# BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 59, Number 2, April 2022, Pages 283–287
https://doi.org/10.1090/bull/1763
Article electronically published on January 4, 2022

*A course on rough paths: With an introduction to regularity structures; Second edition*, by Peter K. Friz and Martin Hairer, Universitext, Springer, Cham, 2020, xvi+346 pp., ISBN 978-3-030-41556-3, ISBN 978-3-030-41555-6

## 1. Why the theory of rough paths?

Stochastic calculus aims at giving meaning to differential systems of the form $\dot{x}_t = f(x_t)\dot{\xi}_t$ driven by signals $\dot{\xi}$ which are random and typically nondifferentiable, such as white noise. Such systems are conveniently rewritten in the integral form

$$x_t = x_0 + \int_0^t f(x_s)\,\mathrm{d}\xi_s, \tag{1}$$

where $\xi$ is defined as a Gaussian noise. Equation (1) generalizes differential equations. One common situation is the case of *stochastic differential equations* (SDEs), where $\xi$ is a Brownian motion. When the solution $x$ takes its values in a functional space, $\xi$ is space-time white noise, and $f(x_t) = Ax_t$ for an unbounded operator $A$, then (1) is called a *stochastic partial differential equation* (SPDEs).

SDEs and SPDEs are prevalent in modeling of noisy phenomenon. Itô's and Stratonovich's calculus are the most common tools to give meaning to the solutions to (1). Although efficient, their use restricts the class of processes $\xi$ to Brownian motion and semimartingales. Considering other processes, such as the fractional Brownian motion, requires ad hoc integration.

By their very constructions, the integral $\int f(x_s)\,\mathrm{d}\xi_s$ is defined as a limit in probability of approximations either by Riemann sums or by ODEs with a regularized noise. This has various implications that make SDEs different from ODEs. For example, one should be careful when designing implicit or adaptive numerical schemes because they use *anticipative* quantities while Itô calculus is designed for *nonanticipative* integrand, i.e., containing no information about the future.

Revisiting the formalism of K.T. Chen on iterated integrals, T. Lyons provided at the end of the 1990 in [9,10] a *pathwise* construction of SDEs with a core feature that the integral is *continuous* with respect to the driving noise, at the price of enhancing it. This continuity property is very useful for modeling and simulating. Besides, the pathwise construction avoids distinction between anticipative and nonanticipative integrands. At last, the construction is no longer restricted to the Brownian motion but may be applied to any other random or deterministic driving signal, provided we suitably enhance the latter.

The core idea of K.T. Chen is to treat ODEs as formal algebraic objects (see, e.g., [1]). If $x : [0, T] \to \mathbb{R}^d$ is a differentiable path and $f_1, \dots, f_d$ is a collection of smooth functions from $\mathbb{R}^m$ to $\mathbb{R}^m$, the $\mathbb{R}^m$-valued solution $y = \{y^\ell\}_{\ell=1,\dots,m}$ of the ODEs

$$y_t^\ell = a + \int_s^t \sum_{i=1}^d f_i^\ell(y_r)\, v dx_r^i,\ s \leqslant t \leqslant T,\ \ell = 1, \dots, m,$$

may be expanded as

$$y_t^\ell = a + \sum_{i=1}^d f_i^\ell(a) \int_s^t \mathrm{d}x_{t_1}^i + \sum_{\substack{i,j=1,\dots,d \\ k=1,\dots,m}} f_j^k(a) \frac{\partial f_i^\ell(a)}{\partial x_k} \int_s^t \int_s^{t_1} \mathrm{d}x_{t_2}^j \, \mathrm{d}x_{t_1}^i + \cdots ,$$

where the "$\cdots$" involve iterated integrals of higher orders as coefficients of the $\mathbb{R}^m$-valued vector field $\sum_{j=1}^m f_i^j \frac{\partial}{\partial x_j}$ applied to themselves. The family of iterated integrals

$$X_{r,t} = 1 + \int_r^t \mathrm{d}x_s + \int_r^t \int_r^{t_1} \mathrm{d}x_{t_2} \otimes \mathrm{d}x_{t_1} + \cdots$$

is itself best seen as the solution of the linear equation in the Banach tensor (non-commutative) algebra $T(\mathbb{R}^d) = \bigoplus_{k \geqslant 0} (\mathbb{R}^d)^{\otimes k}$ defined by

$$X_{r,t} = 1 + \int_r^t X_{r,s} \otimes \mathrm{d}x_s,\ r \leqslant t, \tag{2}$$

as it is easily seen by solving (2) using the Picard principle. In particular, $X = \{X_{r,t}\}_{r \leqslant t}$ satisfies the *Chen relation*: $X_{r,s} \otimes X_{s,t} = X_{r,t}$ for any $r \leqslant s \leqslant t$. This is the algebraic counterpart of the concatenation of the paths $\{x_u\}_{u \in [r,s]}$ and $\{x_u\}_{u \in [s,t]}$. Similarly, time inversion $\{x_{t-u}\}_{u \in [r,t]}$ is translated by inverting $X_{r,t}$ in $T(\mathbb{R}^d)$.

When $x$ is a continuous-by-irregular path, say an $\alpha$-Hölder path with $\alpha < 1$, then one cannot define iterated integrals such as $\int_r^t \int_r^{t_1} \mathrm{d}x_{t_2} \otimes \mathrm{d}x_{t_1}$ unless $\alpha > 1/2$ where Young integration is used.

The main idea of the theory of rough paths is to consider that above an $\alpha$-Hölder driving signal $x$, we are granted $X = \{X_{r,t}\}_{r \leqslant t}$, called a *rough path*, at least in a truncated tensor algebra (in which case, we may reconstruct the whole series) at an order $p = \lfloor 1/\alpha \rfloor$—the lower the regularity, the higher the order. Such an enhancement $X$ above a given driving signal $x$ is *not* unique. Each extension gives a different integrals. For the Brownian motion (paths of Hölder regularity $1/2 - \epsilon$ almost surely), it requires an extension up to order 2. Itô and Stratonovich integrals are recovered using Itô and Stratonovich iterated integrals.

The construction of integrals in the theory of rough paths is completely deterministic. However, such a theory is well suited in stochastic analysis, not only to deal with Brownian motion and martingales, but also to many other processes such as the fractional Brownian motions. Getting a full theory of integration only requires showing the existence of the iterated integrals.

## 2. The theory of rough paths over 20 years

The theory of rough paths is now a vivid field of research that has expanded in many directions and in relationship with several fields of pure and applied mathematics. It is gathered with related theories in the 2020 AMS classification as 60L.

It would be impossible to give here an exhaustive panorama, so instead we focus on some main past and present directions of research.

Beyond the original construction of T. Lyons, many alternative constructions have been proposed, which rely on distinct — albeit related — notions of solutions. The one of A.M. Davie [2], based on numerical schemes, and the one of M. Gubinelli [5], the so-called *controlled rough paths* (used in the book under review), are now the most commonly used. Besides, a complete series of works have shown that rough differential equations are natural extensions of ODEs that share many of their properties: existence, uniqueness, transfer of the regularity of the vector field to the one of the Itô-Lyons map (the map that gives the solution from the driving signal when the solution is unique), generic conditions, existence of global solutions, etc., provided that one uses the suitable topology.

Another body of work has been devoted to extending the *sewing lemma*, which is the technical tool at the heart of the rough paths theory. The name appeared in an article from D. Feyel et al. [4], while the concept generalized ideas proposed by L.C. Young [11] for his integration theory, now partially a subcase of the theory of rough paths. For example, an integral is characterized by the Chasles property $\int_r^t \cdots = \int_r^s \cdots + \int_s^t \cdots$. Assuming we are given an *approximation* $I_{s,t}$ of the integral $S_{r,t} = \int_r^t \cdots$ even *without* having a formal definition of $S_{s,t}$. With the additive sewing lemma, one may *project* $I_{s,t}$ onto $S_{s,t}$ which satisfies $S_{r,s} + S_{s,t} = S_{r,t}$, provided that one may control $I_{r,s} + I_{s,r} - I_{s,t}$. Hence, $S$ is rigorously defined. The sewing lemma is an effective tool with now many declinations: additive, multiplicative (to construct multiplicative integrals or solutions of linear equations, even involving nonbounded operators), stochastic, nonlinear (construction of flows), etc.

Another research direction that appeared early in the literature is to apply this theory to perform stochastic integration for a large class of stochastic driving signals. It is perhaps mostly used for fractional Brownian motion or Gaussian processes, where many results similar to the ones known on SDEs have been proved despite the lack of the Markov property: the Hörmander and support theorems, concentration theorem, etc.

The theory of rough paths is also strongly connected with the theory of geometric integrations. Beyond the extension from tensor algebras to tree algebras, the natural and suitable underlying algebraic structures are related to Hopf algebras (shuffle, quasi-shuffle, etc.) and tools from renormalization theory.

An indirect byproduct of the theory of rough paths is the theory of regularity structures [7], one of the contributions for which M. Hairer was awarded the Fields medal. As for the theory of rough paths, this theory mixes both analysis (a formal Taylor expansion of distributions at each point) and algebra (an algebraic relationship between the objects) to describe the solution of PDEs in the presence of singularities. The *reconstruction lemma* is the counterpart of the sewing lemma. Within this framework, M. Hairer [8] was able to give a rigorous meaning to the Kardar–Parisi–Zhang (KPZ) equation and other singular (S)PDEs arising in mathematical physics, such as the $\Phi^3_4$ equation of quantum field theory. This topic is now strongly connected to algebraic approaches in renormalization theory such as the Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) approach and Connes and Kraimer Hopf algebra.

Finally, a recent trend of research concerns machine learning: rough paths encode some nonlinear features of a signal in a low-dimensional space. Using iterated integrals to classifications and other learning procedures leads to state-of-art techniques, for example in hand-writing recognition.

Many more directions are now explored. Let us cite for example applications to mean-field differential equations, dynamical systems, filtering, Monte Carlo techniques, volatility models (a model in finance that may be tackled with tools from regularity structures), model free finance, and other various theoretical and applications fields.

## 3. The content of the book

The book [3] introduces the theories of rough paths and regularity structures. Basically, its content may be divided into four categories.

- Construction of rough integrals and rough differential equations (RDEs), using mostly the framework of *controlled rough paths* [5]. Chapters 2, 4, 7, and 8 are essential to apprehend the core of the theory: existence, uniqueness, continuity with respect to the driving signal, the impact of the regularity of the driving path, flows, numerical schemes, and many of the properties on RDEs that are similar to those on ODEs.
- Applications to stochastic analysis (Chapters 3, 5, 6, 9, 10, and 11) regarding Brownian motion seen as a rough path, martingales, and Gaussian rough paths. In particular, parallels are drawn between RDEs and SDEs, the recent stochastic sewing lemma is presented, and several recent and original extensions of stochastic calculus, including support and Hörmander theorems for Gaussian rough paths, extension to Burkhölder–Davis–Gundy inequality to rough martingales, etc. Together with their exercises, these chapters contain a grand tour of many of the possible usages of rough analysis to stochastic analysis.
- SPDEs using ideas from rough paths theory (Chapter 12). The basic idea is to use the continuity to pass from a regularized noise to a *true* noise (in time, but also in space and time) to deal with PDEs perturbed by noises. Several types of PDEs are thus covered by this chapter: the rough transport equation for measures; second-order PDEs with linear noise through the Feynman–Kac formula, and then semilinear PDEs using mild notions of solutions which requires one to define some convolutional rough integrals; fully nonlinear equations for which the solutions are rough viscosity solutions in the presence of space-time white noise; stochastic heat equations with space-time white noise. This chapter ends up with several exercises with applications of such SPDEs to filtering.
- SPDEs and regularity structures to deal with singular situations. Chapters 13 and 14 contain some introduction to the analytic and algebraic aspects of the regularity structures with examples of the so-called *polynomial and rough path models*. A new proof of the reconstruction lemma is given. Afterward, operations such as product, differentiation and integration are presented, as they are useful to write down some Schauder estimates, which are estimates on the solutions of the SPDEs. Some applications to rough volatility models are given. Chapter 15 is devoted to the KPZ equation and to showing how the regularity structures give meaning to such equations despite diverging terms. The renormalization strategy is carefully presented.

As we see, the applicative side is focused on stochastic analysis, a topic in which several original and recent results are collected. However, the general sections are of interest for any mathematician willing to apprehend this theory.

As underlined above, the theory of rough paths now evolves in many directions. The book uses the framework of controlled rough paths, albeit there are several alternatives such as the original construction from T. Lyons or the flow *à la Davie* and their extensions. An alternative to regularity structure, the approach through *paracontrolled distributions* [6] is not presented. Results related to the fractional Brownian motion are briefly covered, as well as the treatment of lower-order regularity. Application to data science (*signature methods*), which is a topic by itself, is not mentioned. The authors never had the aim to be exhaustive on the subject but instead to "complement the existing literature on the subject". An exhaustive book would run over many more pages and would be impossible to write, so fast is the pace.

## 4. Conclusion

Written by two leading experts of the field, this book offers a nice and smooth introduction to the theory of rough paths and regularity structures, with an insight into their applications to stochastic calculus and the resolution of singular SPDEs. With detailed proofs and up-to-date developments come many examples and applications. Each chapter ends with a list of exercises. This book is well suited both for newcomers in the field as well as experts. It may be recommended to any mathematician willing to learn more about this fascinating topic.

## References


[1] K.-T. Chen, *Integration of paths—a faithful representation of paths by non-commutative formal power series*, Trans. Amer. Math. Soc. **89** (1958), 395–407, DOI 10.2307/1993193. MR106258

[2] A. M. Davie, *Differential equations driven by rough paths: an approach via discrete approximation*, Appl. Math. Res. Express. AMRX (2008), Art. ID abm009, 40. [Issue information previously given as no. 2 (2007)]. MR2387018

[3] P. K. Friz and M. Hairer, *A course on rough paths: With an introduction to regularity structures; Second edition of [MR3289027]*, Universitext, Springer, Cham, 2020, DOI 10.1007/978-3-030-41556-3. MR4174393

[4] D. Feyel, A. de La Pradelle, and G. Mokobodzki, *A non-commutative sewing lemma*, Electron. Commun. Probab. **13** (2008), 24–34, DOI 10.1214/ECP.v13-1345. MR2372834

[5] M. Gubinelli, *Controlling rough paths*, J. Funct. Anal. **216** (2004), no. 1, 86–140, DOI 10.1016/j.jfa.2004.01.002. MR2091358

[6] M. Gubinelli, P. Imkeller, and N. Perkowski, *Paracontrolled distributions and singular PDEs*, Forum Math. Pi **3** (2015), e6, 75, DOI 10.1017/fmp.2015.2. MR3406823

[7] M. Hairer, *A theory of regularity structures*, Invent. Math. **198** (2014), no. 2, 269–504, DOI 10.1007/s00222-014-0505-4. MR3274562

[8] M. Hairer, *Solving the KPZ equation*, Ann. of Math. (2) **178** (2013), no. 2, 559–664, DOI 10.4007/annals.2013.178.2.4. MR3071506

[9] T. J. Lyons, *Differential equations driven by rough signals*, Rev. Mat. Iberoamericana **14** (1998), no. 2, 215–310, DOI 10.4171/RMI/240. MR1654527

[10] T. Lyons and Z. Qian, *System control and rough paths*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2002. Oxford Science Publications, DOI 10.1093/acprof:oso/9780198506485.001.0001. MR2036784

[11] L. C. Young, *An inequality of the Hölder type, connected with Stieltjes integration*, Acta Math. **67** (1936), no. 1, 251–282, DOI 10.1007/BF02401743. MR1555421



Antoine Lejay

Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France

*Email address*: antoine.lejay@univ-lorraine.fr