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Tensor products of C-algebras and operator spaces*, by Gilles Pisier, London Mathematical Society Student Texts, Vol. 96, Cambridge University Press, Cambridge, 2020, x+484 pp., ISBN 978-1-108-78208-1

The study of tensor products in functional analysis goes back to the work of Grothendieck. Starting with Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, he considered possible norms on the algebraic tensor product $\mathcal{X} \otimes \mathcal{Y}$ which are compatible with the norm structures on $\mathcal{X}$ and $\mathcal{Y}$. The interest comes from the fact that there are many such norms, and the completions with respect to these norms provide a variety of different new Banach spaces that encode information about linear maps of various types. We will be interested in C*-algebra norms on tensor products of C*-algebras.

Grothendieck's study of tensor norms exposed a great number of finer structural properties that a Banach space might enjoy and was the impetus for many of the deeper problems and theories in that area. The study of C*-algebra norms on tensor products of C*-algebras has played a similar role for the theory of C*-algebras. Many structural properties of C*-algebras can be formulated in terms of the behaviour of this tensor theory.

Pisier's book presents much of the tensor theory of C*-algebras which has developed over the past 60-plus years.

In the last decade, connections between this area and questions in quantum information theory and in computational complexity theory have been made. In fact a recent, not-yet published preprint [2] most likely solves one of the biggest problems in the field using methods from computational complexity and the theory of nonlocal games. We are referring to the set of equivalent problems due to Connes, Kirchberg, and Tsirelson.

Motivated by these results and connections, the next decade is likely to see an increased interest in these theories, and Pisier's book is the clearest presentation of the C*-algebraic approach to these ideas.

We now give an overview of this area and of the material covered in Pisier's book.

Recall that a C*-algebra is a Banach $*$-algebra with the special axiom that $\|a^*a\| = \|a\|^2$. Such algebras always have a faithful (isometric) $*$-representation as a closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$, the space of all bounded linear operators on a Hilbert space $\mathcal{H}$. The commutative examples have the form $C_0(X)$ for some locally compact Hausdorff space X.

If G is a discrete group, $\mathrm{C}^*(G)$ is the universal C*-algebra for unitary representations of G. Every unitary representation $\rho : G \to U(\mathcal{H})$ extends to a representation of the group algebra $\mathbb{C}[G]$. Taking the supremum over all group representations yields a normed $*$-algebra whose completion is called $\mathrm{C}^*(G)$. For every representation ρ, there is a unique quotient map of $\mathrm{C}^*(G)$ whose restriction to G is ρ. In particular, if $\mathbb{F}_\infty$ is the free group on countably many generators (g_n), then $\mathrm{C}^*(\mathbb{F}_\infty)$ has the property that given arbitrary unitary operators $U_n \in \mathcal{B}(\mathcal{H})$, there is a representation π of $\mathrm{C}^*(\mathbb{F}_\infty)$ such that $\pi(g_n) = U_n$. Since every C*-algebra is spanned by its unitary elements, there is a quotient of $\mathrm{C}^*(\mathbb{F}_\infty)$ onto every separable C*-algebra.

A von Neumann algebra is a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the strong operator topology (SOT), the topology of pointwise convergence. The double commutant theorem of von Neumann shows that if $A \subset \mathcal{B}(\mathcal{H})$ is a unital C*-algebra, then $\overline{A}^{\mathrm{SOT}} = A''$, where $A' = \{t \in \mathcal{B}(\mathcal{H}) : at = ta \text{ for all } a \in A\}$ and $A'' = (A')'$. A von Neumann algebra is called a *factor* if it has trivial centre. Work in the 1930s by Murray and von Neumann showed that every von Neumann algebra decomposes into a direct sum of three types. Type I algebras are isomorphic to $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Type II_1 algebras have a trace that maps the projections onto $[0, 1]$, and type II_∞ have a semifinite trace that maps the projections onto $[0, \infty]$. In type III von Neumann algebras, every proper projection is the range of an isometry in the algebra. In particular, they constructed a unique II_1 factor $\mathcal{R}$ which is hyperfinite, meaning that there is an increasing sequence of subalgebras isomorphic to full matrix algebras so that $\mathcal{R}$ is the SOT closure of the union. This algebra plays an important role in our story.

Given two C*-algebras A_1 and A_2, we consider possible C*-norms on the algebraic tensor product $A_1 \otimes A_2$. The completion in such a norm yields a C*-algebra tensor product. One way to obtain such a norm is to choose faithful representations $\pi_i : A_i \to \mathcal{B}(\mathcal{H}_i)$. There is a natural way to form a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, and the map

$$\big(\pi_1(a_1) \otimes \pi_2(a_2)\big)(x \otimes y) = \pi_1(a_1)x \otimes \pi_2(a_2)y$$

extends to a bounded linear map on $\mathcal{H}_1 \otimes \mathcal{H}_2$. This yields a representation of $A_1 \otimes A_2$, which can be closed inside $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. This is called the *spatial tensor product*. We will call this $A_1 \otimes_{\min} A_2$ because a theorem of Takesaki showed that this C*-norm does not depend on which faithful representations are used, and that it is smaller than any other C*-tensor norm on $A_1 \otimes A_2$. One condition imposed by Grothendieck was that the norm should be a *cross-norm*: $\|a \otimes b\| = \|a\| \, \|b\|$. Takesaki's result shows that this is automatic for C*-algebra tensor products.

Takesaki also showed that there is a *maximal* tensor product norm. Given A_1 and A_2, consider all pairs of representations $\pi_i : A_i \to \mathcal{B}(\mathcal{H})$ such that $\pi_1(A_1)$ and $\pi_2(A_2)$ commute. This extends to a $*$-representation of $A_1 \otimes A_2$ that we call $\pi_1 \cdot \pi_2$. For $u \in A_1 \otimes A_2$, define $\|u\|_{\max} = \sup \|\pi_1 \cdot \pi_2(u)\|$ where the supremum runs over all such pairs. The completion is called $A_1 \otimes_{\max} A_2$. Every other C*-tensor product is a quotient of the maximal tensor product and has the minimal tensor product as a quotient.

An *operator system* is a self-adjoint unital subspace S of a unital C*-algebra A. For any $n \geq 1$, $M_n(A)$ has a unique C*-algebra structure which induces norms on $M_n(S)$. If $\phi : S \to \mathcal{B}(\mathcal{H})$ is a linear map, there is a natural map $\phi_n : M_n(S) \to M_n(\mathcal{B}(\mathcal{H}))$ by acting by ϕ on each matrix entry. We say that ϕ is positive if it takes positive elements of S to positive elements of $\mathcal{B}(\mathcal{H})$. We say that ϕ is *completely positive* (CP) if each ϕ_n is positive for $n \geq 1$; and is CCP if in addition each ϕ_n is contractive (has norm at most 1); and is UCP if in addition $\phi(1) = 1$. Operator systems and completely positive maps play a central role in C*-algebra theory. Arveson showed that $\mathcal{B}(\mathcal{H})$ is an injective object in the category of operator spaces and CCP maps. That is, if $S \subset T$ are operator systems and $\phi : S \to \mathcal{B}(\mathcal{H})$ is a CCP map, then there is a CCP map $\psi : T \to \mathcal{B}(\mathcal{H})$ that extends ϕ. When a C*-algebra is called injective, it is in this sense, i.e., that it is an injective object in the category whose objects are operator systems and whose morphisms are CCP maps.

An *operator space* is any subspace X of a unital C*-algebra A, endowed with the family of norms on $M_n(X)$ induced by the inclusion $M_n(X) \subseteq M_n(A)$. These are the objects referred to in the title of Pisier's book. There is an analogous theory of *completely bounded* maps for operator spaces, that parallels the theory of operator systems and completely positive maps.

Often it is useful to forget all of the structure of a C*-algebra, except the fact that it is either an operator system or operator space, and study the tensor theory within those two categories. This is one of the themes of Pisier's book.

In a remarkable series of papers in the mid-1970s, Connes revolutionized the study of von Neumann algebras. In particular, he developed a structure theory for the injective factors, and showed that there is a unique injective factor of type II_1, II_∞, and III_λ for $0 < \lambda < 1$. Here λ represents the spectrum of a certain automorphism group. Later Haagerup showed there is a unique injective III_1 factor, and the complicated III_0 case was explored by Araki, Woods, Kreiger, and Connes. Connes asked a question about the structure of II_1 factors in these papers that has become known as the Connes embedding problem. We will describe this in more detail later in the article.

Connes' work was connected to tensor products by work of Lance and of Choi and Effros. Lance called a C*-algebra A *nuclear* if $A \otimes B$ has a unique C*-norm for every C*-algebra B. In the language introduced above, this means that $A \otimes_{\min} B = A \otimes_{\max} B$ for all B. A C*-algebra A has the *completely positive approximation property* (CPAP) if the identity map id_A is a pointwise limit of a net of CCP maps that factor through matrix algebras. Choi and Effros used Connes' work to show that A is nuclear iff A has CPAP iff the second dual A^{**} is an injective von Neumann algebra. Kirchberg found this independently.

Kirchberg called a C*-algebra A *exact* if whenever $0 \to J \to B \to B/J \to 0$ is an exact sequence of C*-algebras, then

$$0 \to J \otimes_{\min} A \to B \otimes_{\min} A \to B/J \otimes_{\min} A \to 0$$

is also exact. This also has a formulation in terms of CCP maps: say that a map between C*-algebras is *nuclear* if it is the point norm limit of a net of CCP maps that factor through matrix algebras. A C*-algebra is exact iff there is a faithful representation which is a nuclear map. So nuclear C*-algebras are exact, as are subalgebras of nuclear C*-algebras. A deep result of Kirchberg which is important in the classification of simple, nuclear C*-algebras is that every separable exact C*-algebra imbeds into a single nuclear C*-algebra known as the Cuntz algebra, $\mathcal{O}_2$. This is a simple nuclear C*-algebra with trivial K-theory.

Lance said that a C*-algebra A has the *weak expectation property* (WEP) if for every representation $\pi : A \to \mathcal{B}(\mathcal{H})$, there is a UCP map $\Phi : \mathcal{B}(\mathcal{H}) \to \pi(A)''$ so that $\phi|_A = \mathrm{id}_A$. When A is nuclear, $\pi(A)''$ is always injective and thus A has WEP. However WEP is considerably weaker than nuclearity. He showed that if A has WEP and $A \subset C$, then $A \otimes_{\max} B \subset C \otimes_{\max} B$ for all B, the important issue being that this is an (isometric) injection.

A pair (A, B) of C*-algebras is a *nuclear pair* if $A \otimes_{\min} B = A \otimes_{\max} B$. Kirchberg showed that $(\mathrm{C}^*(\mathbb{F}_\infty), \mathcal{B}(l_2))$ is a nuclear pair. This led him to show that A has WEP iff $(\mathrm{C}^*(\mathbb{F}_\infty), A)$ is a nuclear pair. Pisier takes this latter property as his definition of WEP. Moreover Kirchberg showed that A is nuclear iff A is both exact and WEP. He conjectured that $\mathrm{C}^*(\mathbb{F}_\infty)$ has WEP, i.e., $\mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\min} \mathrm{C}^*(\mathbb{F}_\infty) = \mathrm{C}^*(\mathbb{F}_\infty) \otimes_{\max} \mathrm{C}^*(\mathbb{F}_\infty)$. Since every separable C*-algebra is a quotient of $\mathrm{C}^*(\mathbb{F}_\infty)$, it would follow that every separable C*-algebra is QWEP (a quotient of a WEP C*-algebra). Kirchberg made a remarkable connection by showing that his conjecture is equivalent to the Connes embedding problem.

Choi and Effros introduced the *lifting property* (LP): a unital C*-algebra A has LP if whenever B is a unital C*-algebra with ideal J and $\phi : A \to B/J$ is UCP, there is a UCP map $\psi : A \to B$ so that $\phi = q\psi$ (where $q : B \to B/J$ is the quotient map). They showed that if A is separable and nuclear, then it has LP. Kirchberg introduced a variant called the *local lifting property* (LLP): a unital C*-algebra A has LLP if whenever B is a unital C*-algebra with ideal J, $\phi : A \to B/J$ is UCP, and $E \subset A$ is a finite-dimensional operator system, the restriction $\phi|_E$ lifts to a UCP map $\psi : E \to B$ so that $\phi|_E = q\psi$. He showed that A has LLP if and only if $(A, \mathcal{B}(l^2))$ is a nuclear pair. Again Pisier takes this as his definition of LLP. Kirchberg's result from the previous paragraph shows that $C^*(\mathbb{F}_\infty)$ has LLP. If Kirchberg's conjecture were correct, this would imply that separable C*-algebras with LLP also have LP. It is suspected that this fails in the nonseparable case even for the C*-algebra of a free group on uncountably many generators.

Junge and Pisier showed that $\mathcal{B}(l^2)$ does not have LLP by showing that $\mathcal{B}(l^2) \otimes_{\max} \mathcal{B}(l^2) \neq \mathcal{B}(l^2) \otimes_{\min} \mathcal{B}(l^2)$. Pisier asked, and recently answered, the question of whether there were nonexact C*-algebras with both WEP and LLP by constructing an explicit separable example. Ozawa showed that $\mathcal{B}(l^2) \otimes_{\min} \mathcal{B}(l^2)$ does not have WEP.

Now we return to the Connes embedding problem. What he asked was whether every II_1 factor in $\mathcal{B}(l^2)$ imbeds into an ultrapower of the unique injective II_1 factor $\mathcal{R}$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. Form the von Neumann algebra

$$l^\infty(\mathcal{R}) = \{a = (a_n) : a_n \in \mathcal{R} \text{ and } \|a\| := \sup_n \|a_n\| < \infty\}.$$

Let τ be the trace on $\mathcal{R}$, and set $J = \{a \in l^\infty(\mathcal{R}) : \lim_{\mathcal{U}} \tau(a_n^* a_n) = 0\}$. Then J is an SOT-closed ideal, and the ultrapower, $\mathcal{R}^{\mathcal{U}} = l^\infty(\mathcal{R})/J$, is a II_1 von Neumann algebra with trace $\tilde{\tau}(a + J) = \lim_{\mathcal{U}} \tau(a_n)$. It is possible to avoid the use of $\mathcal{R}$ and instead use an ultrapower of matrix algebras (M_{i_n}, τ_i), where τ_i is the trace on M_{i_n} normalized so that $\tau_i(I_{i_n}) = 1$.

If the Connes embedding problem has a positive answer, then one can show that every von Neumann algebra has QWEP. From this it follows that every C*-algebra has QWEP and Kirchberg's conjecture is valid. Conversely, if Kirchberg's conjecture is true, then every von Neumann algebra has QWEP. This leads to the fact that the trace factors through a homomorphism into $\mathcal{R}^{\mathcal{U}}$.

In the past decade connections were made between this tensor theory and *Tsirelson's problems* in quantum mechanics. Briefly, imagine that Alice and Bob have separated labs that are in some entangled state. Furthermore, imagine that Alice can perform one of several measurements indexed by X, with outcomes indexed by A and that Bob can also perform one of several measurements indexed by Y with outcomes indexed by B. We will let $p(a, b|x, y)$ denote the conditional probability density that Alice obtains outcome a and Bob obtains outcome b, given that they performed measurements x and y, respectively. Such conditional densities are also referred to as *quantum correlations*.

There are at least four possible mathematical models for describing the set of all possible quantum correlations and the Tsirelson problems are concerned with whether or not various pairs of these models lead to the same sets of densities. In

[3] and [4] it was shown that two of these models produce the same sets of densities if and only if Kirchberg's conjecture about tensor products is valid. The preprint [2] obtains a separation between these two sets of densities, using computational complexity and the theory of nonlocal games. Thus, their paper shows that the Connes embedding problem and all of the various versions of Kirchberg's conjecture have negative answers. It is now known that all four models yield different sets of densities, and it is not known which model, if any, corresponds to the correlations that one can obtain physically.

Pisier's book is a rather complete exploration of questions around tensor products of C*-algebras, and in particular the connections with the Connes embedding problem and the various forms of Kirchberg's conjecture. It covers much of the material in this review, including a study of nuclear and exact C*-algebras and the WEP and LLP properties. The material is very nicely organized, and like his other books, is very well written. A reader is expected to be familiar with a fair bit of material about C*-algebras, von Neumann algebras, and completely positive maps. This book is jam packed with information, and should be an invaluable guide to anyone interested in these ideas.

One other nice book worth mentioning is [1]. This book expects less background of the reader, and covers much of the early material, especially nuclear and exact C*-algebras and the WEP property. For the complete picture and the recent advances, Pisier's book is the place to go.

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