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The Bellman function technique in harmonic analysis, by Vasily Vasyunin and Alexander Volberg, Cambridge Studies in Advanced Mathematics, Vol. 186, Cambridge University Press, 2020, xviii–446 pp.

I first heard about the Bellman function method through the grapevine as a new postdoc, and it seemed to have a fearsome reputation—of being a method too difficult to understand. It was a couple of years later that I dove into it. I will never forget the moment when I understood my first Bellman proof, a mathematical moment of "love at first sight" for me. The beauty and cleverness of the method was so overwhelming that I had to get up and take a walk. So, I could not have been happier to hear that a book on this topic was finally being written. A book on Bellman functions has been greatly, greatly needed—I can say with joy that this book answers the call as a true masterpiece.

The method originates in the works of Richard Bellman in the area of stochastic optimal control theory. Burkholder [1] then brought the method close to harmonic analysis, using it to obtain sharp inequalities for martingale transforms. Then the method really took off in harmonic analysis with the seminal works [3, 4] and subsequent works by Nazarov, Treil, and Volberg, and many others (Petermichl, Osekowski, Bañuelos, Vasyunin, Kovač, Melas, Slavin, and many others).

Rather than just speaking abstractly about the history and influence of the Bellman function method, it is perhaps better to begin with a concrete example to illustrate the basic ideas of the method. The book itself begins with this exact "toy problem" (which can of course be solved using trivial considerations, but nonetheless we treat for now as a serious problem):

> Say we have two nonnegative functions f_1, f_2 on a real interval I, both bounded by 1: $0 \leq f_i \leq 1$, and we ask: how large can their inner product be? Namely, let us maximize the quantity
>
> $$\langle f_1 f_2 \rangle_I := \frac{1}{|I|} \int_I f_1(x) f_2(x)\, dx.$$

A typical Bellman function problem is always an extremal problem and an optimization problem: we always ask how large (or small) can a certain quantity be and, moreover, we wish to find an *optimal*, or *sharp*, upper or lower bound. This is particularly apparent in weighted inequalities, such as bounding an operator $T : L^p(u) \to L^q(v)$ where u and v are weights (locally integrable, almost everywhere positive functions w which automatically define a measure dw via the standard $\int f\, dw := \int f(x)w(x)\, dx$). Of particular interest in harmonic analysis are the *Muckenhoupt* A_p weights [2]—roughly speaking, the weights which characterize boundedness of the maximal function on $L^p(w)$—and the sharp dependence of $\|T\|_{L^p(w)}$ on the Muckenhoupt characteristic of the weight w,

$$[w]_{A_p} := \sup_Q \langle w \rangle_Q \left\langle w^{1-p'} \right\rangle_Q^{p-1},$$

where supremum is over all cubes Q in $\mathbb{R}^n$, $1 < p, p' < \infty$ are Hölder conjugates, and $\langle f \rangle_Q$ denotes the average of f over Q. A large chunk of the activity in current harmonic analysis research revolves around sharp weighted inequalities. Many of these problems are very difficult, and several have been solved efficiently using the Bellman function method.

Returning to our toy problem (which is unweighted), let us see how we would solve it using Bellman functions. The Bellman function of the problem will involve

$$\sup \langle f_1 f_2 \rangle_I ,$$

which makes sense since this is the quantity we wish to maximize. But supremum over what? A first step (inherited from the probabilistic origins of this method) is always to *fix the* (*relevant*)[1] *averages* of all functions involved. In this simple case, we have two functions f_1 and f_2, so we fix

$$\langle f_i \rangle_I =: x_i, \;\; i = 1, 2.$$

Now we may finally define the Bellman function of this problem:

$$\mathbb{B}(x_1, x_2) := \sup \{ (f_1, f_2)_I : 0 \leq f_i \leq 1; \, \langle f_i \rangle = x_i \} .$$

Since our functions satisfy $0 \leq f_i \leq 1$, necessarily $0 \leq x_{1,2} \leq 1$, and since, for example, the constant functions $f_i \equiv x_i$ have $\langle f_i \rangle_I = x_i$ for all $0 \leq x_{1,2} \leq 1$, we have obtained the first fact about our Bellman function, namely its domain: $\Omega = [0, 1]^2$.

You may have noticed that the particular choice of interval I did not appear to matter in the definition of $\mathbb{B}$ above. This is because it does not matter: a simple linear change of variable translates the problem immediately from one real interval I to another one J, *preserving the averages* and inner products, and therefore the set $\mathbb{B}$ takes supremum over is the same. This sort of invariance is a crucial feature of any problem suited for a Bellman approach.

From here on out, the game is to determine enough properties of the Bellman function in order to find it (or to find a "good enough" function, but more on this later). Some types of properties are standard in most Bellman problems—such as the invariance above, something called a *main inequality*, something called an *obstacle condition*, and the so-called *Bellman induction*—but each problem is unique. And herein lies, in my opinion, the most beautiful feature of this method: the objects inhabiting the original problem (may they be weights, square functions, maximal functions, BMO functions, etc.) imprint their own features onto the Bellman function, making each problem a new and interesting challenge.

Here are some examples of properties of $\mathbb{B}$ we may deduce easily for this toy problem.

- **Symmetry.** It is easy to see that $\mathbb{B}(x_1, x_2) = \mathbb{B}(x_2, x_1)$, since interchanging the roles of f_1 and f_2 does not affect the inner product.
- **Boundary conditions.** Say $x_1 = 0$. Since f_1 is nonnegative, this necessarily implies $f_1 = 0$ a.e. So the class of functions in the first slot collapses to just the zero function, making $(f_1, f_2) = 0$ regardless what f_2 is doing. So,

$$\mathbb{B}(0, x_2) = \mathbb{B}(x_1, 0) = 0.$$

[1] In the simple case we are considering here, we have no reason to care about, for example, $\langle f_i^2 \rangle_I$. But say we are working on a problem involving the L^2 norm of some f—in this case we must fix $\langle f^2 \rangle_I =: x$, and x will be one of the variables of the Bellman function.

Similarly, if $x_1 = 1$, this forces $f_1 = 1$ a.e., so once again the role of f_1 is completely eliminated, and $\langle f_1 f_2\rangle_I = \langle f_2\rangle_I = x_2$:

$$\mathbb{B}(1, x_2) = x_2;\ \ \mathbb{B}(x_1, 1) = x_1.$$

Just pause for a moment and absorb how much information we already got about this function, all coming from the structure and setup of the original problem!

Now come the heavier things, mentioned before. The main inequality is a cornerstone of Bellman methods and it usually is a direct consequence of the very simple (but crucial!) invariance property. Simply put, independence of $\mathbb{B}$ from the choice of interval I allows one to pick some interval I, split it into its left and right halves $I_\pm$, and run the Bellman machine *separately* on each half, then put everything together at the end. This yields a *discrete inequality*—in the case of our toy example it is

$$\mathbb{B}(\alpha^- x^- + \alpha^+ x^+) \geq \alpha^- \mathbb{B}(x^-) + \alpha^+ \mathbb{B}(x^+),$$

where x^+, x^- is any pair of points in the domain Ω, and $\alpha^- + \alpha^+ = 1$. In differential language, this means simply that the Hessian of $\mathbb{B}$ is negative semidefinite. And this is ultimately the point of the discrete main inequality, to obtain a differential form which leads to (drum roll, please) the PDE of the problem! This may sound like a good and even simple thing—"Oh, I can just translate this weird harmonic analysis operator inequality into a PDE"—but the PDEs get extremely difficult/impossible, extremely fast.

At this stage, anyone even slightly experienced with Bellman sees that this is indeed a *toy* problem, because the main inequality above is simply a *concavity* condition. In an even slightly serious Bellman problem, this never happens—but it frequently gets maddeningly close to "almost" happening. One usually obtains something very close to concavity (or convexity) but there is always a pesky extra term (or many!) which frequently leave one with a Monge–Ampére equation *with a drift*.

Another standard Bellman aspect mentioned before is the *obstacle condition*, which is typically used in the *Bellman induction*. For instance, in our toy problem, here is how one may come upon an obstacle condition: the constant functions $f_i \equiv x_i$ have the prescribed averages, so they trivially qualify for the supremum (any function that qualifies is usually called an *admissible function*). What can these obvious functions tell us? Well, if they qualify, that means $\mathbb{B}(x_1, x_2) \geq \langle f_1 f_2\rangle_I = x_1 x_2$. And that's the obstacle condition!

$$\mathbb{B}(x_1, x_2) \geq x_1 x_2.$$

What is this used for? This brings us to the last thing mentioned earlier but left in the air, this notion that it is often enough to find a "good enough" function: in a typical problem, we say some function B on the domain of $\mathbb{B}$ is a *least supersolution* if it satisfies the main inequality and the obstacle condition. It is called so because, with the use of Bellman induction—a sort of induction on scales where the obstacle condition usually serves as a stopping condition—one proves an inequality of the type

$$B(x) \geq \mathbb{B}(x).$$

In other words, the "true" Bellman function $\mathbb{B}$ dominates any other function on its domain which satisfies the main inequality and the obstacle condition. This is another extraordinary feature of the method: in many important problems solved via Bellman, the *true Bellman function is NOT known!* Instead, a supersolution was found (by solving some complicated PDE fitting the boundary conditions dictated by the problem), and that is usually enough to prove the sharp inequality one started with. Sometimes, there even is no extremal function but only a sequence of *almost extremizers*.

The book begins however with a chapter describing nine instances of problems where the exact Bellman function can be found, starting with the Bellman function for the toy problem above:

$$\mathbb{B}(x_1, x_2) = \min\{x_1, x_2\}.$$

Each one of the nine problems in this chapter adds building blocks to the reader's understanding and prepares for the road ahead. It also introduces the reader to some of the typical objects one encounters in this area, such as Muckenhoupt weights, BMO function, maximal operators, or square functions. Some needed background on Monge–Ampére equations comes in handy for anyone dealing with Bellman functions but without a solid footing in PDE theory. Aided by many revealing figures, this first chapter is an airtight introduction to the Bellman method. Chapter 2 is the great connector, taking a more bird's eye view, where the connections between applying Bellman in harmonic analysis and the stochastic origins of the method are made perfectly clear. Moreover, it makes an incursion to complex analysis and discusses how some problems there may be tackled with Bellman. A shorter Chapter 3 obtains sharp estimates for conformal martingales and introduces the Ahlfors–Beurling transform, which is expanded on in Chapter 4. Chapter 4 goes deep into dyadic models, continued in the very rich Chapter 5, which contains discussions on many important problems, some of which are still open.

One can find several expository articles on Bellman functions, some more detailed than others, but this is truly a *book* on the subject—it takes its time with explanations, including the little details long known to experts but usually skipped in research papers which can baffle a novice. It builds examples and notions on top of one another, and does so in a carefully planned manner conducive to student learning. This is aided by many exercises and excellent figures alongside explanations. The book is perfectly suited both for a graduate course textbook and as a reference for seasoned mathematicians looking to learn quickly what they need to know about this method in order to attempt applying it to their own problems.

References

[1] D. L. Burkholder, *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Probab. **12** (1984), no. 3, 647–702. MR744226

[2] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226, DOI 10.2307/1995882. MR293384

[3] F. L. Nazarov and S. R. Treil, *The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis* (Russian, with Russian summary), Algebra i Analiz **8** (1996), no. 5, 32–162; English transl., St. Petersburg Math. J. **8** (1997), no. 5, 721–824. MR1428988

[4] F. Nazarov, S. Treil, and A. Volberg, *The Bellman functions and two-weight inequalities for Haar multipliers*, J. Amer. Math. Soc. **12** (1999), no. 4, 909–928, DOI 10.1090/S0894-0347-99-00310-0. MR1685781

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