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*Wigner-type theorems for Hilbert Grassmannians*, by Mark Pankov, London Mathematical Society Lecture Note Series, Vol. 460, Cambridge University Press, Cambridge, 2020, viii+145 pp., ISBN 9781108800327

Wigner's theorem on quantum mechanical symmetry transformations is a cornerstone of the mathematical foundations of quantum mechanics. In this formulation a complex Hilbert space $H$ is used in order to describe experiments at the atomic scale. For instance, the famous Stern–Gerlach experiment can be described using a two-dimensional Hilbert space. Denote by $\mathcal{G}_1(H)$ the projective space of all one-dimensional subspaces, each element of $\mathcal{G}_1(H)$ is interpreted as a quantum pure state. The angle between $P, Q \in \mathcal{G}_1(H)$ is defined by the equation

$$\measuredangle(P,Q) = \arccos |\langle x, y\rangle|,$$

where $x$ and $y$ are unit vectors from $P$ and $Q$, respectively. *Wigner's theorem* states that for every bijective map $\phi\colon \mathcal{G}_1(H) \to \mathcal{G}_1(H)$ which preserves the angle, i.e.,

$$\measuredangle(\phi(P), \phi(Q)) = \measuredangle(P,Q) \quad \text{for all} \quad P, Q \in \mathcal{G}_1(H), \tag{1}$$

there exists either a unitary or an antiunitary operator $U\colon H \to H$ that implements it, i.e., we have

$$\phi(P) = UP \quad \text{for all} \quad P \in \mathcal{G}_1(H). \tag{2}$$

We would like to emphasise that the main (and highly nontrivial) achievement of Wigner's theorem is to conclude this latter form only from the angle-preserving assumption. The theorem plays a crucial role (together with Stone's theorem and some representation theory) in obtaining the general time-dependent Schrödinger equation, that describes quantum systems evolving in time, by purely mathematical means. Wigner originally proved this theorem in his famous book [19], published in the 1930s, and since then many different proofs were given.

Because of its importance there were several attempts to generalize Wigner's theorem. The first ones were obtained by V. Bargmann [1] and U. Uhlhorn [18] at around the same time, in the 1960s. On the one hand, Bargmann gave a new proof of Wigner's theorem that also works for nonbijective maps, hence extends the result. His theorem states that angle-preserving maps are always implemented by linear and conjugate-linear isometries of the underlying Hilbert space. (Note that unitary/antiunitary operators are exactly the bijective linear/conjugate-linear isometries). On the other hand, Uhlhorn generalized Wigner's theorem in a different way. Namely, he proved that in order to obtain the conclusion of Wigner's theorem (2), it is enough to replace (1) by the much weaker assumption that orthogonality of one-dimensional subspaces is preserved, i.e.,

$$\measuredangle(P,Q) = \tfrac{\pi}{2} \quad \iff \quad \measuredangle(\phi(P), \phi(Q)) = \tfrac{\pi}{2},$$

provided that the Hilbert space has dimension at least three. Note that, unlike Wigner's theorem, Uhlhorn's theorem does not generalize for nonbijective maps when the Hilbert space is infinite-dimensional. We also note that the analogous problem for other angles different from $\frac{\pi}{2}$ has been only recently resolved in a series of papers by Li, Plevnik, and Šemrl [13], Gehér [7], and Gehér and Mori [9].

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In the last twenty years there has been a lot of development in generalizing Wigner's theorem for Grassmann spaces. For an integer $1 < k < \dim H$, denote by $\mathcal{G}_k(H)$ the Grassmann space of all $k$-dimensional subspaces. There are many ways to measure the "difference" between two $k$-dimensional subspaces $P, Q$. One such way is to consider the system of all *principal angles*

$$\measuredangle(P, Q) := (\vartheta_1, \dots, \vartheta_k),$$

a notion which originates from Jordan's work [12]. Using modern terminology, one can define the principal angles as the arccosines of the $k$ largest singular values of the (at most rank-$k$) operator $\mathrm{Proj}_P \mathrm{Proj}_Q$, where $\mathrm{Proj}_P$ denotes the orthogonal projection operator onto the subspace $P$. The first generalization of Wigner's theorem on Grassmann spaces was obtained by Molnár in [15], where he described the structure of all (not necessarily bijective) maps on $\mathcal{G}_k(H)$ which preserve the system of all principal angles between subspaces. It turns out that these maps are always induced by linear or conjugate-linear isometries of the underlying Hilbert space, provided that $\dim H \neq 2k$. In the exceptional $2k$-dimensional case there is another option, namely, the transformation which maps every subspace into its orthogonal complement (which in this case is also $k$-dimensional).

Instead of measuring the the difference between two $k$-dimensional subspaces with a sequence of $k$ numbers, it is often more appealing to do that with a single number. This typically happens by considering a function of the principal angles. For instance, one popular choice is to take the sum of the squared sines of the principal angles. This quantity (up to a scalar factor) equals to the Hilbert–Schmidt norm distance of the associated $k$-rank orthogonal projections, and it is also a natural generalization of the notion of transition probability between pure quantum states. The characterization of all (not necessarily bijective) maps on $\mathcal{G}_k(H)$ which preserve this quantity was proved in [8] by the reviewer. It turns out that, despite the significantly weakened assumption on our map, the structure of these transformations is exactly the same as given by Molnár's theorem from the previous paragraph. We also note that in a recent interesting paper [16], Qian, Wang, Wu, and Yuan further generalized this result to the setting of semifinite factors.

Another natural choice to measure the difference between $k$-dimensional subspaces by a single number is to take the maximum of the principal angles. This choice is indeed natural, because the sine of the largest principal angle coincides with the operator/spectral norm distance of the rank-$k$ orthogonal projections whose ranges are these subspaces. The problem of characterizing all bijective maps on $\mathcal{G}_k(H)$ which preserve the largest principal angle was resolved in a series of three papers by Botelho, Jamison, and Molnár [3] and by Gehér and Šemrl [5, 6]. The final conclusion is that these maps are always induced by unitary or antiunitary operators, except in the $2k$-dimensional case where taking the orthogonal complement is a further option. However, unlike in the cases of Molnár's and the reviewer's theorems ([8, 15], respectively), the nonbijective version of this problem is still open. In fact it is considered to be one of the most challenging open problems in the field. We also would like to note that the latter theorem has been recently further generalized to the setting of von Neumann algebras in [14] by Mori.

It is also a natural question whether Uhlhorn's theorem can be generalized for Grassmann spaces. In fact this question plays an important role in the proofs of the above mentioned results. For instance, an essential step in [5] was to show that the maps under question preserve orthogonality of subspaces. Uhlhorn's theorem was first generalized independently by Győry in [11] and by Šemrl in [17] at around the same time in the early 2000s. We mention that the result contained a dimensionality constraint that was eventually removed completely in [5].

Let us also mention a group of exciting open problems whose solutions would signify important advancements in the field. The most general way to measure the difference between $k$-dimensional subspaces $P$ and $Q$ is to consider the function

$$d_g(P,Q) = g(\sin\vartheta_1, \ldots, \sin\vartheta_k),$$

where $(\vartheta_1, \ldots, \vartheta_k)$ are the principal angles and $g\colon \mathbb{R}^k \to \mathbb{R}$ is a so-called symmetric gauge function. In such a way we essentially define the difference between $P$ and $Q$ as a unitarily invariant norm-distance between the orthogonal projections with these ranges. (More information on how symmetric gauge functions and unitarily invariant norms are connected can be found in [2, Chapter IV].) Note that the cases when $g$ is the $\ell^2$ or $\ell^\infty$ norm are exactly the cases discussed above ([3,5,6,8]). We also note that some work has been done recently on the case when the gauge function is the $\ell^p$ norm; see [10]. However, apart from the $\ell^p$-norm cases, nothing is known about the structure of those maps that preserve the distance $d_g(P,Q)$ between $k$-dimensional subspaces. A particularly interesting special case would be when $g$ is a so-called Ky Fan norm:

$$g\left(x_1^\downarrow, \ldots, x_k^\downarrow\right) = \left(\sum_{i=1}^{m} (x_1^\downarrow)^p\right)^{1/p}$$

for some $1 < m < k$, $p \geq 1$, and where $x_1^\downarrow, \ldots, x_k^\downarrow$ is the monotone decreasing rearrangement of $x_1, \ldots, x_k$. Of course the conjecture in general is that these maps are always induced by linear or conjugate linear isometries of the underlying Hilbert space, except in $2k$-dimensions where ortho-complementation is the only additional possibility.

Last but not least, let us introduce the structure of Pankov's book. The book delivers a nice presentation of the topic both from the algebraic and geometric points of view. The first introductory chapter discusses basic properties of the lattice of all subspaces of a vector space, and the orthomodular lattice of all closed subspaces of a complex Hilbert space. The second chapter proves a generalized version of the fundamental theorem of projective geometry and Chow's famous theorem on the fundamental theorem of the geometry of Grassmann spaces (see [4]). At the end of the chapter the author discusses further related results, some of which were obtained by himself. Chapter 3 proves the Kakutani–Mackey theorem, which is another famous generalization of the fundamental theorem of projective geometry for the setting of the lattice of all closed subspaces of an infinite-dimensional Banach space. In Chapter 4, the author proves Wigner's theorem and discusses some of the generalizations. The reader also has the opportunity here to obtain a good understanding of the notion of principal angles. The penultimate chapter discusses automorphisms with respect to the compatibility relation. In particular the author proves three theorems by Molnár and Šemrl, by Plevnik, and by Pankov. The book closes with presenting Kadison's theorem on the automorphisms of the convex set of all quantum states over a Hilbert space, and along this line he discusses some possible generalizations of Wigner's theorem.

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