

*Function theory and  $\ell^p$  spaces*, by Raymond Cheng, Javad Mashreghi, and William T. Ross, University Lecture Series, Vol. 75, American Mathematical Society, Providence, RI, 2020, xvi+219 pp., ISBN 978-1-4704-5593-4

## 1. THE SHIFT ON $\ell^2$ AND THE HARDY SPACE $H^2$

In one variable, the fields of operator theory and complex analysis are so intertwined that many researchers work in both fields, and at conferences nobody is surprised if a talk about one morphs into a talk about the other. There is a three-way street between them: operator theory is used to prove results in complex analysis (e.g., de Branges’s original proof of the Bieberbach conjecture [3]); complex analysis is used to prove results in operator theory (e.g., Stone’s original proof of the spectral theorem [8]); and some of the deepest results cannot even be stated without using both function theory and operator theory (e.g., Löwner’s characterization of matrix monotone functions [6]).

One reason for this is the stunning success in the study of the shift operator and all the function theory that connects to it. The shift operator  $S : \ell^2 \rightarrow \ell^2$  is defined by

$$(1.1) \quad S(a_0, a_1, a_2, \dots) = (0, a_0, a_1, \dots),$$

and is an isometry that is not unitary (since it is not surjective). By the Sz.-Nagy dilation theorem [9], every operator  $T$  on a separable Hilbert space with  $\|T\| < 1$  is of the form

$$T = \bigoplus_J S^* \Big|_{\mathcal{K}},$$

where  $J$  is a countable index set (perhaps finite), and  $\mathcal{K}$  is a closed invariant subspace of  $\bigoplus_J S^*$ . Therefore understanding invariant subspaces of  $\bigoplus_J S^*$  is critically important in operator theory. For simplicity, we shall restrict ourselves in this review to when  $J$  is just a singleton, and, since  $\mathcal{K}$  is invariant for  $S^*$  if and only if  $\mathcal{K}^\perp$  is invariant for  $S$ , we shall discuss the closed  $S$  invariant subspaces.

For each  $k \in \mathbb{N}$ , the set  $\{(a_j)_{j=0}^\infty : a_j = 0 \text{ if } j \leq k\}$  is an obvious invariant subspace. To see more, consider another way to look at the operator. We associate a holomorphic function on the unit disk  $\mathbb{D}$  with its sequence of Maclaurin coefficients. We introduce a new version of the Hilbert space, the Hardy space  $H^2$ , defined as

$$(1.2) \quad H^2 = \left\{ f : \mathbb{D} \rightarrow \mathbb{C} \mid f(z) = \sum_{n=0}^{\infty} a_n z^n, \sum |a_n|^2 < \infty \right\}.$$

We define the norm of a function in  $H^2$  to be the  $\ell^2$  norm of the Maclaurin coefficients. Now, we can reinterpret  $S$  as the operator from  $H^2$  to  $H^2$ ,

$$(1.3) \quad S : f(z) \mapsto zf(z).$$

Immediately, we see that if  $X$  is any subset of  $\mathbb{D}$ , then the subspace of  $H^2$  consisting of all functions that vanish on  $X$  is invariant for  $S$ . Of course, this subspace may be 0. A necessary and sufficient condition for a sequence  $(w_n)$  to be the zero set of

a nonzero function in  $H^2$  is that it satisfy the Blaschke condition

$$(1.4) \quad \sum_n 1 - |w_n| < \infty.$$

There is a third viewpoint. Every function in  $L^2$  of the unit circle  $\mathbb{T}$  has a Fourier series, and by Plancherel's theorem the  $L^2$  norm of the function equals the  $\ell^2$  norm of the Fourier coefficients (which are indexed by  $\mathbb{Z}$ ). We now define

$$(1.5) \quad H^2(\mathbb{T}) = \left\{ F \in L^2(\mathbb{T}) : F(e^{i\theta}) \sim \sum_{n=0}^{\infty} a_n e^{in\theta} \right\}.$$

We then have

$$(1.6) \quad S : F(e^{i\theta}) \mapsto e^{i\theta} F(e^{i\theta}).$$

By a theorem of Fatou [4], for every  $f \in H^2$ , the boundary value

$$F(e^{i\theta}) = \lim_{r \rightarrow 1^-} f(re^{i\theta})$$

exists almost everywhere and gives a function  $F$  in  $H^2(\mathbb{T})$ . Conversely, one can recover  $f$  from  $F$  by integrating against the Szegő kernel

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{F(e^{i\theta})}{1 - e^{-i\theta}z} d\theta.$$

So when studying the operator  $S$  on  $\ell^2$ , one can use the tools of (i) Hilbert space, (ii) complex analysis, and (iii) harmonic analysis. Having three different tool-sets available has allowed for a deep understanding. For example, Beurling [2] characterized the closed invariant subspaces of  $S$ . A function  $u \in H^2$  is called *inner* if its radial limits have modulus one almost everywhere.

**Theorem 1.7** (Beurling). *Every closed invariant subspace of  $S$  in  $H^2$  is of the form  $uH^2$ , where  $u$  is an inner function.*

Every inner function can be factored uniquely as a Blaschke product times a singular inner function. We have

$$u(z) = \left[ z^N \prod_{n \geq 1} \frac{\bar{w}_n}{|w_n|} \frac{w_n - z}{1 - \bar{w}_n z} \right] \left[ \exp\left(-\int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\mu_s(\xi)\right) \right],$$

where  $\mu_s$  is a singular measure on the circle. The first factor, the Blaschke product, takes into account  $u$ 's zeroes on  $\mathbb{D}$  (a sequence that satisfies (1.4)). The second factor, the singular inner function, reflects where  $u$  vanishes to infinite order on the circle. Every function in  $H^2$  can be uniquely factored as an inner function times an outer function, which is a function  $g$  that satisfies

$$g(z) = \exp \left[ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |g(e^{i\theta})| d\theta \right].$$

It follows from Beurling's theorem that the outer functions are precisely the functions that are cyclic for  $S$  (i.e., functions such that the smallest closed invariant subspace containing them is the whole space  $H^2$ ).

There are many books about function theory in the Hardy space and the operator theory associated with it; see for example [7].

2. THE SHIFT ON  $\ell^p$ ,  $p \neq 2$ 

Just as in (1.2), for any  $p \in (0, \infty]$  we can define a space of analytic functions on  $\mathbb{D}$ , which we shall call  $\ell_A^p$ ,

$$\ell_A^p = \left\{ f : \mathbb{D} \rightarrow \mathbb{C} \mid f(z) = \sum_{n=0}^{\infty} a_n z^n, (a_n) \in \ell^p \right\}.$$

For  $p \geq 1$  this is a Banach space with the  $\ell^p$  norm, and for  $0 < p < 1$  it is a Fréchet space.

When  $p \neq 2$  the shift operator (1.1) is still an isometry from  $\ell^p$  to  $\ell^p$ , but is much harder to understand, since we no longer have the tools of either Hilbert space or harmonic analysis. Although there are many beautiful results, the theory is far from complete. Beurling noted in his paper that his results can easily be modified to the case of  $H^p$  (where we take (1.5) and replace  $L^2$  by  $L^p$ ) but not  $\ell_A^p$ , “a case of considerably greater interest”. He asked whether being outer is relevant to being cyclic for the shift in  $\ell_A^p$  for  $1 < p < 2$ ; this question is still unresolved.

These spaces  $\ell_A^p$  are the principal topic of the book under review. The first three chapters give a comprehensive introduction to functional analysis of  $\ell^p$ . Chapter 4 is about weak parallelogram laws. These are inequalities in a Banach space  $\mathcal{X}$  of the following form: for fixed constants  $C > 0$  and  $r > 1$ , the lower parallelogram law is

$$(2.1) \quad \|x + y\|^r + C\|x - y\|^r \leq 2^{r-1}(\|x\|^r + \|y\|^r) \quad \forall x, y \in \mathcal{X}.$$

The upper parallelogram law is when  $\leq$  is replaced by  $\geq$ . There is a complete analysis of which weak parallelogram laws hold for each  $\ell^p$ ,  $1 < p < \infty$ .

Chapter 5 gives a quick summary of some known results in the Hardy space and the Bergman space (the space of holomorphic functions on the disk that are square-integrable with respect to area measure). If  $f \in H^2$ , and  $\hat{f}$  is the projection of  $f$  onto the closed linear span of  $\{z^n f(z) : n \geq 1\}$ , it follows from Beurling’s theorem that  $f - \hat{f}$  is a scalar multiple of an inner function; indeed

$$f - \hat{f} = \frac{f(0)}{u(0)}u,$$

where  $u$  is the inner factor of  $f$ . A critical insight of Hedenmalm [5] was that this could be used to *define* inner functions in a much broader class of spaces, namely  $u$  is inner if and only if it is of unit norm and is orthogonal to  $\{z^n u(z) : n \geq 1\}$ . This revolutionized the theory of Bergman spaces; among other things it led to the proof by Aleman, Richter, and Sundberg [1] that every function  $f$  in the Bergman space can be factored (not necessarily uniquely) as an inner function times a function that is cyclic (for multiplication by the independent variable). Moreover, the inner factor is, up to a constant,  $f - \hat{f}$ .

The spaces  $\ell_A^p$  are introduced, along with their basic properties, in Chapter 6. The range  $0 < p \leq 1$  is quite different from the case  $p > 1$ , since for  $p \leq 1$  each space  $\ell_A^p$  is an algebra, and every function in it has an absolutely convergent Taylor series on  $\overline{\mathbb{D}}$  so is continuous there. For  $p = 1$ , we have the analytic part of the Wiener algebra.

Chapter 7 discusses various operators on  $\ell_A^p$ —the shift, the difference quotient, composition operators, and Lamperti’s characterization of the isometries. The first major difference from the Hardy and Bergman results is exhibited in Chapter 8,

where it is proved that for  $p = 4/3$  there is a polynomial  $f$  so that the function  $J = f - \hat{f}$  has one more zero than  $f$  in  $\mathbb{D}$ . Here we define  $\hat{f}$  to be the closest point to  $f$  in the span  $\{z^n f(z) : n \geq 1\}$ . In particular, we cannot have  $f = Jg$  for any holomorphic  $g$ .

Chapter 9 discusses zero sets of  $\ell_A^p$  functions. Among other results, it is proved that for every  $p > 2$  there is a zero set that does not satisfy the Blaschke condition (1.4), and that for every  $p < 2$  there is a Blaschke sequence that is not a zero set. The weak parallelogram laws are used in lieu of Hilbert space constructions. Chapters 10 and 11 discuss what is known about invariant subspaces for the shift on  $\ell_A^p$  and cyclic vectors for the backward shift. Chapter 12 is about multipliers, and the final Chapter 13 is on the Wiener algebra.

The book is accessible to anyone who has had basic graduate level exposure to complex and functional analysis. The proofs are clear and well motivated. Later in the book statements of some results are included without giving the proofs; this is a good decision, as including the full details would make the book much longer and harder to read.

Many of the theorems about  $\ell_A^p$  are quite delightful, but the theory is still far from complete. This book provides a lovely introduction to the field, and should serve as an invitation to those who wish to delve further and to help fill out the theory.

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