BOOK REVIEWS

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Accuracy of mathematical models: Dimension reduction, homogenization, and simplification, by Sergey I. Repin and Stefan A. Sauter, EMS Tracts in Mathematics, Vol. 33, European Mathematical Sociey, Zürich, xvi+333 pp., ISBN 978-3-03719-206-1

1. Accuracy of models

How good is a mathematical model? To answer this question, it is necessary to define "good" and to have in mind what the object being modeled is. Good models yield solutions that approximate well "reality", in the sense that the model and exact solutions or data are close to each other and important qualitative properties are preserved. Models might approximate real-life phenomena or other equations. In fact, the Stokes equations are simultaneously a model for "real" fluids and for the Navier-Stokes equations. Whether it is an accurate model depends on how approximations are measured and on the fluid regime itself: OK for slow moving viscous flows, not OK for turbulence.

To narrow down the discussion and secure ourselves on comfortable mathematical grounds, let us consider a model as an approximation of some complicated mathematical problem. And to stay in the realm of problems under consideration in the book, consider partial differential equations (PDEs). Actually, most of the book discusses only steady state problems—a wise decision, given the breadth of the topic.

The subtitle of the book hints at some modeling techniques: *Dimension reduction, homogenization, and simplification*. In the present context, a typical example of dimension reduction is given by elastic shells, thin three-dimensional domains under elastic deformation. It is only natural to replace shells by their two-dimensional middle surfaces, reducing the dimension from three to two (note that dimensional reduction has other meanings in mathematics, especially when reducing the dimension of ODE models; in neuroscience for instance, it is convenient to reduce the Hodgkin–Huxley four-dimensional ODE model to two dimensions [8, 10]). Homogenization may likely be a better-known field of research, and refers to the idea that it is possible to replace an oscillatory coefficient by a "smooth" one. Finally, "simplification" may be a less clear term that fits everything else (dimension reduction and homogenization are simplifications, after all). The idea is to "simplify" parts of the problem—for example replacing the rugose surface of a golf ball with a smooth one when modeling flow dynamics.

Of course, it not possible to change the geometry, coefficients, or data of a complicated differential equation without causing harm, and the book goes to great lengths to develop abstract tools, mostly based in functional analysis, to measure how solutions may change under such modifications. It is no surprise that asymptotic analysis tools are also important, since models are often based on considering the effects of small parameters—the famous $\epsilon \ll 1$ becomes the shell thickness, the period of oscillatory coefficients, or the characteristic length of the golf ball dimples.

One of the most interesting topics among the above, included in the book, is dimension reduction. When considering PDEs posed in slender domains (beams, rods, plates, or shells), it might be advantageous to pose new equations in the middle surface. Of course, one must be careful here since it is not clear which equations yield solutions that are close to the original three-dimensional problem. It is reasonable to expect that the model solution and the original solution are close to each other if the thickness is small. We consider the plate problem next and show how these concepts hold.

Let $P^\varepsilon = \Omega \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^3$ be a homogeneous and isotropic plate of thickness $2\varepsilon > 0$, where $\Omega \subset \mathbb{R}^2$ is an open bounded domain with Lipschitz boundary $\partial\Omega$. Consider the linearized elasticity problem of finding the stress $\boldsymbol{\sigma}^\varepsilon : P^\varepsilon \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$, and the displacement $\boldsymbol{u}^\varepsilon : P^\varepsilon \to \mathbb{R}^3$ of a plate clamped along its lateral boundary and under transverse traction load density εg_3, where g_3 is constant and independent of ε, for simplicity. Thus

$$\begin{aligned} \mathcal{A}\boldsymbol{\sigma}^\varepsilon = \boldsymbol{e}(\boldsymbol{u}^\varepsilon), \qquad -\operatorname{div} \boldsymbol{\sigma}^\varepsilon = 0 \quad \text{in } P^\varepsilon, \\ \boldsymbol{u}^\varepsilon = 0 \quad \text{on } \partial\Omega \times (-\varepsilon, \varepsilon), \qquad \boldsymbol{\sigma}^\varepsilon \boldsymbol{n} = (0, 0, \varepsilon g_3)^T \quad \text{on } \Omega \times \{-\varepsilon, \varepsilon\}. \end{aligned} \tag{1}$$

Above, $\mathcal{A}$ is the usual compliance tensor, and $\boldsymbol{e}(\boldsymbol{u}^\varepsilon)$ is the infinitesimal strain tensor given by the symmetric part of the gradient of $\boldsymbol{u}^\varepsilon$.

Although (1) defines the equation of the "real" plate P^ε, when mentioning plate model I believe most mathematicians think about the biharmonic model

$$D\,\Delta^2\,\omega = g_3 \quad \text{in } \Omega, \qquad \omega = \frac{\partial \omega}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where the flexural rigidity D depends on the plate's Lamé coefficients. Such a model is often associated with Marie-Sophie Germain and Joseph-Louis Lagrange and is also called the Kirchhoff–Love model.

At this point it is not clear how the two-dimensional solution $\omega : \Omega \to \mathbb{R}$ is related to the three-dimensional $\boldsymbol{u}^\varepsilon$. It is even less clear how one can possibly derive (2) from (1). But before discussing that, we note that (2) is easier to solve than (1), being more amenable to complex analysis or Fourier expansion techniques, for instance. It is also easier to design numerical methods for (2) than for (1), in particular for small thicknesses ($\varepsilon \ll 1$).

Now, given the biharmonic solution ω, we define the biharmonic approximation

$$\boldsymbol{u}^0(x_1, x_2, x_3) = \begin{pmatrix} -x_3\,\mathbf{grad}\,\omega(x_1, x_2) \\ \omega(x_1, x_2) \end{pmatrix}$$

for $(x_1, x_2) \in \Omega$ and $x_3 \in (-\varepsilon, \varepsilon)$. Also, the gradient operator above acts on (x_1, x_2) only. The biharmonic model is accurate in the sense that

$$\frac{\|\boldsymbol{u}^\varepsilon - \boldsymbol{u}^0\|_E}{\|\boldsymbol{u}^\varepsilon\|_E} \le C\sqrt{\varepsilon},$$

where C depends on Ω and g_3 only, but not on ε, and $\|\cdot\|_E$ is the energy norm. We can thus conclude that the biharmonic model is a good one, in the sense that its solution converges to the original solution as the plate thickness goes to zero. A proof that the biharmonic model converges was derived by Morgenstern [15] using the *two energies principle*, and the rate of convergence can be derived as in [1].

The question of how to obtain the biharmonic model remains, and there are different ways to do so. The most most popular is based on physical ("reasonable") arguments. It is possible however to derive the model using functional analysis or asymptotic expansions [5–7]. One concludes that the biharmonic approximation is simply the asymptotic limit of the solution of (1).

Taking asymptotic limits is not the only modeling technique. A mathematically sound way to obtain models is by characterizing exact solutions as critical points of variational formulations in functional spaces and search for critical points in *carefully chosen* subspaces (aka the Galerkin method). There are myriad ways to do so, sometimes, but not always, leading to provably good models. In plate theory, this yields Reissner–Mindlin plate models, which have better approximation properties than the biharmonic model [1, 2].

Models based on asymptotic limits are not always useful. That is the case when considering PDEs posed on domains with "periodic" rough boundaries, where the asymptotic limit is given by the same PDE posed on a domain with smooth boundaries. However, that might lead to a poor description of reality, since the geometry of the wrinkles play a surprising and nontrivial role on the solution's behavior. For example, dimples of a golf ball are designed to enhance its aerodynamics — a smooth golf ball would travel around a 100–130 yards when hit by a skilled golfer, while a "real" golf ball travels around 250–290 yards [14, 17]. So, modeling the flight of a golf ball using asymptotic limits yields a deceivingly nice solution.

A neat way to incorporate the geometry of wrinkles is to impose new Robin boundary conditions on the smooth underlying surface, the so-called effective boundary condition or "wall-law". The new conditions involve constants that show up when developing asymptotic expansions and are related to the geometry of the wrinkles. The resulting model is able to capture the behavior of the original solutions in the interior of the domain [3, 11].

The book considers PDE problems posed in domains with complicated but not necessarily periodic boundaries, and it proposes *geometric simplifications*, replacing the troublesome domain by a larger one. The authors show how to estimate the modeling error. By completely disregarding the geometric details of the complicated boundary, there is a significant loss of information, but this should not come as a surprise when considering such a general case. Modeling is rarely a walk in the park.

The book discusses in detail the effect of replacing coefficients of PDEs to simplify its computation. For instance, consider the *single pore* case, when the coefficient jumps at a single ball of radius ε. Is it possible to pretend that the jump does not exist? Yes, but at a price. What about if the domain has a small hole in its interior? That is related to *topological derivatives*, which computes the asymptotic change of functionals if one pokes "infinitesimal" holes in the domain [12, 16].

2. The book

When mathematical problems are too computationally expensive, it is a good idea to model the original problem by changing the geometry and data. How accurate are the models? The book under review has a laudable goal: to discuss and establish estimates. To a large extent, the goal is achieved. Of course, many cases and details were not considered, but that is alleviated by the extensive bibliography. Also, there is an extensive list of books that deal with related topics, including the ones already cited and [4, 9, 13], to mention only a few books published by the American Mathematical Society.

The authors chose to present, at the begining of the book, a number of interesting functional analysis results that are useful elsewhere. That was probably a wise decision, but it comes with a price. It is not always easy nor seamless to jump into a latter section or example and understand some of its arguments. Also, I feel that the notation is somewhat heavy and not intuitive at some points.

The theory developed in Chapter 2 ("Distance to exact solutions") is very interesting and broad enough to be useful in several applications. The applications developed in Chapters 3, 4, and 5 are illuminating. And the topics considered in Chapter 6, which are closer to actual numerical techniques, open doors for further explorations by interested readers.

I believe that the book is a significant contribution to the mathematical literature and, in particular, is of great importance for those interested in numerical analysis of PDEs. Every good mathematical library should have a copy of this book on its shelves.

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