributed informally, often with no name or source written on them and always without proof. Referring to a character table in a research article was awkward at times. The general theory of Brauer gave many arithmetic conditions on the character table which in "easy" cases allowed one to fill in many blank entries for the table of a particular group. This was not always the case. For instance, David Hunt's work on the tables for the Fischer 3-transposition groups took an especially long time and involved extensive computer work and a study of induct-restrict tables for subgroups with known character tables.

In sum, the five authors have collected some of this early and unpublished work, then greatly extended it and put it in a form suitable for easy modern applications.

The book is organized as follows: (I) Introduction and explanations (28 pages), (II) The character tables (235 pages), (III) Supplementary tables (6 pages), (IV) References (8 pages) and Index (1 page).

(I): Sections 1, 2 and 3 contain a rapid introduction to the families of finite simple groups. It is clear and telegraphic in style and not intended for someone who is looking for full discussions and constructions.

Sections 4 through 7 discuss the multiplier, automorphism groups, isoclinism and the group extension theory which is relevant to interpreting the blocks (and broken-edge blocks) in the tables, notation for conjugacy classes, algebraic numbers and algebraic conjugates of these two concepts. We comment on the tables themselves in (II). The authors' notations for algebraic integers are very successful for character tables, e.g., $z = z_N = \exp(2\pi i/N)$, $b_N = \frac{1}{2}\sum_{t=1}^{N-1} z^{t^2}$, $c_N = \frac{1}{3}\sum_{t=1}^{N-1} z^{t^3}$ (for $N \equiv 1 \pmod 3$), etc.

One fault with the exposition is that the authors use terms and notation without explanation, then define them later. In the above sequence of definitions, for z_N, b_N, $c_N, \cdots$, one finds "n_2", but not a definition until further down the column. The notation $^*k$ is used in Section 7.3 but no hint is given for where to look for the definition. It would help if an index of notations and definitions were included to help the reader who starts reading in the middle.

The authors discuss the several existing systems of notation for the simple groups. Parts of the system used in the *Atlas* make the reviewer uncomfortable.

The most glaring item is the use of "O" for the simple composition factor of the n-dimensional orthogonal group of type ϵ over $\mathbf{F}_q$. In other systems, this group would be $\mathrm{P}\Omega^\epsilon(n,q)$ or one of $D_m(q)$, ${}^2D_m(q)$ (when $n = 2m$) or $B_m(q)$ when $n = 2m+1$. The authors reject these notations because they want one letter for the basic name of all these simple groups.

The second comment is about names assigned to sporadic groups; see Table 1, page viii. The principle generally used by group theorists has been to name a sporadic group after its discoverers and use a symbol related to these names. The sometime exceptions to this have been the Conway groups (denoted by .0, .1, .2 and .3 since 1968 but by Co_0, Co_1, Co_2, and Co_3 in this volume), the Fischer groups (denoted by $M(22)$, $M(23)$ and $M(24)'$ originally, but later by Fi_{22}, Fi_{23} and Fi_{24}') and the Monster (the group discovered by Fischer and the reviewer in November 1973; the *Atlas* symbols are M, FG and F_1) and the Baby Monster (the $\{3,4\}^+$-transposition group discovered by Fischer earlier in 1973; the *Atlas* symbols are B and F_2) and the Harada group (called the Harada-Norton group in the *Atlas*; the *Atlas* symbols are HN and F_5).

The system of F's with subscripts has several nice group-theoretic features. However, there seems to be no natural systems covering all sporadics. Why not keep the names and remember the history, at least? Perhaps later developments will suggest a good solution.

Finally some comments about notation for other finite groups. Several recommendations in 5.2 really are at variance with general usage. The authors mention C_m for a cyclic group of order m but not $\mathbf{Z}_m$! Their term "diagonal product" $A \triangle B$ is otherwise known as a pullback or a fiber product. The most common notation for an extraspecial group is p^{1+2n} or p_ϵ^{1+2n}. Since notation for an extension $A \cdot B$ reads left-to-right along an ascending series, it would be more appropriate to write $(A \times B)\frac{1}{2}$ than $\frac{1}{2}(A \times B)$.

(II): The organization of the individual tables is discussed in Section 6. See page xxiv for a well-diagrammed example. Let G be the simple group. The tables come in blocks with each block corresponding to an extension of the form $m.G.a$, where m is a cyclic quotient of the Schur multiplier and a is a cyclic subgroup of the outer automorphism group; for reaons why these cases suffice (nearly), see 6.5 and 6.6.

To the left of the block is the downward running list of characters ($\chi_1 = 1, \chi_2, \chi_3, \cdots$) and their indicators (0, + or − as the character is not real-valued, afforded by a real representation, or real-valued but not afforded by a real representation). Across the top is a band with several rows of information about the columns (indexed by the conjugacy classes, $C_i,\ i = 1, \cdots, k$). The experience of the last 25 years has shown the importance of enriching the traditional "classic" character table to include power maps (i.e., for $n \in \mathbf{Z}$, which classes contain the nth powers of elements from a fixed class), factorizations (i.e. if $g \in C_i$ and π is a set of primes and $g = g_\pi g_{\pi'}$ is the unique commuting factorization of g into a π-element and a π'-element, which C_j contains g_π), and so on. A simple application of this information, which is not possible to execute with a strictly classical table, is to find the dimension of the space of cubic invariants on a module V affording the character $g \mapsto \frac{1}{6}\{\chi(g)^3 + 3\chi(g)\chi(g^2) + 2\chi(g)^3\}$ and so its inner product with the trivial character of G gives the answer.

The difficulty of getting these blocks correct increases generally according to the sequence $m = 1,\ a = 1;\ a = 1;\ m, a$ arbitrary. Indeed the authors acknowledge errors which turned up as the book went to press (see page xxxii, bottom). How the notations extend across the several upward and downward extensions is articulated well.

(III): The final part of the *Atlas* text consists of three tables and a list of references. (1) Partitions and classes of characters for S_n, useful, say, in working out particular invariants of the group in question. (2) Involvement of sporadic groups in one another (the single "?" in this *Atlas* table is now claimed to be "−" in recent work of R. A. Wilson). (3) Orders of over 250 simple groups, with orders in base 10 and in factorized forms and with Schur multiplier and outer automorphism group.

(IV) The bibliography is restricted to (i) some very general works on the families of finite simple groups and (ii) lengthy lists of articles on each of the 26 sporadic groups.

Survey articles (no proofs) for absolute beginners are worth mentioning and could go in (i), e.g., a paper by R. Carter [J. London Math. Soc. **40** (1965), 193–240; MR0174655] for groups of Lie type and a paper by the reviewer [in *Vertex operators in mathematics and physics* (Berkeley, Calif., 1983), 217–229, Springer, New York, 1985; MR0781380] for sporadic groups. Also, references for Schur multiplier and automorphism groups would be of general interest.

Tables of numerical information are notorious for errors and it does pay to compare; for example, the order of McLaughlin's group is incorrectly given on page 136 of D. Gorenstein's *Finite simple groups* [Plenum, New York, 1982; MR0698782]. After the Higman-Sims group, G, was discovered in 1968, it was deduced that G must have subgroups $K \leq H \leq G$ with $H \cong \mathrm{PSU}(3, 5)$ and $K \cong \mathrm{Alt}_7$. Of course, the characters of G must restrict sensibly to characters of K and H but the character tables then at hand produced a contradiction! The error in the tables was found.

Should a researcher, urgently needing to prove a theorem, trust the *Atlas*? The question is like that of whether to accept the classification of finite simple groups. Both efforts are widely respected, the participants in both have worked at high levels to reach the goal, yet have admitted that errors exist. In both cases, the group theory community feels that probably only local adjustments would be needed in the ambient program to deal with errors. So, the answer is: "Yes, but…".

Only a purist would turn his or her back on either claim of completion. To make progress, we must accept them as essentially correct but pay attention for some time and look for alternate arguments whenever possible. One can treat them as axioms when writing arguments down formally.

Norton has shown a list of errors discovered since publication. One is a nonsquare character table! It is worth mentioning that Chat-Yin Ho recently found a maximal 7-local subgroup of the Monster not on the *Atlas* list. There may be a problem with the list of maximal subgroups for Co_1.

{Reviewer's remarks: The reviewer is disappointed at the incorrectness of the scholarship in a few instances (notwithstanding the disclaimer on page xxxii, Section 8.5.1). The correctness of the Monster character table is not completely proved (though not doubted). (a) The determination of the conjugacy classes requires sufficient knowledge of centralizers of elements in a subgroup of $\mathbf{M}$ of the form $2^{1+24}\cdot\mathrm{Co}_1$; the authors guessed the basic information, then proceeded. (b) The existence of the irreducible character of degree 196883 was taken as a hypothesis (196883 is the smallest number which could be the degree of a nonprincipal character); a proof that such a character exists was claimed by Norton in 1981 but no manuscript has appeared, and its relationship with (a) has not been explicitly stated; existence of such a character is necessary to complete the program devised by J. G. Thompson [Bull. London Math. Soc. **11** (1979), no. 3, 340–346; MR0554400] for proving uniqueness of $\mathbf{M}$.

{It would have been helpful to have some recent references, e.g. to the reviewer's recent work on code loops. The reviewer understands that future editions will contain no new references.

{The book is attractive in appearance. The cover is a cherry red with white writing on stiff cardboard. The authors' names form a neat matrix listed vertically in alphabetical order (which agrees with their respective ages, apparently), each with two initials and a 6-letter last name. The price is extremely fair. The authors are to be commended for their influence on the price and for getting the publisher to replace the originally intended soft binding.

{The book is large—too large for most briefcases. The wire binding on the reviewer's copy became deformed right away and interfered with easy closing and opening of the book to lie flat on a table. The edges of the pages near the binding have begun to suffer due to struggles with the binding. One idea is to make the tables available on tape, potentially a big saving of effort for the user who intends computer calculations.

{The mathematics community (and physics community) should be grateful to the creators of the *Atlas* for their extremely fine service. An appreciation and use of the finite simple groups might be expected to spread noticeably faster as a result.}

R. L. Griess

From MathSciNet, May 2021

MR0245518 (39 #6824) 10.20

Conway, J. H.

A characterisation of Leech's lattice.

Inventiones Mathematicae **7** (1969), 137–142.

A unimodular lattice Λ is called "even" if the square of the distance between two points of Λ is always an even integer. Hence $u_1 = u_3 = \cdots = 0$, where u_n denotes the number of points of Λ at squared distance n from the origin. In a beautifully written paper, the author shows that the even unimodular lattice Λ discovered by J. Leech [Canad. J. Math. **16** (1964), 657–682, see pp. 670–671; MR0167901; ibid. **19** (1967), 251–267; MR0209983] is the only one of dimension $d < 32$ with $u_2 = 0$ (Theorem 6), that the order of the automorphism group of Λ is $(u_8/48)2^{12}|M_{24}|$, where M_{24} is the simple Mathieu group, $u_8 = 398,034,000$, and that the condition "$u_2 = 0$" defining Leech's lattice (from the even unimodular family of dimension $d < 32$) may be replaced by any one of "$u_{2n} = 0$ for some n", "Λ is not directly congruent to its mirror image", "no reflection leaves Λ invariant" (Theorem 7).

This initial result (Theorem 1) shows that any even unimodular lattice with $d < 32$ which, like Leech's, has $u_2 = 0$, then has $d = 24$, $u_4 = 196,560$, $u_6 = 16,773,120$, $u_8 = 398,034,000$. To do so, the author utilizes the known result that for an even unimodular lattice, $d = 8n$ and $\sum u_n \exp(2\pi i n\tau)$ is a modular form of weight $d/4$ for the full modular group; from the latter he deduces $u_2 > 0$ for $d = 8, 16$, and a formula giving u_{2n} for the case $d = 24, u_2 = 0$. This gives him his values u_4, u_6, u_8, which will be crucial later on (e.g., Theorems 2, 4) to convert inequalities into equalities. To prove Theorem 6, it remains for him to show that $d = 24, u = 0$ defines Leech's lattice, and this task comprises the bulk of the paper (Theorems 2–6).

The first step (Theorem 2) is to show that each of the 2^{24} equivalence classes $\Lambda/2\Lambda$ contains either precisely one vector of length $< (\sqrt{}8)(u_0/1 + u_4/2) + u_6/2$ equivalence classes), or precisely 24 mutually orthogonal vectors of length $(\sqrt{}8)$ ($u_8/48$ equivalence classes). From any one of the latter, coordinates $x = c(x_1, \cdots, x_n)$ are chosen with $c = 1/\sqrt{}8$ such that the vectors in the orthogonal set have either one $x_i = \pm 8$ or two $x_i = \pm 4$, with all other $x_i = 0$.

Then $x \in \Lambda$ implies $x_i \in Z$, $x_i - x_j \in 2Z$, and those x with all $x_i \equiv 0 \bmod 2$ may be mapped homomorphically into the subsets $C(x) = \{i \colon x_i \equiv 2(4)\}$ of $\{1, \cdots, 24\}$, $C(x + y) = C(x) + C(y)$, with sums of vectors corresponding to symmetric differences of sets. The range $\mathcal{C} = \{C(x)\}$ is generated (by symmetric difference) by exactly $759 = \binom{24}{5} / \binom{8}{5}$ subsets of $|C| = 8$ elements, with each 5-element subset of $\{1, \cdots, 24\}$ in exactly one of these; no $C \in \mathcal{C}$ has $0 < |C| < 8$, and there are exactly 2^{12} (of 2^{24} possible) subsets $C(x)$ in $\mathcal{C}$ (Theorems 3, 4).

The stage is now set to characterize the $x = c(x_1, \cdots, x_{24})$ in Λ and thence deduce Λ must be Leech's lattice. In Theorem 5 the author shows $x \in \Lambda$ if (A) all $x_i \in 2Z$ with $\sum x_i \equiv 0 \bmod 8$, or all $x_i \in 2Z + 1$ with $\sum x_i \equiv 4 \bmod 8$, and (B) $\{i \colon x_i \equiv k(4)\} \in \mathcal{C}$. Thus Λ depends only on $\mathcal{C}$ which in turn, being generated by 8-subsets in which every 5-subset appears exactly once, is a Steiner system $S(5, 8, 24)$ and hence unique up to permuting $\{1, \cdots, 24\}$ [E. Witt, Abh. Math. Sem. Univ. Hamburg **12** (1938), 265–275]; thus Λ is unique up to isomorphism (Theorem 6).

The paper concludes with the determination of the size of the automorphism group of Λ, and the proof of Theorem 7. There is also a reference to related work of H.-V. Neimeier ["Definite quadratische Formen der Dimension 24 und Diskriminante 1", Ph.D. Thesis, Univ. Göttingen, Göttingen, 1968], who has since enumerated the even uni-modular lattices of dimension $d = 24$, finding 24 of these, only one (Leech's) having $u_2 = 0$. Neimeier's method appears unamenable to extension, however, as even for the next case $d = 32$ there are known to be at least 10^8 such lattices.

G. K. White

From MathSciNet, May 2021

MR4080553 00A08; 03B42

Conway, J. H.; Paterson, M. S.; Moscow, U. S. S. R.

A headache-causing problem.

American Mathematical Monthly **127** (2020), *no.* 4, 291–296.

This amusing note was originally published in celebration of Hendrik Lenstra's Ph.D. defense in Amsterdam on May 18, 1977. Since this note now appears in the *Monthly*'s April 2020 issue, it may be viewed as an April fool's joke, a tradition in keeping with such a note as "Remembering F. O. Vechs" by Ann Dalmak, better known as Dan Kalman, in [D. Kalman, Math Horiz. **22** (2015), no. 4, 12–13; MR3335030]. Since J. H. Conway died on 11 April 2020, the article may also be viewed as an unscheduled yet fitting goodbye from a master mathematician who fully embodies the playful aspect of mathematics.

The *headache-causing problem* presented in this note is a logical paradox: Consider N men each with a nonnegative integer on their forehead, each of whom can see everyone's number except their own. A blind umpire has written a number of numbers on the blackboard, one of which is the sum of all the forehead integers. For example, in succinct notation—reminiscent of the exhaustive catalogue of mostly impartial two-person games in Berlekamp, Conway, and Guy's monumental *Winning ways* [*Winning ways for your mathematical plays. Vol. 1*, second edition, A K Peters, Natick, MA, 2001; MR1808891, *Winning ways for your mathematical plays. Vol. 2*, second edition, A K Peters, Natick, MA, 2003; MR1959113, *Winning ways for your mathematical plays. Vol. 3*, second edition, A K Peters, Natick, MA, 2003; MR2006327, *Winning ways for your mathematical plays. Vol. 4*, second edition, A K Peters, Wellesley, MA, 2004; MR2051076]—the notation $(2, 2, 2|6, 7, 8)$ means that $N = 3$, that each man bears a 2, and that the umpire has written 6, 7, and 8 on the blackboard. Play begins with the umpire asking each man in succession if he "can deduce solely from this information what number is written on his forehead". Play continues, cycling through the panel of men, until someone *intelligently and honourably* responds, "Yes."

The celebrated theorem referred to as P—which is both true and false—as Conway and Paterson proceed to demonstrate—is that if the number of numbers on the blackboard is less than or equal to N then the game terminates after a finite number of umpire queries. Some of the twists and turns of their proof involve semantics, and it is very much in the spirit of the *surprise examination paradox* as introduced by Martin Gardner in his March 1963 *Scientific American* puzzle column, analyzed in detail in T. Y. Chow's "The surprise examination or unexpected hanging paradox" in [Amer. Math. Monthly **105** (1998), no. 1, 41–51; MR1614002], where a group of students are told that they will have a surprise quiz during the next five days. Of course, the quiz cannot be on the last day, nor the day before that, and so on all the way to the first day—which means that the surprise cannot occur. However, when the quiz does occur, it surprises everyone.

As a fun application, since the phrase (P and not-P) is both true and false, then the phrase (P and not-P) $\Longrightarrow (0 = 1)$ is true, which means that $(0 = 1)$ is true, which in turn means that $(1^3 = 1^3 + 1^3)$ is true, so unveiling a counterexample to Fermat's last theorem. Is the article all nonsense? Multiple meaning is deeply embedded within human dialogue, and the allure of transforming poetry into mathematical sense is overwhelming at times for those with playful natures.

Andrew James Simoson

From MathSciNet, May 2021

MR782233 (86h:20019) 20D08

Conway, J. H.

A simple construction for the Fischer–Griess monster group.

Inventiones Mathematicae **79** (1985), *no.* 3, 513–540.

Since the appearance of the paper of R. L. Griess, Jr. [same journal **69** (1982), no. 1, 1–102; MR0671653] in which the Fischer–Griess monster simple group (here denoted by M) was first constructed, much attention has been devoted to the possibility of alternate, or at least simpler, constructions of M. The present paper is a major contribution in this direction; moreover, the author provides some facts about the structure of the Norton–Griess algebra A, a certain commutative, nonassociative algebra of dimension 196 884 whose automorphism group is M. These latter results, many due to S. P. Norton, give the first nontrivial results concerning the structure of A and also offer a means of effectively computing the action of elements of M on A.

The author begins with some properties of a certain remarkable loop P of order 2^{13} (introduced by the author and R. A. Parker); P might be thought of as an "extra-special" loop, since P has a center $\{\pm 1\}$ and $P/\{\pm 1\}$ can be naturally identified with the ubiquitous Golay code C. In particular, P has a group of "standard" automorphisms of shape $2^{12} \cdot M_{24}$. We also let P_0 be obtained from P by the adjunction of a zero 0.

A certain notion of "triality" seems to be a crucial ingredient in recent work on M, and in the present paper it intervenes in the form of a certain group N of functions from $P_0 \times P_0 \times P_0$ to itself. This group has the shape $(2^2 \times 2^2) \cdot 2^{11} \cdot 2^{22} \cdot \Sigma_3 \times M_{24}$ and its utilization is one of the key ideas of the paper. It turns out that N has a normal subgroup $K_0 \cong Z_2 \times Z_2$ such that N/K_0 is isomorphic to the normalizer of a certain 4-group in M. Moreover, N contains three involutions x, y, z which are transitively permuted by N, and $C_N(x)/K_0$ is canonically identified with a maximal subgroup of the centralizer G_{x0} of a certain involution in M. The author is now able to construct, without too much difficulty, a representation of $C_N(x)$ on a space $196\,884_x$ as a sum of constituents $300_x + 98\,280_x + 98\,304_x$ and show that this representation extends to one of the group G_{x0}. Similarly, one gets representation spaces $196\,884_y$ and $196\,884_z$ for groups G_{y0}, G_{z0} respectively. Via triality one can identify the three representation spaces and in fact the author shows that such identifications may be taken to be N-invariant isometries (with K_0 acting trivially). In this way one gets a space of dimension 196 884 (with an algebra structure A) admitting three distinct groups G_{x0}, G_{y0} and G_{z0}. G_0 is defined as the join of these three groups, and finiteness of G_0 is obtained without difficulty as a consequence of certain facts concerning the action of G_0 on A. Then a result of S. D. Smith [J. Algebra **58** (1979), no. 2, 251–281; MR0540638] yields $G_0 \cong M$. The author concludes with results concerning A alluded to above.

Geoffrey Mason

From MathSciNet, May 2021

MR2253008 (2007k:20005) 20B20; 05B25, 20B25, 20D08, 51E20, 94B25

Conway, John H.; Elkies, Noam D.; Martin, Jeremy L.

The Mathieu group M_{12} and its pseudogroup extension M_{13}. (English)

Experimental Mathematics **15** (2006), *no.* 2, 223–236.

The set of permutations that result from arbitrary sequences of moves is called M_{13}. It is not a group, but it is a union of cosets of M_{12} in Sym(13) and it has size $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$, exactly the size that it would have if it were a sharply six-fold transitive permutation group of degree 13. Section 5 of the paper reports on an investigation of how M_{13} may be thought of as being six-fold transitive.

The paper deals also with extensions of the basic game. The first of these is a "signed game" in which the counters may be turned over. This leads to the nontrivial double cover $2M_{12}$ of the Mathieu group and an analogous double cover of M_{13}. The second is a "dualized game" in which a second set of twelve counters is placed on the set of lines of $\mathbb{P}_3$ with the proviso that the point-hole lies on the line-hole. This leads to another proof that $G_{\text{bas}} \cong M_{12}$ and an interesting interpretation of an outer automorphism of M_{12}.

Section 6 deals with metric properties of M_{12} and M_{13}. One may define the distance $d(\sigma, \tau)$ between elements of M_{13} to be the length of the shortest sequence of basic moves that changes σ into τ. Computed information provides the starting point for investigation of various metric and statistical aspects of M_{12} , M_{13} and their double covers.

Peter M. Neumann

From MathSciNet, May 2021

MR1172696 (94f:11030) 11F22; 17A70, 17B67, 20D08

Borcherds, Richard E.

Monstrous moonshine and monstrous Lie superalgebras.

Inventiones Mathematicae **109** (1992), *no.* 2, 405–444.

The representation theory of the monster simple group has a fascinating history, and the subject, dubbed "monstrous moonshine", touches upon many corners of modern mathematics. In 1979, Conway and Norton conjectured the existence of a graded module for the monster whose Thompson series are certain Hauptmoduls. Frenkel et al. constructed a likely candidate in the mid-1980s, and in this paper the author proves that this vertex algebra V satisfies the conjecture of Conway and Norton. In a thorough and satisfying paper, he provides ample history, background results, detail and motivation. He gives not simply a proof of the conjecture, but many insightful remarks, valuable results and open questions.

Central to this work is the construction from V of a generalized Kac-Moody Lie algebra M, appropriately named the "monster Lie algebra". The monster acts on M, which is graded by the two-dimensional Lorentzian lattice $\mathrm{II}_{1,1}$. This is not the first monster Lie algebra constructed by the author. In 1990, he constructed an algebra M_Λ from V_Λ, a vertex algebra associated to the Leech lattice Λ. This algebra M, itself called the monster Lie algebra at the time, now sports the moniker "fake monster Lie algebra". Similarities between V and V_Λ suggested the construction of M from V.

By calculating the twisted denominator formula for M, the author shows that the Thompson series for V satisfy a set of identities known to hold for the Hauptmoduls of Conway and Norton. He then finds sufficient initial conditions for these relations to identify a series uniquely and shows that the Thompson series and the Hauptmoduls of Conway and Norton satisfy the same initial conditions.

The author builds on the techniques used in his proof to go beyond the main theorem. Rounding out this paper are examples of monstrous Lie superalgebras of even rank between 2 and 26 (including 10 and 26), whose denominator formulas give new Macdonald-type infinite product identities.

Steven N. Kass
From MathSciNet, May 2021

MR1198809 (94d:57010) 57M25
Birman, Joan S; Lin, Xiao-Song
Knot polynomials and Vassiliev's invariants.
Inventiones Mathematicae **111** (1993), *no.* 2, 225–270.

The authors establish the relationship between the invariants of knots of Jones type (Jones, skein (Homfly-pt) and Kauffman polynomials) and invariants of finite type (i.e. Vassiliev invariants). In particular they show that there exist Vassiliev invariants of any order. They show that the spaces of Vassiliev invariants of degree 2, 3 and 4 are 1-, 1- and 3-dimensional, respectively. They also prove that the subspaces generated by the skein polynomial are 1-, 1- and 2-dimensional, respectively. Furthermore they show that the unknotting number of a knot is not a finite-type invariant.

{Reviewer's remarks: D. Bar-Natan ["On the Vassiliev knot invariants", Topology, to appear] extended the computations of the paper and found the space of rational Vassiliev invariants up to degree 9; in particular he found that the space of rational Vassiliev invariants of degree 9 is 44-dimensional. T. Stanford ["Finite-type invariants of knots, links and graphs", Preprint, Columbia Univ., New York, 1992; per bibl.] found the space of integer Vassiliev invariants up to degree 7 (he verified that at least up to degree 7 integral Vassiliev invariants coincide with the rational invariants). All Vassiliev invariants found up to now can be obtained using cablings and the skein and Kauffman polynomials (Bar-Natan verified this for all rational Vassiliev invariants of degree no more than 9). In particular no Vassiliev invariant was found to distinguish the knot 8_{17} from its reverse -8_{17}. On the other hand there are Vassiliev invariants of degree less than 100 (and probably much less) which distinguish mutants (e.g. the Conway knot and the Kinoshita-Terasaka knot). Bar-Natan ["Weights of Feynman diagrams and the Vassiliev knot invariants", Preprint, 1991; per bibl.] proved that the coefficients of the Alexander-Conway polynomial are finite-type invariants (in a different context, that of n-trivial knots, this was also observed by Y. Ohyama [Topology Appl. **37** (1990), no. 3, 249–255; MR1082935]). The fact that the skein and Kauffman polynomials can be obtained from the Vassiliev invariants was proved independently by M. N. Gusarov [Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **193** (1991), Geom. i Topol. 1, 4–9, 161; MR1157140]. Lin ["Vertex models, quantum groups and Vassiliev's knot invariants", Preprint, Columbia Univ., New York, 1991; per bibl.] proved generally that any invariant obtained in the setting of quantum groups can be obtained as a limit of finite-type invariants. O. Viro suggested [Preprint, Problem 6, 1991; per revr.] that this is the case because of the use of two compatible filtrations in the semigroup algebra over the space of knots. The Vassiliev filtration $\{C_i\}$ is generated by the singular knots with i double points and the second filtration is an I-adic filtration in the ring of polynomials (for the Jones polynomial $I = (t-1)$ [see J. H. Przytycki, "Vassiliev-Gusarov skein modules of 3-manifolds and criteria for knot's periodicity", in *Low-dimensional topology (Knoxville, TN, 1992)*, Johannson Internat. Press, Cambridge, MA, to appear]). For a recent survey on Vassiliev invariants we refer to an article by Birman [Bull. Amer. Math. Soc. (N.S.) **28** (1993), no. 2, 253–287; MR1191478].

Józef H. Przytycki
From MathSciNet, May 2021

MR0880512 (88b:57012) 57M25
Lickorish, W. B. R.; Millett, Kenneth C
A polynomial invariant of oriented links.
Topology. An International Journal of Mathematics **26** (1987), *no.* 1, 107–141.

Let L_+, L_- and L_0 be diagrams of three oriented links that are exactly the same except near one crossing point (see Figure 1 on p. 108 in the paper). Alexander found (1928) that his polynomial invariant of links can be characterized by using its values on L_+, L_- and L_0. Conway introduced in 1969 the normalized form of the Alexander polynomial which satisfies $\Delta_{L_+}(t)-\Delta_{L_-}(t)+(t^{1/2}-t^{-1/2})\Delta_{L_0}(t) = 0$. In the spring of 1984 Jones found a new polynomial invariant of links and later he and Lickorish and Millett found that the Jones polynomial satisfies a relation similar to that of the Alexander polynomial: $t^{-1}V_{L_+}(t) - tV_{L_-}(t) + (t^{-1/2} - t^{1/2})V_{L_0}(t) = 0$. The similarity between the formulae for $\Delta_L(t)$ and $V_L(t)$ was too great to be a coincidence and in fact it was found by Lickorish and Millett (August 1984) that there exists a polynomial invariant of links, $P_L(m,l)$, which satisfies $lP_{L_+}(l,m) + l^{-1}P_{L_-}(l,m) + mP_{L_0}(l,m) = 0$. The same result was found independently by Freyd and Yetter, Hoste, Ocneanu (August–September 1984) and the reviewer and Traczyk (early December 1984). The paper under review introduces the $P_L(l,m)$ polynomial. (There is no agreement on what name should be used for P_L. The following are in use: Jones–Conway, generalized Jones, two-variables Jones, twisted Alexander, HOMFLY and FLYPMOTH.) Many important properties of P_L are proved in the paper; in particular the formula which allows one to compute P_L for the link being the sum of two tangles. The authors also develop the linear skein theory (using ideas of Conway and Giller). Linear skein theory is an important concept when one tries to go from polynomial invariants of links in the 3-sphere to algebraic invariants of any 3-manifold.

Józef H. Przytycki
From MathSciNet, May 2021

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Piccirillo, Lisa

The Conway knot is not slice.

Annals of Mathematics. Second Series **191** (2020), *no.* 2, 581–591.

The Conway knot of the title is an 11-crossing knot which together with the Kinoshita-Terasaka knot forms the smallest pair of positive mutant knots, as Conway discovered in 1970. The Kinoshita-Terasaka knot is smoothly slice and both knots have Alexander polynomial 1 and so are topologically slice. For nearly fifty years the question of whether the Conway knot is slice has remained open.

As the collection of knot invariants which detect non-sliceness have increased over the years, they have all failed to show that the Conway knot is not slice. The author starts with an old observation about the trace of the 0-framed surgery on a knot k, $X_0(k)$. This is formed by attaching a 2-handle to a 4-ball along k: it is a simply connected smooth 4 manifold with boundary. The boundary is a homology $S^1 \times S^2$ and $X_0(k)$ has the homotopy type of S^2. There is a combinatorial embedding of S^2 in $X_0(k)$, $S^2_k \subset X_0(k)$, with one singular point given by coning k in the 4-ball which represents the generator of $H_2(X_0(k))$.

Next it is observed that k is slice if and only if $X_0(k)$ embeds in S^4. If k is slice, use the slice disk to embed the 2-handle in $S^4 - B^4$ where B^4 is the 4-ball in $X_0(k)$ and a hemisphere in S^4. The other direction follows immediately from [R. H. Fox and J. W. Milnor, Osaka Math. J. **3** (1966), 257–267; MR0211392], Theorem 1.

To conclude the proof one needs to find a non-slice knot k with $X_0(k)$ diffeomorphic to X_0(Conway knot). There exist techniques for beginning with a knot and producing other knots with diffeomorphic X_0's which have been developed and exploited by the author and others. The author recalls the dualizing patterns construction, which can be applied to a knot k to produce another knot k^*, and proves that $X_0(k)$ is diffeomorphic to $X_0(k^*)$.

The author finds an explicit dualizing pattern for the Conway knot to get a new knot, k', and then applies the Rasmussen s-invariant to show that k' is not slice. This part is the most technical part of the paper. There is a spectral sequence which computes a bi-graded group known to be $\mathbb{Q} \oplus \mathbb{Q}$ and the gradings of the generators determine the s-invariant. Thanks to the sparseness of the differentials, $s(k')$ is determined to be 2 whereas if k' were slice, the s-invariant would be 0.

As a corollary the author obtains that the s-invariant is not determined by X_0 since s(Conway knot) $= 0$.

The paper is fun to read and the arguments are carefully laid out. An interwoven discussion of the history of the problem makes it clear just how difficult the problem appears to be until one finds just the right approach.

Laurence R. Taylor
From MathSciNet, May 2021

MR1707296 (2001c:20028) 20D08

Griess, Robert L., Jr.

Twelve sporadic groups. (English)

Springer Monographs in Mathematics.
Springer-Verlag, *Berlin*, 1998, iv+169 pp., \$79.95, ISBN 3-540-62778-2

Following the announcement of the proof of the classification theorem for finite simple groups (CFSG) in 1980, much work has been done on many fronts to try to "understand" the 26 sporadic simple groups, which do not fit into any of the well-behaved and well-understood infinite families, and exhibit often bizarre behaviour. While the more extravagant hopes have not been realised, our understanding has reached the point where definitive monographs and introductory textbooks are sorely needed. The book under review joins two books by M. G. Aschbacher [*Sporadic groups*, Cambridge Univ. Press, Cambridge, 1994; MR1269103; *3-transposition groups*, Cambridge Univ. Press, Cambridge, 1997; MR1423599] and one by A. A. Ivanov [*Geometry of sporadic groups. I*, Cambridge Univ. Press, Cambridge, 1999; MR1705272] in attempting to fill this gap.

The problem which immediately confronts the author of such a book is that there is a huge amount of material to be covered, and no book can hope to do justice to the subject as a whole. Aschbacher limits himself (roughly) to proofs of existence and uniqueness of the 26 groups, and in his two books so far has dealt with only a handful of them. Of necessity, his books are very technical, and provide little of the kind of information which users of the CFSG require. Ivanov similarly is concerned with the definitive exposition of the solution of a classification problem for certain geometries.

The book under review is quite different. It aims to be a textbook for graduate students, rather than a monograph. It is only 169 pages long, and attempts to develop the theory more or less from first principles up to the construction and basic properties of the twelve sporadic groups which are related to the Golay codes and the Leech lattice—that is, the five Mathieu groups, the three Conway groups, and the groups of Hall-Janko, Higman-Sims, Suzuki and McLaughlin.

It is inevitable in a book like this that one must take certain background material on trust—one cannot develop in detail all the theory required and still hope to reach interesting conclusions. Griess's choice of what to leave out obviously reflects his individual tastes. He treats codes and lattices in some detail, and emphasizes cohomological methods and results, while omitting proofs of results in "pure" group theory (such as Schur-Zassenhaus, Thompson's transfer theorem, etc.) and the theory of classical groups and groups of Lie type.

This background is collected into the first two chapters, after which there are three chapters on codes, culminating in detailed analysis of the (extended binary) Golay code, including existence and uniqueness proofs. The treatment here follows the "hexacode" approach of Benson and Conway, but is a good deal more axiomatic than usual. In Chapter 6 we turn from the code to its automorphism group, and find detailed and explicit descriptions of all its maximal subgroups. No attempt is made however to prove that the list of maximal subgroups is complete. Chapter 7 presents the analogous material for the small Mathieu groups, again following Conway in using the tetracode to construct the (extended) ternary Golay code, and it is more or less independent of the other chapters.

Most of the rest of the book is concerned with the Leech lattice, and subgroups of its automorphism group. Here the pace increases noticeably, and some proofs are rather sketchy, or relegated to exercises. Basic properties of the Leech lattice, more often obtained by elementary counting arguments, are here derived by quoting properties of theta functions from the theory of modular forms. The entire construction of the Hall-Janko group is left as an exercise.

The book ends with a mixed bag of appendices whose purpose is not entirely clear, and a chapter giving the briefest of introductions to the remaining 14 sporadic groups, as well as a highly individual view of certain historical events.

While an Atlas author in his glass house should be wary of throwing stones, I should mention some of the more unfortunate errors in this book, besides the frequent misprints and inaccurate cross-referencing. In Exercise 2.8, $\dim H^1 = 1$, not 0. On page 16, the Schur multiplier of ${}^2D_l(q)$ is cyclic of order $(4, q^l + 1)$, not $(4, q+1)$. On page 17, but not on page 169, the orders of four of the sporadic groups are wrong. In Table 2.16, for the natural modules of $\mathrm{SL}(2, 2^m) = \mathrm{Sp}(2, 2^m)$ for all $m > 1$, and of $\mathrm{Sp}(4, 2^m)$ for all m, the dimension of H^1 is 1 not 0, while for $\Omega^-(6, 2)$ the dimension of H^1 is 0, not 1. In Table (2.17) the non-split group $5^{3\cdot}\mathrm{SL}(3, 5)$ is omitted, as are the groups $2^{2m\cdot}\mathrm{SL}(2, 2^m)$ for $m > 2$ and $2^{4\cdot}\mathrm{Sp}(4, 2)$—there may be others. Note also that the dimensions of H^2 for the two given representations of M_{24} are 0 and 1 respectively, proved by D. J. Jackson [“Some problems in finite group theory”, Ph.D. Thesis, Cambridge Univ., Cambridge, 1982; per revr.]. In Table 10C Griess has unfortunately copied the misprints from Conway’s paper as well as introducing new ones.

These reservations notwithstanding, this book succeeds in its aim of making a fascinating but difficult area of mathematics more readily accessible, and of the books currently available it is the one which I would recommend as an introduction to sporadic groups for a beginning graduate student.

Robert A. Wilson
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MR1478672 (98k:11035) 11E12; 11-01

Conway, John H.

The sensual (quadratic) form. (English)

Mathematical Association of America, *Washington, DC*, 1997, xiv+152 pp., \$32.95, ISBN 0-88385-030-3

The principal theme of this little book is the classical topic of the theory of integral quadratic forms over the rational integers $\mathbf{Z}$. It is however a rather unorthodox and highly personal account of the subject.

The first chapter begins by defining a tree associated with a rank 2 $\mathbf{Z}$-module, and then using a binary integral quadratic form to label the tree with the integers represented primitively by the form. Conway calls this the “topograph” of the form. Certain invariants called “vonorms” and “conorms” are defined for (binary) positive definite forms. The topographs of the possible kinds of binary forms (positive or negative definite, semidefinite, indefinite,...) are investigated, leading to algorithms for determining the numbers represented by such forms, and their equivalence classes (although the latter is not made completely explicit). He also touches very briefly on the question of determining the isometry groups of these forms for their topographs. In some “Afterthoughts” to the first chapter, topographs are related to properties of the upper half plane.

Chapter 2 begins by posing Mark Kac’s question of “hearing the shape of a drum”, and the author relates the higher-dimensional analogue of this idea on tori—quotients of $\mathbf{R}^n$ by a lattice—to the question of whether a positive definite integral quadratic form is determined by the numbers it represents. A property of such a form is called “audible” if the property is determined by these numbers, or equivalently, by the theta function of the quadratic form. As examples, he shows that the determinant of the form and the theta function of the dual form are audible. He also provides counterexamples to the higher-dimensional Kac question, the first of which were found by J. Milnor. The Afterthoughts of this chapter deal with Kneser's "gluing method" for constructing lattices, and this is applied to a description of the 24-dimensional even lattices and Witt's characterization of lattices generated by vectors of norm 2.

Chapter 3 defines the Voronoĭ cells of a lattice (positive definite form), and thence the Voronoĭ vectors, the Voronoĭ norms (vonorms), and the conorms. These are related to the ideas of Chapter 1 (for binary forms) and then they are analyzed for ternary forms. The shapes of the Voronoĭ cells are determined in dimensions 2 and 3, and a discussion of them in dimension 4 appears in the Afterthoughts.

The fourth and final chapter is concerned first with the author's version of the Hasse-Minkowski theorem and then with the local invariants of integral forms (the Jordan decomposition and the genus). It ends with a proof of Kitaoka's theorem that the genus of a form of rank ≤ 4 is audible, and an example of 2 non-equivalent 5-dimensional lattices with the same theta functions—therefore providing a counterexample to the audibility of the genus in dimension ≥ 5. A postscript describes some mostly classical facts about integral quadratic forms, such as Legendre's three squares theorem.

This is a book rich in ideas. They seem to burst forth from almost every page, and it is perhaps not surprising that it seems a little disorganized at times. I suspect that the author's hope that "even the experts in quadratic forms will find some new enlightenment here" will be realized. The parts of the book dealing with topographs, and with Voronoĭ cells, vonorms and conorms are especially interesting. One can only hope that this book will help to bring them into the argot of quadratic forms.

Carl Riehm

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