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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
HAROLD WIDOM

MR0331107 (48 #9441) 47B35; 45E10

Widom, Harold

Toeplitz determinants with singular generating functions.

American Journal of Mathematics **95** (1973), 333–383.

The present paper represents a jump of several quanta in depth and sophistication in an area which is not only of great interest to mathematicians, but to theoretical physicists as well. Since the introduction of this paper gives a uniquely lucid summary of its contents, the review below is simply part of this introduction.

From the author's introduction: "The strong Szegő limit theorem states that if the generating function σ is sufficiently smooth, nonzero, and satisfies the condition (1) $\Delta_{-\pi\le\theta\le\pi}\arg\sigma(\theta) = 0$ then for the Toeplitz determinant $D_n[\sigma] = \det(\frac{1}{2}\pi^{-1}\int_{-\pi}^{\pi} e^{-i(j-k)\theta}\sigma(\theta)\,d\theta)$ $(j,k=0,\cdots,n)$ one has an asymptotic formula $D_n[\sigma] \sim E[\sigma]G[\sigma]^{n+1}$ as $n\to\infty$. Here $G[\sigma]$ and $E[\sigma]$ are defined by $G[\sigma] = \exp s_0$ and $E[\sigma] = \exp\sum_{k=1}^{\infty} ks_k s_{-k}$, respectively, where $\log\sigma(\theta) = \sum_{k=-\infty}^{\infty} s_k e^{ik\theta}$. This was originally proved by G. Szegő [Medd. Lunds Univ. Mat. Sem. Tome Suppl. (1952), 228–238; MR0051961] under the assumption that σ is positive and has a derivative satisfying a Lipschitz condition with some positive exponent, although these conditions have by now been weakened considerably. If σ has zeros or singularities, this formula breaks down. A. Lenard [see the preceding review, MR0331106] conjectured that if $\sigma(\theta) = \tau(\theta)\prod_{r=1}^{R}(2-2\cos(\theta-\theta_\tau))^{a_r}$ with each $a_r > -\frac{1}{2}$ and τ satisfying (1) then the correct asymptotic formula takes the form $D_n[\sigma] \sim E[\tau,\alpha_1,\cdots,\alpha_r,\theta_1,\cdots,\theta_r]n^{\sum\alpha_r{}^2}G[\tau]^{n+1}$ for some constant $E[\tau,\alpha_1,\cdots,\alpha_r,\theta_1,\cdots,\theta_r]$. Moreover, he was able to verify this conjecture in the case $R=2$, $\theta_1=0$, $\theta_2=\pi$, $\tau(\theta)=1$. M. E. Fisher and R. E. Hartwig [Adv. Chem. Phys. **15** (1968), 333–353] gave a heuristic argument leading to a conjecture concerning the form of the constant $E[\tau,\cdots,\theta_r]$.

"In this paper we shall present a proof of the general conjecture under the assumption that τ satisfies (1) and has a derivative satisfying a Lipschitz condition with some positive exponent. We assume that $G[\tau]=1$ which is clearly no loss of generality. Then we shall find that, with $E[\tau]$ defined as before, $E[\tau,\alpha_1,\cdots,\ \alpha_r,\theta_1,\cdots,\theta_r] = E[\tau]\prod_{r\ne s}|e^{i\theta_r}-e^{i\theta_s}|^{-\alpha_r\alpha_s}\prod_r\tau(\theta_r)^{-\alpha_r}\prod_r E\alpha_r$; here $E_\alpha = e^{\alpha(\alpha-1)/2}\pi^{\alpha/2}2^{-2\alpha^2+\alpha}\Gamma(\alpha+\frac{1}{2})^{-\alpha}/\prod_k\frac{1}{2}\pi x_k J_{\alpha+1/2}(x_k)^2$, where the x_k are the positive zeros of the Bessel function $J_{\alpha-1/2}(x)$. The general form of the result confirms the conjecture of Fisher and Hartwig. If one applies the result to the cases treated by Lenard, then one is led to an alternate expression for E_α, namely, $E_\alpha = G(\alpha+1)^2/G(2\alpha+1)$, where G is the Barnes function. We mention here only the facts $G(1)=1$, $G(z+1)=\Gamma(z)G(z)$. The equivalence of the two expressions for E_α is a very curious identity indeed, established here by what is certainly a roundabout procedure. A very pretty direct proof for $\alpha=2$, the simplest nontrivial case, was found by Stanton Philipp (private communication).

"We shall now indicate in general terms how the result is obtained. The Toeplitz matrix $T_n[\sigma]$, which has the (j,k)-entry $(1/2\pi)\int_{-\pi}^{\pi} e^{-i(j-k)\theta}\sigma(\theta)\,d\theta$ $(j,k=0,\cdots,n)$, may be thought of as an operator of the space of sequences $\{a_0,\cdots,a_n\}$ or, equivalently, the space of polynomials $a_0+\cdots+a_nz^n$ of degree at most n. If p is the solution of $T_n[\sigma]p=1$ then p has constant term $D_{n-1}[\sigma]/D_n[\sigma]$, by Cramer's rule. A deeper fact is the following [see I. I. Hirschman, Jr., J. Math. Mech. **16** (1966), 167–196; MR0208279]: If q is the polynomial analogous to p with $\sigma(\theta)$ replaced by $\sigma(\theta)^*$ (the asterisk, as well as a bar, will denote complex conjugate), and if all the zeros of p and q lie outside the unit circle, then $D_n[\sigma]=(D_n[\sigma]/D_{n-1}[\sigma]^{n+1}E(n))$, where $E(n)=\exp\{-(2\pi i)^{-1}\int_{-\pi}^{\pi}\log p(e^{i\theta})\,d\log q(e^{i\theta})^*\}$. Thus sufficiently precise asymptotic information about p (and q) will give the asymptotic form of $D_n[\sigma]$; this in turn depends on the asymptotic inversion of $T_n[\sigma]$ which must, and will be, the starting point. In case σ is nonnegative, $p(z)=z^n\phi_n(z^{-1})$, where ϕ_n is the suitably normalized nth orthogonal polynomial associated with $\sigma(-\theta)$. Thus our work will imply asymptotic formulas for these orthogonal polynomials, which is of considerable independent interest. In the case $R=1$, $\tau(\theta)=1$ the orthogonal polynomials are simple combinations of Jacobi polynomials for which there are Hilb-type asymptotic formulas in terms of Bessel functions. It should therefore come as no surprise that Bessel functions arise in what follows. To obtain an approximation to $T_n[\sigma]^{-1}$ we begin with a device akin to one that seems to have been originated by R. Latter [Quart. Appl. Math. **16** (1958), 21–31; MR0104128], who indicated how a certain pair of integral equations could be used to find approximate solutions of finite convolution equations. The idea was subsequently used by V. Hutson [Proc. Edinburgh Math. Soc. (2) **14** (1964/65), 5–19; MR0166570] to obtain, quite simply, asymptotic formulas for the eigenvalue and eigenfunctions of finite convolution operators; previous results on this question had been obtained only with great difficulty."

I. I. Hirschman Jr.

From MathSciNet, February 2022

MR0440724 (55 #13595) 60K05; 60K15

Kesten, Harry

Random difference equations and renewal theory for products of random matrices.

Acta Mathematica **131** (1973), 207–248.

In this paper the author studies the asymptotic behavior of the limiting distribution of the solution Y_n of the difference equation $Y_n=M_nY_{n-1}+Q_n$, $n\ge 1$, where $\{(M_n,Q_n);n\ge 1\}$ is a sequence of independent and identically distributed random matrices. The M_n are $d\times d$ and the Q_n are $d\times 1$ and all are assumed to have non-negative components. The matrices M_n are not assumed to commute. Put $\|M\|=\sup\{\|xM\|\colon \|x\|=1\}$, $\|x\|=$ Euclidean norm of x.

Under the assumption $E\log^+\|M_1\|<\infty$, it is known that $\alpha=\lim_n(1/n)\log\|D_n\|$ exists and is constant with probability 1, where $D_n=M_1M_2\cdots M_n$, $D_0=I$ [see H. Furstenburg and the author, Ann. Math. Statist. **31** (1960), 457–469; MR0121828]. When $\alpha<0$, $\|D_n\|\to 0$ exponentially fast and the solution Y_n converges in distribution to the distribution of the random vector $R=\sum_{k=1}^{\infty}D_{k-1}Q_k$. The study of the tail behavior of R is reduced to the study of $P[\sup_{n\geq 1}\sum_{k=1}^{n}\log(\|xD_k\|/\|xD_{k-1}\|)>\log t]=P[\sup_{n\geq 1}\|xD_n\|>t]$ for large t, where x is a fixed (row) vector in R^d. This necessitates the development of renewal theory for the sequence of partial sums $\sum_{k=1}^{n}\log(\|xD_k\|/\|xD_{k-1}\|)$. Here is a summary of the main results. Put $N_x(t)=\min\{n\geq 0\colon \log\|xD_n\|>t\}$, $W_x(t)=\log\|xD_{N_x(t)}\|-t$, $Z_x(t)=(xD_{N_x(t)})/\|xD_{N_x(t)}\|$. Theorem A: Assume that $P[M_1\geq 0]=1$, $P[M_1$ has a zero row$]=0$ and $E\log^+\|M_1\|<\infty$. Assume also that the group generated by $\bigcup_{n=1}^{\infty}\{\log r(m_1m_2\cdots m_n)\colon m_i\in \text{support}(M_1)\}$ is dense in R^1, where $r(m)$ is the largest positive eigenvalue of matrix m. If $\alpha>0$, then $Z_x(t)$ and $W_x(t)$ have limit distributions as $t\to\infty$ which are independent of $x\in S_+=\{x\in R^d\colon \|x\|=1, x\geq 0\}$. Also, for $x\in S_+$ and $h\geq 1$, $\lim_{t\to\infty}E[\#\{n\colon t\leq\|xD_n\|\leq th\}]=\log h/\alpha$. If $\alpha<0$ and there exists a $k_0>0$ for which $E[\min_i(\sum_j M_1(i,j))]^{k_0}\geq d^{k_0/2}$ and $E\|M_1\|^{k_0}\log^+\|M_1\|<\infty$, then there is a $k_1\in(0,k_0)$ such that $\lim_{t\to\infty}t^{k_1}P[\sup_{n\geq 1}\|xD_n\|>t]=v(x)$ exists and is strictly positive for $x\in S_+$. Theorem B: Assume that all of the preceding hypotheses hold and that $\alpha<0$. Assume also that $P[Q_1\geq 0]=1$, $P[Q_1=0]<1$ and $E\|Q_1\|^{k_1}<\infty$ (k_1 from above). Then $\lim_{t\to\infty}t^{k_1}P[x\cdot R>t]$ exists and is finite for all unit x. The limit is positive for $x\in S_+$.

These renewal theorems are a significant generalization of the classical renewal theorems for sums of i.i.d. real-valued random variables. The existence of $\lim_{t\to\infty} E[\#\{n\colon t<\|xD_n\|\leq th\}]$ is the analogue of the Blackwell-Feller-Orey renewal theorem and the existence of a limit distribution for $W_x(t)$ is the analogue of the existence of the limit distribution of the residual waiting time [see W. Feller, *An introduction to probability theory and its applications, Vol. II*, second edition, Chapter 11, Wiley, New York, 1971; MR0270403]. The author bases his results on a renewal theorem of his for semi-Markov chains [Ann. of Probability **2** (1974), 355–386; MR0365740]. In his proof he has discovered for the matrix case the analogue of the powerful method of associated random walks in the real case, i.e., the transformation $F\{dx\}\to e^{\beta x}F\{dx\}$, where $\int_{-\infty}^{\infty}e^{\beta x}F\{dx\}=1$, which converts a distribution with negative mean into one with positive mean [see Feller, op. cit., Chapter 12].

The following question has been asked before but needs to be asked again. How does one, to borrow a phrase from H. P. McKean, effectively compute the quantity $\alpha=\lim_n(1/n)\log\|D_n\|=\lim_n(1/n)E\log\|D_n\|$?

K. B. Erickson

From MathSciNet, February 2022

MR1257246 (95e:82003) 82B05; 33C90, 47A75, 47G10, 47N55, 82B10

Tracy, Craig A.; Widom, Harold

Level-spacing distributions and the Airy kernel.

Communications in Mathematical Physics **159** (1994), *no.* 1, 151–174.

In the theory of random matrices, one tries to understand the distribution of eigenvalues of some class of operators, often Hermitian or unitary. The idea is to replace infinite-dimensional operators with finite $N\times N$ matrices (N large), try to answer questions about the eigenvalues of the finite matrices and then with a "rescaling", let N approach infinity. This approach works remarkably well, as can be seen from numerical evidence, and has created a host of interesting problems.

The paper is concerned with the statistical behavior of the eigenvalues for Hermitian matrices and the connections of this problem to the Airy kernel, defined by

$$K(x,y) = \frac{A(x)A'(y) - A'(x)A(y)}{x-y}$$

and $A(x) = \sqrt{\lambda}Ai(x)$. This kernel arises when the eigenvalue density for finite N is rescaled so that the density is 1 near $\pm\sqrt{2N}$. This is called scaling at the "edge of the spectrum". The importance of the kernel lies in the fact that all of the eigenvalue behavior can be derived from operators associated to it. For example, if $J = (s, \infty)$, then the probability that exactly n eigenvalues will be found in the interval J is given by

$$\frac{(-1)^n}{n!}\frac{\partial^n}{\partial\lambda^n}\mathrm{Det}(I-T)|_{\lambda=1},$$

where T has kernel $K(x,y)\chi_J(y)$ and "Det" means the Fredholm determinant.

If the scaling is done differently, that is, in the "bulk of the spectrum", then the above holds with the Airy kernel replaced by the sine kernel, $\sin\pi(x-y)/\pi(x-y)$. Several results are proved in the paper that are analogues of those for the sine kernel. The first is the existence of a completely integrable system of PDEs that involves the operator T and the endpoints of the interval J, now generalized to be a finite union of open intervals. These were found by Jimbo, Miwa, Môri, and Sato in the sine-kernel case using different techniques. The methods used in this paper are based on some simple commutator identities for operators and make the exposition quite clear.

In the special case where $J = (s, \infty)$, the PDEs can be used to show that if we define $q(s,\lambda) = ((I-T)^{-1}A)(s)$ and let $'$ be d/ds, then q satisfies $q'' = sq + 2q^3$ with $q(s,\lambda) \sim \sqrt{\lambda}Ai(s)$ as $s \to \infty$. This is a special case of a Painlevé II differential equation. For the sine-kernel case, the corresponding result involves Painlevé V equations.

The final part of the paper is devoted to describing the asymptotics (as $s \to -\infty$) of the probability that exactly n eigenvalues lie in (s, ∞). This is done by proving the existence of a second-order differential operator which commutes with the operator T for $\lambda = 1$. It is shown how the asymptotics of the eigenvalues of the commuting differential operator can be used to determine the probability in question.

This paper illustrates and emphasizes the close connection between operator theory and random matrices. The outcome of this approach is a paper that contains interesting results, is nicely organized and is very pleasant to read.

{For further related work see the following two reviews [MR1266485; MR1277933].}

Estelle L. Basor

From MathSciNet, February 2022

MR1277933 (95e:82005) 82B05; 33C90, 47A75, 47G10, 47N55, 82B10

Tracy, Craig A.; Widom, Harold

Fredholm determinants, differential equations and matrix models.

Communications in Mathematical Physics **163** (1994), *no.* 1, 33–72.

Since the pioneering work in the late 50's and early 60's of E. P. Wigner, F. J. Dyson and others, the eigenvalues of random matrices have been used to model the statistical properties of the energy levels of classically chaotic quantum systems. Basic quantities in the statistical analysis of the energy levels are the probabilities $E(n;J)$ that there are exactly n levels in the interval (or union of intervals) J. In particular, the distribution of the spacing between consecutive levels can be obtained from $E(0;J)$ by differentiation. For a large class of orthogonal polynomial random matrix models of $N\times N$ Hermitian matrices with unitary symmetry, as well as scaling limits of these models, the formula

$$E(n;J)=\frac{(-1)^n}{n!}\frac{d^n}{d\lambda^n}\det(1-\lambda K)\big|_{\lambda=1},$$

where K is the integral operator on J with kernel

$$K(x,y)=\frac{\varphi(x)\psi(y)-\psi(x)\varphi(y)}{x-y}, \tag{1}$$

holds [see, e.g., M. L. Mehta, *Random matrices*, second edition, Academic Press, Boston, MA, 1991; MR1083764].

The paper under review is concerned with Fredholm determinants of integral operators having kernel of this form with emphasis on the determinants thought of as functions of the endpoints of $J=\bigcup_{j=1}^m(a_{2j-1},a_{2j})$. In the special case of the bulk scaling limit of the Gaussian unitary ensemble (GUE)

$$K(x,y)=\frac{\sin(x-y)}{x-y},$$

a characterisation of this type was first given by M. Jimbo et al. [Phys. D **1** (1980), no. 1, 80–158; MR0573370]. (The GUE is the probability space of $N\times N$ Hermitian matrices with independent, complex Gaussian, mean zero elements. The bulk scaling limit is the limit $N\to\infty$ where distance is rescaled to make the mean spacing between consecutive eigenvalues in the bulk of the spectrum one.) Furthermore, in the case of a single interval of length s these PDE's were shown to reduce to a Painlevé equation of the fifth kind for the quantity

$$\sigma(s;\lambda)=-s\frac{d}{ds}\log\det(1-\lambda K).$$

Tracy and Widom significantly generalize the results of Jimbo et al. [op. cit.] to any Fredholm determinant with kernel of the type (1) for which φ and ψ satisfy a linear differential equation of the form

$$\frac{d}{dx}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}=\Omega(x)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}, \tag{2}$$

where $\Omega(x)$ is a 2×2 traceless matrix with rational entries. They show that the (φ,ψ) pairs arising from orthogonal polynomial Hermitian matrix models with weight functions $w(x)=\exp(-V(x))$, $-\infty<x<\infty$, $w(x)=x^\alpha\exp(-V(x))$, $0<x<\infty$, and $w(x)=(1-x)^\alpha(1+x)^\beta\exp(-V(x))$, $-1<x<1$, V polynomial, satisfy (2) with explicit formulas given for the matrix elements of Ω. (The simplest cases are GUE with $V(x) = x^2$, Laguerre with $V(x) = x$, and Jacobi with $V(x) = 1$, respectively.) Furthermore, they show that the (φ, ψ) pairs in the "edge scaling limits" of GUE, Laguerre and Jacobi and the "double scaling limit" of 2D matrix models of quantum gravity satisfy (2). Thus their PDE's apply to a large class of problems of current interest. In particular, they obtain the distribution function for the largest eigenvalue of finite N GUE in terms of Painlevé IV and the limiting distribution for the scaled largest eigenvalue in terms of Painlevé II. (Further details of this last case are in the earlier paper [C. A. Tracy and H. Widom, Comm. Math. Phys. **159** (1994), no. 1, 151–174; see the preceding second review; MR1257246].)

There is also an exponential variant of the kernel in which the denominator in (1) is replaced by $e^{bx} - e^{by}$, where b is an arbitrary complex number. The authors find an analogous system of differential equations in this setting. If $b = i$, then one can interpret this operator as acting on the unit circle in the complex plane. As an application of this the authors write down a system of PDE's for Dyson's circular ensemble of $N \times N$ unitary matrices, and then an ODE if J is an arc of the circle.

Another significant feature of the present work, in addition to the generality of the final results, is the accessibility of the derivation to the non-expert. Indeed, the derivation is self-contained and is based mostly on some simple operator formulas. These features make a detailed study of the work—something I strongly recommend to those with interests in random matrices or applications of nonlinear equations—very feasible.

Peter J. Forrester

From MathSciNet, February 2022

MR1469319 47G10; 15A52, 30E25, 34A55, 34E20, 41A60, 82B23, 82B44

Deift, Percy A.; Its, Alexander R.; Zhou, Xin

A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics..

Annals of Mathematics. Second Series **146** (1997), *no.* 1, 146–235.

The central problem considered in this paper is the $x \to +\infty$ asymptotic analysis of a Fredholm determinant $P_x = \det(1 - K_x)$, where K_x is the integral operator with kernel $\sin x(z - z')/\pi(z - z')$ acting on the disjoint union of intervals $J = \bigcup_{j=1}^n (a_j, b_j)$. In random matrix theory P_x arises as the bulk scaling limit of the probability that no eigenvalues lie in the interval collection $(x/\pi)J$ for the Gaussian unitary ensemble (GUE). The principal result in the paper is a theorem that gives the asymptotics of $(d/dx) \log P_x$ as $x \to \infty$: Theorem 1.31. Let J be the union of disjoint intervals given above. Then for any $j = 1, 2, \cdots$ one has the $x \to \infty$ asymptotic development

$$\frac{d}{dx} \log P_x = -2\alpha x + \frac{d}{dx} \log \theta(xV) + \frac{G_1(x)}{x} + \cdots + \frac{G_j(x)}{x^j} + O\left(\frac{1}{x^{j+1}}\right),$$

where $\alpha > 0$ is a positive constant, θ is a theta function, V is a vector in $\mathbf{R}^n$, and the functions $G_j(x)$ are bounded quasiperiodic functions of x.

The quantities α, θ, V, and G_j are explicitly described in the paper, and this theorem is a refinement of a result originally proved by H. Widom [Comm. Math. Phys. **171** (1995), no. 1, 159–180; MR1341698; J. Approx. Theory **77** (1994), no. 1, 51–64; MR1273699; Advances in Math. **3** (1969), 127–232; MR0239059].

Widom's results are, in turn, a mathematically rigorous generalization of asymptotics obtained by J. des Cloizeaux and M. L. Mehta [J. Mathematical Phys. **14** (1973), 1648–1650; MR0328158] and F. J. Dyson [Comm. Math. Phys. **47** (1976), no. 2, 171–183; MR0406201] for the single interval case, $n = 1$. In this case the theta function, θ, degenerates and one has $(d/dx)\log P_x = -x - 1/4x + O(1/x^2)$, as a special case of Theorem 1.31. Widom's analysis, which is a tour de force of classical analysis and approximation theory, makes it clear that the rigorous investigation of such asymptotics is a difficult and subtle problem. The main point of the paper under review is that the general collection of techniques which one might collectively refer to as "Riemann-Hilbert steepest descent methods" have an illuminating application to the study of such asymptotics. Riemann-Hilbert problems involve the characterization of a (matrix-valued) analytic function defined on the complement of some collection of contours in the complex projective plane in terms of (matrix-valued) multipliers which relate the boundary values of the analytic function on the two sides of the contours. That there is a Riemann-Hilbert characterization of the logarithmic derivative of $\det(1-K_x)$ goes back to M. Jimbo et al. [Phys. D **1** (1980), no. 1, 80–158; MR0573370], where it appears as an early instance of what later became a general theory of τ-functions. A simplified version of this characterization is the starting point in the Deift-Its-Zhou analysis, which then proceeds by introducing a cleverly chosen multiplier which at least formally reduces the Riemann-Hilbert problem to an asymptotically simpler and "explicitly solvable" form. Both the choice of the multiplier and the explicit solution of the asymptotic Riemann-Hilbert problem involve detailed function theory on the hyperelliptic curve associated with the equation $y^2 = \prod_{j=1}^{n}(z-a_j)(z-b_j)$ (the critical function appearing in the multiplier is also a central figure in Widom's analysis). These developments formally produce the asymptotics in Theorem 1.31 but some non-uniformity in the convergence stops matters short of a proof.

When a system of contours has finite length and the multipliers are uniformly close to the identity it is a relatively elementary fact that one can solve an integral equation to find an analytic function that solves the Riemann-Hilbert problem which has this system of multipliers. It is satisfying that this simple result is the principal tool used to justify the approximations that lead to the asymptotics in Theorem 1.31. However, it is by no means a trivial matter to see how to find the approximations to which this result applies. Deift, Its, and Zhou employ ideas from steepest descent methods, monodromy preserving deformation theory, WKB asymptotics, and turning point analysis just to arrive at an appropriate approximation. All the while the focus of the method is on producing an approximation characterized by Riemann-Hilbert data—for example, in the course of the analysis, a linear differential equation is first approximated, then asymptotically solved near a turning point and finally discarded when only a certain aspect of the local behavior of the solution is required for the Riemann-Hilbert data.

We will conclude this review with a few words about the scope of Riemann-Hilbert methods in the analysis of nonlinear "exactly solvable" problems. The success of these methods includes a growing number of problems such as: the analysis of connection problems for Painlevé transcendents [A. R. Its, A. S. Fokas and A. A. Kapaev, Nonlinearity **7** (1994), no. 5, 1291–1325; MR1294544], an analysis of the asymptotics of solutions to equations like the mKdV equation [P. A. Deift and X. Zhou, Ann. of Math. (2) **137** (1993), no. 2, 295–368; MR1207209], examples of universality in bulk scaling for non-Gaussian matrix models [A. R. Its and P. Bleher, "Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model", Preprint 97-2, Dept. Math. Sci., Indiana Univ.–Purdue Univ. Indianapolis (IUPUI), Indianapolis, IN; per revr.], and the analysis of the asymptotics of correlation functions in some solvable models in statistical mechanics [A. R. Its et al., Internat. J. Modern Phys. B **4** (1990), no. 5, 1003–1037; MR1064758]. These examples and more are surveyed in the introduction to the paper under review and in the extensive references there. The method itself is evolving. In this paper the "method of steepest descent" differs from the earlier Deift-Zhou triangular factorization-contour shifting scheme and the role of WKB methods is also changing. Considerable art is required to apply the method to problems as subtle as the one considered here but in the hands of skilled practitioners the technique is opening a mathematical window on a wide range of difficult asymptotic problems with importance in both mathematics and physics.

John N. Palmer

From MathSciNet, February 2022

MR1780118 47B35; 05E10, 15A15, 33C05, 33C15, 33C60, 47A68, 60C05
Borodin, Alexei; Okounkov, Andrei
A Fredholm determinant formula for Toeplitz determinants.
Integral Equations and Operator Theory **37** (2000), *no.* 4, 386–396.

MR1780119 47B35; 05E10, 15A15, 33C05, 33C15, 33C60, 47A68, 60C05
Basor, Estelle L.; Widom, Harold
On a Toeplitz determinant identity of Borodin and Okounkov.
Integral Equations and Operator Theory **37** (2000), *no.* 4, 397–401.

In June 1999, during the MSRI Workshop on Random Matrices, A. Its and P. Deift raised the question whether there is a general formula that expresses the determinant of a Toeplitz matrix $T_n(a) = (a_{j-k})_{j,k=0}^{n-1}$ as the Fredholm determinant of an operator $I-K$ where K acts on $l^2(\{n, n+1, \cdots\})$ and admits an integral representation in terms of the symbol $a(e^{i\theta}) = \sum_{k\in\mathbf{Z}} a_k e^{ik\theta}$. Borodin and Okounkov showed that such a formula indeed exists, thus genuinely highlighting the entire story of Toeplitz determinants, which started with G. Szegő in 1915.

In notation suggested by Widom, the Borodin-Okounkov formula reads

$$\frac{\det T_n(a)}{G(a)^n} = \frac{\det(I - Q_n H(b) H(\tilde{c}) Q_n)}{\det(I - H(b)H(\tilde{c}))}.$$

Here a is a sufficiently smooth function without zeros on the unit circle and with winding number zero. This guarantees the existence of a Wiener-Hopf factorization $a = a_- a_+$. The functions b and c are defined by $b = a_- a_+^{-1}$ and $c = a_-^{-1} a_+$, and $H(b)$ and $H(\tilde{c})$ are the Hankel matrices $H(b) = (b_{j+k+1})_{j,k=0}^{\infty}$ and $H(\tilde{c}) = (c_{-j-k-1})_{j,k=0}^{\infty}$. The smoothness required ensures that $H(b)$ and $H(\tilde{c})$ are Hilbert-Schmidt operators, so that $H(b)H(\tilde{c})$ is in the trace class. By Q_n we denote the orthogonal projection of $l^2(\mathbf{Z}_+)$ onto $l^2(\{n, n+1, \cdots\})$. Finally, $G(a)$ is the geometric mean of a, $G(a) = \exp(\log a)_0$. Furthermore, it is well known that

$$1/\det(I - H(b)H(\tilde{c})) = \exp \sum_{k=1}^{\infty} k(\log a)_k (\log a)_{-k} =: E(a).$$

Since $Q_n H(b)H(\tilde{c})Q_n \to 0$ in the trace norm, the Borodin-Okounkov formula clearly implies at once the Szegő-Widom limit theorem:

$$\lim_{n\to\infty} \det T_n(a)/G(a)^n = E(a).$$

Borodin and Okounkov's proof of their remarkable formula is based on representation theory and combinatorics, in particular on results by Okounkov on infinite wedge and random partitions and a theorem by I. M. Gessel expressing a Toeplitz determinant as a sum over partitions of products of Schur functions. Two other proofs are given in the paper by Basor and Widom, who also extend the Borodin-Okounkov formula to block Toeplitz matrices. The first of these proofs uses an identity for $\det T_{n-1}(a)/\det T_n(a)$ containing just $H(b)H(\tilde{c})$ which was established by Widom in 1973, and the second is a further development of the argument employed by Basor and J. W. Helton in 1980 to prove the Szegő-Widom limit theorem. The two proofs by Basor and Widom are operator-theoretic and very transparent.

A third operator-theoretic proof, also working in the block case, was recently found by the reviewer. It follows from the identity

$$T_n^{-1}(a) = T_n(a_+^{-1})(I - P_n T(c) Q_n X Q_n T(b) P_n) T_n(a_-^{-1}),$$

where $X = (I - Q_n H(b)H(\tilde{c})Q_n)^{-1}$, $P_n = I - Q_n$, and $T(f)$ stands for the infinite Toeplitz matrix $(f_{j-k})_{j,k=0}^{\infty}$. This identity, which lifts the Borodin-Okounkov formula from the determinant level to the matrix level, was obtained in 1980 by B. Silbermann and the reviewer.

Finally, here is the probably shortest proof of the Borodin-Okounkov formula. It is a modification of the second proof given by Basor and Widom. For an arbitrary trace class operator K we have

$$\det P_n(I-K)^{-1}P_n = \frac{\det(I - Q_n K Q_n)}{\det(I-K)}.$$

With K replaced by $P_m K P_m$, this is Jacobi's formula for the principal $n \times n$ minor of the inverse of a finite matrix, and for general K the identity results from the fact that $P_m K P_m \to K$ in the trace norm as $m \to \infty$. Now put $K = H(b)H(\tilde{c})$. Then $I - K = T(b)T(c)$, and since $P_n T^{-1}(c) T^{-1}(b) P_n$ equals

$$P_n T(a_+^{-1}) T(a_-) T(a_+) T(a_-^{-1}) P_n = T_n(a_+^{-1}) T_n(a) T_n(a_-^{-1})$$

and $\det T_n(a_+^{-1}) T_n(a_-^{-1}) = G(a)^{-n}$, we arrive at the Borodin-Okounkov formula.

Although now at least five proofs of the Borodin-Okounkov formula are known and two or three of these proofs are extremely simple, it should be emphasized that this formula is nevertheless an admirable discovery. After the reviewer had communicated his own proof to Harold Widom, he replied: "So you can add yourself to the list of people who can kick themselves for not having found the formula when they were so close."

{Reviewer's remark: For a continuation of the story see also the preprints by J. Baik, P. Deift and E. Rains ["A Fredholm determinant identity and the convergence of moments for random Young tableaux", http://arXiv.org/abs/math.CO/0012117] and the reviewer ["On the determinant formulas by Borodin, Okounkov, Baik, Deift and Rains", http://arXiv.org/abs/math.FA/0101008].}

{Addendum (September, 2003): In July 2003, the people involved since 1999 in what was then called the Borodin-Okounkov formula and its proof received an email from Percy Deift. This email was as follows: "Recently Jeff Geronimo showed me a 1979 paper of his with Ken Case in which they wrote down the Borodin-Okounkov formula in the context of proving strong Szegő. The reference is [J. Math. Phys. **20** (1979), no. 2, 299–310; MR0519213]. See, in particular, formula VII.28 on page 308. It's quite remarkable that the formula was already known in 1979. The proof of the formula by Geronimo-Case is inverse-scattering theoretic and is the analog of Dyson's second-derivative log det formula for the Schrödinger case."

{Without diminishing the outstanding achievement of Borodin and Okounkov, the reviewer congratulates Jeff Geronimo and Ken Case on their brilliant feat and the eventual recognition of their great success and expresses his regret that their names are missing in the above review.}

A. Böttcher

From MathSciNet, February 2022

MR2831118 47B35; 11M50, 15A15, 30E25, 42C05, 47B38

Deift, Percy; Its, Alexander; Krasovsky, Igor

Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher-Hartwig singularities.

Annals of Mathematics. Second Series **174** (2011), *no.* 2, 1243–1299.

The fascination of Toeplitz matrices and determinants is that looking at their entries does not yield anything whereas looking at the function whose Fourier coefficients are the entries tells us almost everything. Given a complex-valued function $f \in L^1(\mathbf{T})$, where $\mathbf{T}$ is the unit circle, put

$$f_k = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ik\theta}d\theta$$

and let $D_n(f)$ denote the determinant of the $n \times n$ Toeplitz matrix $(f_{j-k})_{j,k=0}^{n-1}$. Szegő's theorem states that, after normalization, $D_n(g)$ converges to a finite nonzero limit as $n \to \infty$ provided g is smooth enough, has no zeros on $\mathbf{T}$, and has winding number zero about the origin.

In 1968, M. E. Fisher and R.E. Hartwig raised a conjecture on the asymptotic behavior of $D_n(f)$ in case f violates the assumptions of Szegő. They assumed that

$$f(t) = g(t)\prod_{j=1}^{m}(-t/t_j)^{\beta_j}|t-t_j|^{2\alpha_j}, \quad t \in \mathbf{T},$$

where g is as in Szegő's theorem, t_j are distinct points on $\mathbf{T}$, and α_j, β_j are complex numbers satisfying $\operatorname{Re}\alpha_j > -1/2$ (which ensures that $f \in L^1(\mathbf{T})$). The conjecture says that, possibly under additional restrictions on the α_j, β_j and after appropriate normalization, $D_n(f)$ is asymptotically a constant times n^σ with $\sigma = \sum_{j=1}^m(\alpha_j^2 - \beta_j^2)$. Thus, the conjectured miracle is that one can predict the essential asymptotics of the determinants from the sole knowledge of the singularities of f, without knowing f itself, and that the parameters of each singularity make their contribution to the exponent σ in the beautiful form $\alpha^2 - \beta^2$. This conjecture, which is of relevance in statistical physics, random matrix theory, and has connection with L-functions, has caused considerable effort in the mathematics and physics communities.

In the last third of the last century, the conjecture was proved in important cases, while in other cases counterexamples were found for certain constellations of the α_j, β_j. The development culminated with T. Ehrhardt, who in his status report [in *Recent advances in operator theory (Groningen, 1998)*, 217–241, Oper. Theory Adv. Appl., 124, Birkhäuser, Basel, 2001; MR1839838] proved the conjecture under the only additional assumptions that $|||\beta||| := \max |\operatorname{Re}\beta_j - \operatorname{Re}\beta_k| < 1$ and that none of the numbers $\alpha_j \pm \beta_j$ is a negative integer.

These additional assumptions are removed in the paper under review. As already observed by B. Silbermann and the reviewer [Math. Nachr. **102** (1981), 79–105; MR0642143], H. Widom [in *Topics in operator theory: Ernst D. Hellinger memorial volume*, 387–421, Oper. Theory Adv. Appl., 48, Birkhäuser, Basel, 1990; MR1207410], and E. L. Basor and C. A. Tracy [Phys. A **177** (1991), no. 1-3, 167–173; MR1137031], in this general case the determinants $D_n(f)$ are asymptotically a sum of oscillating terms. Basor and Tracy [op. cit.] stated a precise conjecture in the case $|||\beta||| \geq 1$, which included the form of the exponent and of the constants in the oscillating terms, and this conjecture is proved in the present paper.

The proof is based on techniques developed by the authors and their collaborators over the last 20 years. They use the close connection between the determinants $D_n(f)$ and the orthogonal polynomials with respect to the (complex-valued) weight $f(t)$, $t \in \mathbf{T}$. These polynomials satisfy a Riemann-Hilbert problem, and this problem can be solved asymptotically by a steepest-descent method. The authors are in particular able to extend the results of A. Martínez-Finkelshtein, K. D. T.-R. McLaughlin and E. B. Saff [Int. Math. Res. Not. **2006**, Art. ID 91426; MR2250012] to the case where the weight $f(t)$ still includes jumps, which happens in the presence of the β_j's.

Armed with their deep understanding of the asymptotic behavior of orthogonal polynomials, the authors move on to Hankel and Toeplitz-plus-Hankel determinants. The Hankel determinant $D_n^H(w)$ generated by a complex-valued function $w \in L^1(-1,1)$ is the determinant of the $n \times n$ Hankel matrix $(w_{j+k})_{j,k=0}^{n-1}$ where $w_k = \int_{-1}^{1} x^k w(x)dx$. Under certain circumstances, Toeplitz and Hankel determinants are related by the equality $D_n^H(w) = C_n D_{2n}(f)$ where C_n involves quantities known from orthogonal polynomials and $f(e^{i\theta}) = w(\cos\theta)|\sin\theta|$. Similar relations exist between certain Toeplitz-plus-Hankel determinants, such as $D_n^{TH}(f) := \det(f_{j-k} + f_{j+k})_{j,k=0}^{n-1}$ or $D_n^{TH}(f) := \det(f_{j-k} \pm f_{j+k+1})_{j,k=0}^{n-1}$, and Hankel determinants. These relations are employed to establish asymptotic formulas for $D_n^H(w)$ and $D_n^{TH}(f)$ in case f has Fisher-Hartwig singularities.

At the end of the very instructive introductory section, the authors briefly discuss some applications of their results. For example, they point out that the mean values of certain L-functions in orthogonal and symplectic families studied by H. M. Bui and J. P. Keating [Proc. Lond. Math. Soc. (3) **96** (2008), no. 2, 335–366; MR2396123] are Toeplitz-plus-Hankel determinants with a Fisher-Hartwig singularity, so that the results of this paper confirm conjectures by Bui and Keating.

This paper is indeed a great achievement and an outstanding piece of mathematical analysis. The very well-written introduction encompasses 16 pages, the remaining 37 pages are hard analysis combined with ingenious play with determinants and orthogonal polynomials.

A. Böttcher
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MR2784665 60B20

Tao, Terence; Vu, Van

Random matrices: universality of local eigenvalue statistics.

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Recall that a random $n \times n$ Hermitian matrix H belongs to the Gaussian Unitary Ensemble when $\{H_{ii}, H_{ij}\}_{1\le i<j\le n}$ are independent with $H_{ii} \sim \mathcal{N}(0, 1/n)$ and $H_{ij} \sim \mathcal{N}(0, \frac{1}{2n} I_2)$ for $i \neq j$; I_2 is the 2×2 identity matrix. Let $\lambda_1(H) \geq \cdots \geq \lambda_n(H)$ be the eigenvalues of H. The unitary invariance and the Gaussian nature of the law of H allow many explicit computations. In particular, the law of the eigenvalue vector is known explicitly. This allows one to deduce for instance that the empirical spectral distribution $\frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i(H)}$ tends as $n \to \infty$ to the semi-circle distribution. This phenomenon, concerning a global statistics of the spectrum, is actually universal, in the sense that it still holds if one drops the Gaussian assumption while keeping the moments up to order two (Wigner matrices). It was conjectured in the 1960s that most local spectral statistics such as the law of spacings or the k-point correlations are universal, possibly under additional moment assumptions. Many aspects of this conjecture, including in particular the behaviour at the edge of the spectrum, have been studied in the literature. This paper provides proofs of the universality conjecture for local statistics.

The main result of this important paper is a fourth moment theorem, which states roughly that if W and W' are two Wigner matrices such that the entries of W and W' match up to order two on the diagonal and up to order four outside the diagonal, then for any $\varepsilon > 0$ and any $I \subset \{\lfloor \varepsilon n\rfloor, \ldots, \lfloor (1-\varepsilon)n\rfloor\}$, the laws of the random vectors $\lambda_I(W)$ and $\lambda_I(W')$ get closer and closer as $n \to \infty$. This asserts that the fine spacing of eigenvalues in the bulk of the spectrum for Wigner matrices is only sensitive to the first four moments. It seems that the method cannot go beyond the fourth moment. The fourth moment theorem is strong enough quantitatively to allow the deduction of universality results for various local statistics, using what is known for the Gaussian Unitary Ensemble. For instance, it provides the universality of Gaussian fluctuations of any specific eigenvalue $\lambda_{i(n)}$ where $i/n \to c$ with $0 < c < 1$. Another example is the universality of the fluctuation of the least eigenvalue.

The method is based on a Lindeberg strategy of replacing non-Gaussian entries with Gaussian entries, developed at the origin for the classical central limit theorem. Note that this strategy is already known to be efficient for global statistics. The main proof, highly technical, involves many ingredients used in random matrix theory such as the Cauchy-Stieltjes transform and concentration of measure techniques.

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