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Singularities of mappings: The local behaviour of smooth and complex analytic mappings, by David Mond and Juan J. Nuño-Ballesteros, Grundlehren der mathematischen Wissenschaften, Vol. 357, Springer, Cham, xv+567 pp., ISBN 978-3-030-34439-9

1. Introduction to singularity theory

This book concerns the title topic of singularities. The terms *singularities* or *singular points* have a number of different meanings depending on the context. A singular point x_0 of a mapping between manifolds $f : N \to P$ is one where the derivative of f at x_0, $df(x_0)$, has nonmaximum rank. So if $(\dim N, \dim P) = (n, p)$, then if $n \geq p$ the singular points are also called critical points (i.e., $\mathrm{rank}(df(x_0)) < p$). If instead $n < p$, then all points are critical points, while the singular points form a subset of N. Then the role of singularities in the book concerns the *structure of mappings* f between manifolds and also the structure of the *solution sets* $f^{-1}(y_0)$ for any point y_0. The singular points of $f^{-1}(y_0)$ are those where it is not a manifold. Here the rub is the rather vague notion of *structure*, which includes both local and global structural properties, where the global properties result from the interaction of the local properties.

Naively, when the case f is a smooth mapping with $f(x_0) = y_0$, one might hope to find local coordinates with x_0 and y_0 corresponding to the origin so that f may be written in local coordinates as a special polynomial mapping $f : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ which can be identified using properties of a finite number of higher derivatives of f. This illustrates *finite determinacy* and the polynomial is a *normal form*. The polynomial form then allows the local geometric and topological properties of f near x_0 to be analyzed. This representation of f is purely local, meaning that this is only defined in a neighborhood of 0. If two local mappings agree in a smaller neighborhood of 0, we say they define the same *germ of a mapping*, and the local theory is developed using this terminology; but it does concern local properties in sufficiently small neighborhoods.

Furthermore, if we slightly deform such a mapping, either it may retain the same local representation, but possibly at a nearby neighboring point x_0', or there may be a fundamental change in structure. It would be desirable to give a simple criterion when the first case occurs, in which case we say the germ is *locally stable*, and to enumerate the local stable germs for given dimensions (n, p). In the second case, we would like to describe in some sense the possible ways that a small deformation can change the structural properties.

If we think globally, then there may be a set of points $S = \{x_1, x_2, \ldots, x_k\}$ that map under f to a common point y_0. Then we would also like to understand how the different local properties of f at each x_i interact in the image at y_0. This is the *multilocal problem* of the form $f : \mathbb{R}^n, S \to \mathbb{R}^p, 0$, and we would still like to have comparable analysis. Taken together these questions constitute the basic ones for singularity theory. Furthermore, we could equally well consider holomorphic mappings between complex manifolds and ask the corresponding multilocal questions for $f : \mathbb{C}^n, S \to \mathbb{C}^p, 0$, where local holomorphic changes of coordinates are used.

The conceptual framework for attacking these questions for smooth mappings originates with René Thom as explained in his 1958 Bonn Lectures written by Harold Levine [LT]. This was motivated by earlier results of Marston Morse and especially the invaluable insights from the work of Hassler Whitney.

Morse [Ms1] (see also [Ms2]) considered smooth mappings $f : N \to \mathbb{R}$ having only singular points x_0 where the Hessian of f is nonsingular. For an appropriate choice of local coordinates about such a singular point, he showed f can be written $c + x_1^2 + \cdots + x_{n-k}^2 - x_{n-k+1}^2 - \cdots - x_n^2$. The number of negative terms k is the index of the critical point, and the topology of a compact manifold N can be determined by attaching a cell (disk) of dimension k for each critical point.

Whitney [Wh1], [Wh2] determined the *generic singularities* which arise when trying to embed manifolds in Euclidean space and for mappings of surfaces to the plane. He identifies the singularities that may occur for stable mappings, which form an open dense set of the space of smooth mappings. For example, his work shows that the simplest possible singularity which occurs when trying to smoothly embed the (nonorientable) real projective plane in $\mathbb{R}^3$ has, in appropriate coordinates, the form $g(x, y) = (x^2, xy, y)$, whose image is a *crosscap* in $\mathbb{R}^3$ (also called a *Whitney umbrella*). We might expect that we could deform g to simplify the singular behavior. It turns out that under any small smooth deformation of g, this type of singular point is stable by the work of Whitney, and so persists. For mappings between surfaces, he shows the generic singularities are *fold points* and *cusp points*, given by (1.1), and the images of the fold points are curves which may intersect transversely but will miss the images of cusp points (multilocal behavior).

$$(x_1, x_2) \mapsto (x_1^2, x_2) \qquad \text{and} \qquad (x_1, x_2) \mapsto (x_1^3 + x_2 x_1, x_2). \tag{1.1}$$

This can be contrasted with the real polynomial $f_0(x, y) = y^2 - x^3$. It has an isolated singular point at 0, which defines the curve through 0, $y^2 = x^3$, with a singular cusp point at the origin. If we deform f to $F(x, y) = y^2 - x^3 + ax + b$ for small a and b, then the singular point will either break into two singular points $(x, y) = (\pm\sqrt{\frac{a}{3}}, 0)$ when $a > 0$ or no singular points when $a < 0$. In neither case does the cusp singular point persist, hence f_0 is not stable. We may ask what other types of small deformations are possible in a neighborhood of 0 (potentially an infinite number of possibilities) and how we may characterize them, given f_0.

One point to be stressed regarding the work of Morse and Whitney is how their analyses of the singularities which arise were carried out separately for each case. The types of singular points were identified using specific criteria on lower-order derivatives, and these criteria show that the (multi)local singularities are stable under small perturbations. Also, for each case the local change of coordinates were performed by hand, using the local derivative information, and then successively applying the inverse function theorem multiple times. The polynomial forms they obtained are examples of normal forms for the singularities.

Moreover, at the same time as Morse, Solomon Lefschetz [Lf] was studying the topology of smooth complex projective varieties $X \subset \mathbb{C}P^N$ using a pencil $\mathcal{C}$ (i.e., a complex projective line) of hyperplanes $\{H_y : y \in \mathcal{C}\}$ in $\mathbb{C}P^N$ containing a common codimension 2 linear subspace W which meets X in general position. He extends the projection $X \backslash W \to \mathcal{C}$ to a modification of X on $X \cap W$; and for an appropriate choice of local coordinates the singularities of an associated projection mapping can be put into standard form $z_1^2 + \cdots + z_n^2$, for $n = \dim_{\mathbb{C}} X$. This is used to show that X is obtained from $X \cap H_y$ for generic y, by attaching cells of dimension n. Hence, they have the same homotopy type up through dimension $n-2$, yielding the Lefschetz hyperplane theorem; see, e.g., [Lm]. This is the first instance of what is now called *complex Morse theory.* Morse's and Lefschetz's results show that the real and complex cases can have analogous results, taking into account the additional cases arising over the reals involving signs.

2. Thom–Mather theory

The framework for obtaining analogous results for arbitrary dimensions was carried out in a remarkable series of papers by John Mather [M-I]–[M-VI], by quite different methods which build on the ideas of Thom, and which use and extend the fundamental theorems of Thom concerning transversality and those of Bernard Malgrange introducing the use of algebraic infinitesimal methods. This allows him to avoid trying to repeat the hands-on methods for general dimensions (n, p).

Although this was carried out for smooth mappings, Mather rather offhandedly mentioned in his papers that his (multi)local methods worked as well in the complex case for holomorphic mappings. Quite "coincidently", these results agree with and considerably expand upon those obtained independently by algebraic geometers for the class of complex analytic germs in the case $n > p$ which define *isolated complete intersection singularities* (ICIS). A number of the main results for local Thom–Mather theory which are covered in the book will be explained below.

Furthermore, for (proper) global mappings, Thom showed that in certain dimensions the smooth stable mappings could not be dense in the space of smooth mappings. Using the above results, Mather explicitly identified the pairs of dimensions (n, p) where the stable mappings are dense. These he refers to as the *nice dimensions* (see below). A more general approach, which includes all dimensions, was proposed by Thom [Th1] and refined by Mather [M3], [M4], [M5], using instead *stratifications of mappings* via Whitney stratifications [Wh3]; also see [GLDW]. This yields instead the density of topologically stable mappings in all dimensions (i.e., under the weaker notion of topological equivalence using homeomorphisms). These global results can only hold for smooth mappings (due to the more rigid properties of global holomorphic mappings on compact complex manifolds).

This complete collection of both local and global results which address the above questions constitute *Thom–Mather theory.* In the first part of the reviewed book, the results for local Thom–Mather theory are presented, allowing both real and complex cases to be considered together, with an explanation (without proofs) of how the local theory provides an answer to the global theory in the case of smooth mappings. In addition, it incorporates several refinements and extensions for the theory by various people. Then, using these results, the second part of the book develops results concerning the topology and geometry of complex analytic mappings for the case $n < p$. Because of the ultimate use for second part, there is more of an emphasis on the complex analytic results, even though historically the smooth theory was developed first.

We should point out that beyond the expected differences when considering smooth versus holomorphic mappings, the situation can take many additional different forms where singularity theory is applied. For example, around the same time as Thom and Mather's work, Vladimir Arnold established the Moscow school where methods were developed for analyzing structures in physical problems, such as, e.g., the structure of caustics from light propagation and singularities occurring in wave front evolution, among a number of other contributions; see, e.g., [AGV, vol. I] and [A2]. These have close connections to the approach of Thom and Mather.

What features the applications of singularity theory share is that important questions for them can be phrased in terms of properties of specific associated mappings. However, these mappings may be of a restricted type. Then basic Thom–Mather theory has been extended in many directions for a number of applied areas. These further developments are not treated in the book, but a brief survey can be found, e.g., in [D2].

2.1. **Algebraic and geometric frameworks.** Thom–Mather theory begins by using the actions of certain (infinite-dimensional) groups of diffeomorphisms to capture equivalence of mappings (which only works in the smooth case) or local diffeomorphisms for local mappings, including deformations. The local theory is also valid in the complex analytic category.

For the local case, we use the group of germs of (smooth, resp., holomorphic) diffeomorphisms $\mathcal{D}_n \times \mathcal{D}_p$ acting on $f_0 : \mathbb{F}^n, 0 \to \mathbb{F}^p, 0$, via composition $(\psi, \varphi) \cdot f_0 = \varphi \circ f_0 \circ \psi^{-1}$, with f_0 smooth if $\mathbb{F} = \mathbb{R}$ and holomorphic if $\mathbb{F} = \mathbb{C}$. This group is denoted by $\mathcal{A}$ and called the group of *left-right equivalence*. In the special case when $p = 1$, we may only use $\mathcal{D}_n$, which is denoted by $\mathcal{R}$ and called *right equivalence*; and if $n < p$, we might only use $\mathcal{D}_p$, denoted by $\mathcal{L}$, called *left-equivalence*. There is one additional local group $\mathcal{K}$, the *contact group*, which Mather introduced and which plays a crucial role for classification problems. It induces an ambient diffeomorphism between the germs $f_0^{-1}(0)$ for f_0 in a common orbit. For considering deformations there are corresponding groups acting on *unfoldings* $F : \mathbb{F}^{n+q}, 0 \to \mathbb{F}^{p+q}, 0$ of the form $F(x, u) = (\bar{F}(x, u), u)$ of germs $f_0(x) = \bar{F}(x, 0)$ with parameters u.

2.1.1. *Local infinitesimal methods.* Then infinitesimal methods can be used to analyze the actions of these diffeomorphism groups. The tangent spaces at the identity for the groups and spaces of mappings or germs of mappings can be determined by differentiating smoothly varying families of germs of the diffeomorphisms and functions. In the local case, for example for $\mathcal{D}_n$, we obtain the space of germs of vector fields (in the appropriate category) on $\mathbb{F}^n, 0$ vanishing at 0. Conversely, integrating such a vector field yields a local one-parameter family in $\mathcal{D}_n$. This tangent space has a richer algebraic structure as a module over the ring of smooth germs $\mathcal{E}_n$ (resp., holomorphic function germs $\mathcal{O}_n$) on $\mathbb{F}^n, 0$. In each case, these are local rings with unique maximal ideals denoted m_n consisting of germs vanishing at 0; however, $\mathcal{O}_n$ has better algebraic properties. If θ_n denotes the module of all germs of vector fields on $\mathbb{F}^n, 0$, then $T\mathcal{D}_n = m_n\theta_n$.

For each of the above groups $\mathcal{G}$, the derivative of the group action on a germ f_0 then yields a tangent space $T\mathcal{G} \cdot f_0 \subseteq m_n\theta(f_0)$, where $\theta(f_0)$ denotes the $\mathcal{E}_n$-module (resp., $\mathcal{O}_n$-module) of germs of vector fields $\zeta : \mathbb{F}^n, 0 \to T\mathbb{F}^p, 0$ lying over f_0. This tangent space is then a sum of explicit modules over these rings. For deformations, the local diffeomorphisms are allowed to move the origin, so the full modules of germs of vector fields (e.g., $\theta(f_0)$, θ_n, etc.) are used, and we obtain instead the *extended tangent space* $T\mathcal{G}_e \cdot f_0 \subseteq \theta(f_0)$. Then for any of the groups $\mathcal{G}$ listed above, the germ f_0 is said to have *finite $\mathcal{G}$-codimension* if $\dim_{\mathbb{F}}(m_n\theta(f_0)/T\mathcal{G}\cdot f_0) < \infty$ (or equivalently $\dim_{\mathbb{F}}(\theta(f_0)/T\mathcal{G}_e\cdot f_0) < \infty$).

In the smooth case, the use of infinitesimal methods is made possible via the work of Bernard Malgrange [Mg], which overcomes the weaker algebraic properties of the ring $\mathcal{E}_n$. An essential ingredient is the *Malgrange preparation theorem* for smooth germs $f_0 : \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, which states that if M is a finitely generated $\mathcal{E}_n$-module, then it is finitely generated by $\{\varphi_1, \dots, \varphi_k\}$ as an $\mathcal{E}_p$-module (via $f_0^* : \mathcal{E}_p \to \mathcal{E}_n$) iff their images span $M/f_0^*m_p \cdot M$ as a real vector space. This is the smooth version of the Weierstrass preparation theorem for the complex analytic case. This allows for the use of methods of commutative algebra criteria involving the powers of the maximal ideal m_n^k, in place of local analytic estimates based on $O(\|x\|^k)$. It provides the foundation for all of the algebraic infinitesimal methods used in local Thom–Mather theory.

A first basic consequence is that the germ f_0 has finite $\mathcal{G}$-codimension iff there is an integer $\ell > 0$ so that $m_n^\ell\theta(f_0) \subset T\mathcal{G}\cdot f_0$ (and hence for $T\mathcal{G}_e\cdot f_0$). Then at least infinitesimally, the orbit contains all variations of order $\geq \ell$. Then for a one-parameter family (or unfolding) $F(x,t) = (\bar{F}(x,t), t)$ with $\bar{F}(x,0) = f_0(x)$, conditions on $\frac{\partial \bar{F}}{\partial t}_{|t=0}$ and the preparation theorem allow the infinitesimal conditions for triviality of F to be solved and the solution vector field integrated to yield a local trivialization of F, i.e., being $\mathcal{G}$ equivalent to $f_0 \times id$. The last step involves locally solving differential equations and is the only direct use of analysis. Then ingenious uses of this idea combined with the preparation theorem allow for the proof of the finite determinacy theorem and the versal unfolding theorem stated below.

First, "f_0 is finitely $\mathcal{G}$-determined" means there is a k so that if f_1 has the same k-order Taylor expansion as f_0, then they are $\mathcal{G}$-equivalent (for $\mathcal{G}$ one of the above groups). Then the *finite determinacy theorem* asserts f_0 is finitely determined iff f_0 has finite $\mathcal{G}$-codimension. Also, for each $\mathcal{G}$ a specific k is given from the ℓ above. Tougeron [Tg] gives an alternative approach to $\mathcal{K}$-equivalence. A number of improvements for the order of determinacy and classification follow from the work of Gaffney, Bruce, du Plessis, and Wall, as explained in the book.

There are also geometric characterizations of finite determinacy which play a crucial role in the second part of the book. For holomorphic map germs, finite $\mathcal{K}$-determinacy of f is equivalent to the critical set of f only intersecting $f^{-1}(0)$ at 0. In the case $n \geq p$, this says that f defines an ICIS, which has been importantly studied for its geometry and topology (see §3.1); and if $n < p$, then the critical set is $\mathbb{C}^n$ so $f^{-1}(0) = \{0\}$, and f defines a *fat point*, an isolated point with additional algebraic structure. There is also the very useful *Mather–Gaffney criterion* characterizing finite $\mathcal{A}$-determined f which states that there is a neighborhood U of 0 so that $f|U\backslash\{0\}$ is (infinitesimally) stable.

A second key result for finite $\mathcal{G}$-codimension germs f_0 is the characterization of all small deformations of f_0 under $\mathcal{G}$-equivalence via $\mathcal{G}$-versal unfoldings. The unfolding $F(x,u) = (\bar{F}(x,u), u)$ on q parameters $u = (u_1, \dots, u_q)$ being $\mathcal{G}$-versal means that for any other unfolding $G(x,v) = (g(x,v), v) : \mathbb{F}^{n+r}, 0 \to \mathbb{F}^{p+r}, 0$ of $f_0(x)$ factors through F via a germ $\lambda : \mathbb{F}^r, 0 \to \mathbb{F}^q, 0$ so that $F(x, \lambda(v))$ is $\mathcal{G}$-equivalent to G as an unfolding via an unfolding equivalence $\varphi(x,v) = (\varphi_1(x,v), v)$ with $\varphi_1(x,0) = x$. Then the *versal unfolding theorem* asserts F is $\mathcal{G}$-versal iff $\{\frac{\partial \bar{F}}{\partial u_i}|u = 0 : i = 1, \dots, q\}$ span $\theta(f_0)/T\mathcal{G}_e\cdot f_0$. In particular, if $\{g_i(x) : i = 1, \dots, q\}$ span $\theta(f_0)/T\mathcal{G}_e\cdot f_0$, then the *linear unfolding* (2.1) is $\mathcal{G}$-versal,

$$F(x,u) = (f_0(x) + \sum_{j=1}^{q} u_j g_j, u_1, \ldots, u_q)\,. \tag{2.1}$$

This result for $\mathcal{R}$-equivalence is due to Mather in unpublished notes [M2], which provided a rigorous foundation for Thom's *elementary catastrophe theory* [Th2]. Later, Martinet gave an especially clear proof for $\mathcal{A}$ and $\mathcal{K}$ (see [Mt1]) which serves as a model for a much more general form valid for many other groups; see [D1] and references therein.

We contrast this with the approach of Kodaira–Spencer theory [KoS] as applied to deformations of local singular spaces. Allowable deformations require *flatness*, by which the deformations are not only of the defining equations given infinitesimally by a module T^1, but also require that there be deformations of the relations between the equations. These give obstructions to deformations that are given infinitesimally by a module T^2. Consequently, the parameter space may not be smooth. However, for ICIS singularities defined by a germ f, Tjurina [Tj] showed that the versal deformation is unobstructed (i.e., $T^2 = 0$) so the parameter space is smooth, with dimension now called the *Tjurina number* τ. Another very pleasant coincidence is that in this case the versal deformation corresponds to the minimal $\mathcal{K}$-versal unfolding of the germ f.

By contrast, we remark that for $n < p$ the Kodaira–Spencer theory is generally obstructed, while the versal unfoldings in general Thom–Mather theory are always unobstructed, including for many additional groups $\mathcal{G}$ (see more generally [D1], [D2] and references therein).

2.2. **Geometric transversality framework.** Geometric properties of mappings are obtained using a result of René Thom [LT], involving transversality to submanifolds of jet space, which geometrically characterize Taylor expansion properties of mappings. The k-jet space $J^k(N,P)$ provides a method for simultaneously considering all kth order Taylor expansions for all pairs of points in $N \times P$. The most relevant submanifolds in jet spaces are those invariant under the appropriate Lie group actions on the jet spaces induced from the actions of the various groups of diffeomorphisms. The resulting Lie group actions in both real and complex cases are algebraic so the orbits are actually submanifolds, and these already capture geometric properties.

A mapping f extends to a mapping $j^k(f) : N \to J^k(N,P)$ into k-jet space sending a point $x \in N$ to the kth order Taylor expansion of f at x. Then, for example, for a compact manifold N, the *Thom transversality theorem* asserts that, for a submanifold $W \subset J^k(N,P)$, the set of mappings f with $j^k(f)$ transverse to W is a residual subset of $C^\infty(N,P)$ and hence dense (as $C^\infty(N,P)$ is a Baire space). Then if $j^k(f)$ is transverse to W, it follows that $j^k(f)^{-1}(W)$ is a submanifold of N of the same codimension as W in $J^k(N,P)$ or empty.

As a consequence, if f is globally stable and $W \subset J^k(N,P)$ is invariant under the jet Lie group of $\mathcal{A}$, then $j^k(f)$ is transverse to W. Thus, a globally stable f must be transverse to all such manifolds. This provides considerable local geometric structure for stable mappings. For the global geometric characterization of stability, Mather extended the Thom transversality theorem to a *multitransversality theorem* [M-V], which captures multilocal geometric behavior at multiple points.

He furthermore used this approach to characterize global stability in terms of multitransversality conditions [M-V, Thm 4.1]. A more precise geometric form he gave is that a germ $f_0 : \mathbb{F}^n, 0 \to \mathbb{F}^p, 0$ is stable iff $j^{p+1}(f_0)$ is transverse at 0 to its $\mathcal{A}$-jet orbit (resp., its $\mathcal{K}$-jet orbit) in $J^{p+1}(\mathbb{F}^n, \mathbb{F}^p)$ and then f_0 is $p+1$-determined (this is the best possible general estimate as seen for the cusp singularity in (1.1).

The unexpected role here of $\mathcal{K}$ is further explained by its role in the classification of stable germs. Besides $\mathcal{K}$-equivalence inducing ambient diffeomorphisms of $f_0^{-1}(0)$, it also induces an automorphism of the algebra $\mathcal{E}_n$ (resp., $\mathcal{O}_n$ if $\mathbb{F} = \mathbb{C}$), sending the ideal $I(f_0) = f_0^*(m_p)\mathcal{E}_n$, spanned by the coordinate functions of f_0, to the corresponding ideal of the $\mathcal{K}$-equivalent germ. It follows that the local algebra $Q(f_0) = \mathcal{E}_n/I(f_0)$ is an invariant of $\mathcal{K}$-equivalence, as are the truncated algebras $Q_\ell(f_0) = Q(f_0)/m_n^{\ell+1}Q(f_0)$ (with corresponding version for multigerms). These algebras for $n \leq p$ are finite dimensional and are isomorphic to quotients of formal power series rings.

General germs can have isomorphic algebras without being $\mathcal{A}$-equivalent as, for example, the germ $f(x, y) = (x^3, y)$ has the same local algebra as the cusp map germ in (1.1). Mather proved the very unexpected classification result that stable germs $f_0 : \mathbb{F}^n, 0 \to \mathbb{F}^p, 0$ are classified up to $\mathcal{A}$-equivalence by the isomorphism class of their local algebras $Q_{p+1}(f_0)$ (or $Q(f_0)$ if $n \leq p$).

As a consequence, Mather further proves that if $I(f_0) \subset m_n^2$, so the coordinates of f_0 have no linear terms, and if $\{g_1, \ldots, g_q\}$ spans $m_n\theta(f)/T\mathcal{K}_e \cdot f_0$, then the germ $F(x, u) : \mathbb{F}^{n+q}, 0 \to \mathbb{F}^{p+q}, 0$, given by

$$F(x, u) = (f_0(x) + \sum_{j=1}^{q} u_j g_j, u_1, \ldots, u_q)\,, \tag{2.2}$$

is a stable germ, a *stable unfolding* of f_0, with local algebra $Q(F) \simeq Q(f_0)$. This provides a method to construct normal forms for stable germs with a given local algebra.

Then the combination of the multitransversality characterization of stability together with the classification of stable germs by local algebras and the explicit method for constructing stable germs with a given local algebra by stable unfoldings allow Mather to precisely determine the range of dimensions (n, p) where the set of globally stable mappings forms an open dense subset of the space of (proper) smooth mappings. The failure of density results from the transversality characterization of stability, when there is a submanifold S of jet space $J^k(N, P)$ of codimension $\leq n$ foliated by $\mathcal{K}$-jet group orbits in S of constant positive codimension in S. The orbits are captured by smooth *moduli parameters.* Mather shows there are germs with jets transverse to S but which fail to be transverse to all of the orbits in S.

He then uses invariant theory for the jet Lie group actions of $\mathcal{K}$, to obtain for each pair (n, p) the lowest codimension submanifold $\Pi(n, p)$ in jet space where this occurs. With $\sigma(n, p)$ denoting its codimension, Mather then shows that $n < \sigma(n, p)$ exactly characterizes the nice dimensions, where stable mappings are dense, and the above results allow to him to explicitly give for such an (n, p) the stable germs with specific normal forms. The nice dimensions play an important role in Part II of the book, where $n < p$, and they are given by the inequalities

$$n < \begin{cases} \frac{6}{7}p + \frac{6}{7} & p - n > 3, \\ \frac{6}{7}p + \frac{9}{7} & p - n \leq 3\,. \end{cases} \tag{2.3}$$

3. Geometry and topology of finitely determined map germs

With the foundation provided by the results of local Thom–Mather theory in place, then the stage is set for analyzing the topological and geometric properties of both stable germs and the germs of *finite codimension.* Although a few results have been obtained in the real case, this has been principally carried out in the holomorphic case taking advantage of the richness of complex structures.

3.1. **Geometry and topology in the complex analytic case $n > p$.** Previously, the study of singularities of complex algebraic varieties V had centered on replacing the singularities by their resolutions, to give a nonsingular variety $\tilde{V}$ mapping diffeomorphically to V off the singular set $\mathrm{sing}(V)$ and using the properties of $\tilde{V}$ and the *exceptional divisor*, i.e., the inverse image of the singular set to understand V. Two classes of singularities that have been carefully studied this way are isolated curve and surface singularities.

The approach for the local geometric and topological structure of complex analytic singularities dramatically changed beginning in the early 1970s. This began with the fibration theorem of Milnor [Mi] for hypersurface germs $f : \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$. It is formed for sufficiently small $0 < \eta \ll \varepsilon$, a ball about 0 of radius ε, $B_\varepsilon \subset \mathbb{C}^{n+1}$ and punctured disk $B^*_\eta \subset \mathbb{C}$, so that $f : f^{-1}(B^*_\eta) \cap B_\varepsilon \to B^*_\eta$ is a fibration. Moreover, if V has an isolated singularity at 0, a fiber $F_w = f^{-1}(w)$ for $w \in B^*_\eta$ is homotopy equivalent to a *bouquet of spheres* of dimension n, $\bigvee_{i=1}^{\mu} S^n$ (i.e., a union of such spheres sharing a common point). The number of such spheres μ is called the Milnor number and is computed as the $\mathcal{R}_e$-codimension of f (i.e., $\dim_{\mathbb{C}}(\mathcal{O}_{n+1}/\,\mathrm{Jac}(f))$, where $\mathrm{Jac}(f)$ is the ideal generated by the partials $\frac{\partial f}{\partial x_i}$, $i = 1, \ldots, n+1$). Milnor actually constructed his fibration in S_ε, the boundary sphere of B_ε, but it is equivalent to the one given here. The closure of F_w is a manifold with boundary diffeomorphic to the link $L(V) = f^{-1}(0) \cap S_\varepsilon$.

Example 3.1. As an illustration, we consider the germ of the complex polynomial $f_0(x, y) = y^2 - x^3$ at 0. It has an isolated singular point at 0 and in this case the Milnor fiber is diffeomorphic to the global complex curve $y^2 - x^3 = b$ for $b \neq 0$, that can be compactified with a point at infinity to yield an elliptic curve, which is topologically a torus. Thus, the Milnor fiber is diffeomorphic to a torus with a point removed. It has as a deformation retract the one point union of two S^1's (the Milnor number is 2). An analysis also shows that the link consists of a knotted curve in a torus which wraps three times in one direction while just twice in the other, forming a $3-2$ *torus knot*, which provides a boundary to the Milnor fiber. This is the simplest example of the *algebraic knots* as links of complex plane curve singularities.

This was extended by Hamm [Ha] to a Milnor fibration for ICIS, defined by a germ $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$. He shows that the Milnor fiber has the homotopy type of a bouquet of spheres of dimension $n - p$. Here the number of spheres involves a more subtle computation, except in the weighted homogeneous case where by Greuel (see, e.g., [L1]) it is given by the Tjurina number τ, i.e., $\mathcal{K}_e$-codimension of

f_0. Furthermore, Lê [Lê1] extended this to germs $f : X, 0 \to \mathbb{C}, 0$ on a complete intersection X, when f has an *isolated singularity* in an appropriate sense.

An immense body of work by a long list of distinguished researchers, beginning with Brieskorn and Arnold, have investigated structures built upon the Milnor fibration. This includes an intersection pairing defined on the Milnor fiber; properties of the monodromy of the Milnor fibration about the origin; a relative de Rham complex to compute the complex cohomology of the Milnor fiber; an integral lattice defined by the integral homology; a distinguished basis of *vanishing cycles* obtained from the Morse singularities arising from a stabilization of f_0 (also called a Morsification), a resulting action of the full monodromy group on the integral lattice via Picard–Lefschetz theory; a mixed Hodge structure by Steenbrink (adapting the theory introduced by Deligne); a local Gauss–Manin connection on the complex cohomology of the Milnor fiber as a vector bundle, and the related theory of $\mathcal{D}$-modules (modules over the ring of holomorphic differential operators); etc.

These were further applied in view of the important classification results discovered by Arnold extending the germs appearing in the nice dimensions and revealing rich geometric characterizations of these singularities [A1]. For the *simple singularities*, their properties relate to the simple Lie groups A_k, D_k, and E_6, E_7, E_8, and for surface singularities they arise as quotients by Kleinian groups acting on $\mathbb{C}^2$. These further extend to quotients by Schwarz triangle groups, yielding classes of singularities involving moduli, which are ignored in the nice dimensions. All of these developments highlight the importance of the case $n > p$, and are explained in the papers of Brieskorn [Br1]–[Br3], the books [L1], [AGV, Vol II], and [PeSt], and the many references therein.

3.2. **Topology of finite map germs and discriminants.** For the second part of the book, the above would provide a virtual wish list of types of results one might hope to obtain for the topology and geometry for local complex analytic mappings with $n < p$. The results which can be obtained build in fundamental ways upon the results from Thom–Mather theory from the first part. However, the approach to the topology for the case $n > p$ has to be modified in a number of significant ways.

First, finitely $\mathcal{K}$-determined map germs $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, with $n < p$, are finite-to-one mappings. There is no immediate analogue of a Milnor fibration because the image of f_0 is a (germ of a) complex analytic subspace of dimension $n < p$. Thus, for almost all $y \in \mathbb{C}^p$ in any neighborhood of 0, $f_0^{-1}(y) = \emptyset$, while for y in the image of f_0, $f_0^{-1}(y)$ is a finite set which will vary in cardinality depending on y. This raises a number of questions:

(1) Is there a substitute for the Milnor fibration?
(2) How does the difference $p - n$ affect the approach to the topology?
(3) How does the variability of $\mathrm{card}(f_0^{-1}(y))$ enter into the topological analysis?
(4) Is there a substitute for de Rham cohomology of fibers?
(5) What computational methods are available for (4)?

The goal of the second part of the book is to begin to answer a number of these questions and identify parts for which additional future work is needed. By comparison with the algebraic results needed in the first part of the book for Thom–Mather theory, now deeper algebraic and geometric properties of the various complex analytic sets are needed, bringing into play additional results from local analytic geometry, sheaf theory, and commutative and homological algebra.

3.3. **Stabilization of finitely ($\mathcal{A}$-) determined map germs.** The answers to these questions began with an important idea introduced by David Mond when he considered finitely determined germs $f_0 : \mathbb{C}^2, 0 \to \mathbb{C}^3, 0$ [Mo]. He restricts to germs which are finitely $\mathcal{A}$-determined. By the Mather–Gaffney criterion there is then a neighborhood U of 0 so that $f_0|U\backslash\{0\}$ is stable. As these germs are in the nice dimensions, there is a one-parameter perturbation f_t of f_0 which is stable on a neighborhood $B_\varepsilon \subset U$ when $0 < t < \delta$ for some $\delta > 0$.

We should note there is an analogous statement for isolated hypersurface singularities $f_0 : \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$. Such germs are finitely $\mathcal{A}$-determined, and they have a stabilization $f_t : B_\varepsilon \to \mathbb{C}$ which yields for $t \neq 0$ a mapping with only complex Morse singularities. The number of such singularities equals the $\mathcal{R}_e$-codimension of f_0 which leads to a proof, using complex Morse theory, for the number of spheres $\mu(f_0)$ in the homotopy type of the Milnor fiber. Then $\mathcal{R}_e$-codimension $\geq$ $\mathcal{K}_e$-codimension, which is the Tjurina number τ, with equality if f_0 is weighted homogeneous. Thus, $\mu(f_0) \geq \tau$, with equality if f_0 is weighted homogeneous. By a result of Greuel, the analogue also holds for ICIS. Rather surprisingly there is an analogous result discovered by Mond for the germs $f_0 : \mathbb{C}^2, 0 \to \mathbb{C}^3, 0$.

3.3.1. *The Mond conjecture.* First, the work of Mond with Marer, and with de Jong and van Straten showed that if $f_0 : \mathbb{C}^2, 0 \to \mathbb{C}^3, 0$ is weighted homogeneous, the $\mathcal{A}_e$-codimension equals the *vanishing Euler characteristic* of the *deformed image* $f_t(B_\varepsilon)$. Later it was recognized by Mond that $f_t(B_\varepsilon)$ is homotopy equivalent to a bouquet of two-dimensional spheres. The deformed image is referred to as a *disentanglement of the image* and the number of S^2's is called the *image Milnor number* and denoted $\mu_I(f_0)$ (this equals the vanishing Euler characteristic). However, now for this case, we let instead $\tau = \mathcal{A}_e$-codimension. Then the full statement becomes $\mu_I(f_0) \geq \tau$ with equality if f_0 is weighted homogeneous. A crucial condition for the result is that the image is a hypersurface.

This led to the *Mond conjecture* that the corresponding statement is true for all n with $(n, n+1)$ in the nice dimensions. The proof of this requires understanding at a deeper level the algebraic structure of the normal space $\theta(f_0)/T\mathcal{A}_e \cdot f_0$ to control the properties under deformation. As tempting as this result is, at this point it has resisted a general proof, although there are many partial results mentioned in the book. What this result points to is the importance of how we should understand the algebraic and geometric properties of the image as a highly singular complex analytic germ and the relation with its topology.

3.4. **Vanishing topology of discriminants.** There is another important case of a highly singular hypersurface, namely the discriminant $D(f_0)$ for a finitely $\mathcal{A}$-determined germ $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ for $n \geq p$. If (n, p) lies in the nice dimensions, then as above we can deform f_0 to a stable mapping $f_t : B_\varepsilon \to \mathbb{C}^p$ for $t \neq 0$ and consider instead the discriminant of $D(f_t)$. The theorem of [Lê1] mentioned earlier can be shown to apply in this very different situation so that $D(f_t)$ is homotopy equivalent to a bouquet of spheres of dimension $p - 1$; see, e.g., Figure 1. This is now referred to., as the *discriminant Milnor fiber* and the number of spheres $\mu_\Delta(f_0)$ as the *discriminant Milnor number*. However, unlike the case of the image

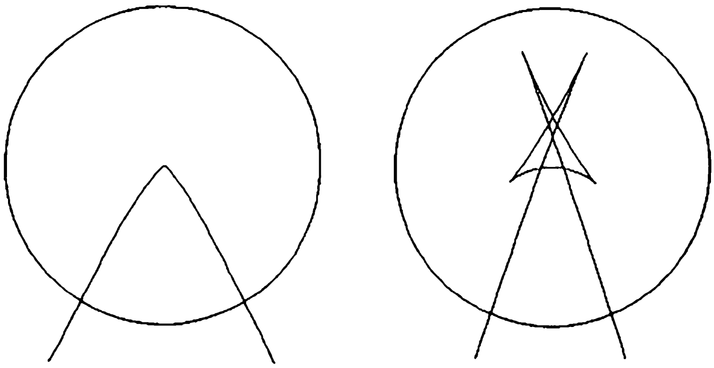

FIGURE 1. Illustrations for (a) the discriminant of a germ of $\mathcal{A}_e$-codimension 6 ($f_{\mathbb{R},0}(xy, y) = (x, xy + y^6)$) and (b) the resulting stabilized discriminant ($f_{\mathbb{R},t}$ of $f_{\mathbb{R},0}$), which even in the real case is homotopy equivalent to a bouquet of six S^1's.

for $n = p - 1$ first considered by Mond, it is possible to explicitly compute the number of such spheres in the weighted homogeneous case in the nice dimensions.

This results from algebraic properties of the discriminant $D(F)$ for F a stable unfolding of f_0, due to the combined work of Teissier [Te1] (using the 0th Fitting ideal), K. Saito [Sa], and Looijenga [L1] leading to the resulting freeness of the module of germs of ambient vector fields tangent to $D(F)$ (then $D(F)$ is a *free divisor*). Then $\theta(f_0)/T\mathcal{A}_e \cdot f_0$ can be isomorphically identified with another module for capturing how $\mathbb{C}^p$ intersects $D(F)$. The corresponding module for the stabilization is Cohen–Macaulay. This allows counting the number of critical points of the restriction G_t of the defining equation G for $D(F)$ to the image of $\mathbb{C}^p$ under the deformation by t, yielding the the $\mathcal{A}_e$-codimension of f_0. Then the number of spheres is given by complex Morse theory using a result of Siersma [Si]. Then with $\tau = \mathcal{A}_e$-codimension of f_0, $\mu_\Delta(f_0) \geq \tau$ with equality if f_0 is weighted homogeneous. Thus, the form of the original $\mu \geq \tau$ conjecture is true in this case but with a different meaning for τ. In this form, these topological results closely align with the topological properties for ICIS.

3.4.1. *The Mond conjecture revisited.* In the cases $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^{n+1}, 0$ for $n = 1, 2$ for the Mond conjecture, a similar approach is used except now the discriminant is replaced by the image, denoted $D(f_0)$. Now for the stable unfolding F, the structure of $D(F)$ is more complicated. Instead, $\theta(f_0)/T\mathcal{A}_e \cdot f_0$ is isomorphic to an algebraic structure which now captures additional properties of the multiple point sets in the image (where multiple x_i map to a common y). This time Fitting ideals (for $\mathcal{O}_n$ as an $\mathcal{O}_{n+1}$-module via f_0^*) are used, but principally to algebraically capture the *double point set* in $\mathbb{C}^{n+1}$. Its properties result from the special form obtained by work of Mond and Pellikaan. For example, in the case $n = 2$, this yields an algebraic structure whose dimension differs from $\mathcal{A}_e$-codim(f_0) by the number of Whitney umbrella points in the stabilization f_t. On the other hand, this structure counts the number of critical points of g_t but includes critical points on the image $D(f_t)$. The singular points on $D(f_t)$ include curves of double points, isolated triple points, and

Whitney umbrella points, but only the latter are counted via the algebra, so the corrections cancel, and the result follows. Parts of the method extend to a more general situation and higher n, except complications arise from higher multiple point sets and increased singularity types in f_t. For this reason only partial results exist at this time for higher dimensions, a number of which are referred to in the book.

3.5. **Geometry and topology of multiple point spaces.** If we consider the cases $n < p - 1$, the images are no longer even hypersurfaces. Their algebraic structure is more complicated, so the structure of the deformed image now requires a careful analysis of the multiple point spaces for varying cardinality.

The machinery for capturing the structure of the k-multiple point spaces is initially introduced for a local finite complex analytic mapping $f : X \to Y$ as an analytic subset $\mathcal{D}^k_{\text{cl}} \subset X^k$ consisting of distinct $x_1, x_2, \ldots, x_k$ with all $f(x_i) = y_0$, but removing closed analytic components where some x_i are equal. In the case of a finitely $\mathcal{A}$-determined germ $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ with $n < p$, this is refined by considering $\mathcal{D}^k_{\text{cl}}$ for a stable unfolding F of f_0 and then restricting to $(\mathbb{C}^n)^k$. This version has better algebraic properties and is denoted by $\mathcal{D}^k(f_0)$, and for a finitely $\mathcal{A}$-determined f_0 these may differ by the addition of $\{0\}$. There are analogous $\mathcal{D}^k(f_t)$ for a stabilization of f_0.

There is a natural action of the symmetric group S_k on $\mathcal{D}^k(f_0)$ by permuting the points, so the approach is to study these sets with this group action. The image under f_0 collapses $\mathcal{D}^k(f_0)$ to the image of points y with $f_0^{-1}(y)$ consisting of k or more points. There is also a projection $\varepsilon^{k+1} : \mathcal{D}^{k+1}(f_0) \to \mathcal{D}^k(f_0)$ by forgetting the $(k+1)$-st point. The image contains the subset which is the closure of the set where there are more then k points mapping to the same point. Then the symmetric group S_k acts freely on $\mathcal{D}^k(f_0)\backslash\varepsilon^{k+1}(\mathcal{D}^{k+1}(f_0))$, and its image under f_0 consists of the set of $y \in \mathbb{C}^p$ with $f_0^{-1}(y)$ consisting of exactly k points.

The image is made up of the disjoint union of the images of these differences. The goal in analyzing the image is to understand the algebraic and geometric structure of each $\mathcal{D}^k(f_0)$ and to introduce methods for deducing properties of the image from these analytic sets and their relations. All of this is likewise true for the deformed image using the $\mathcal{D}^k(f_t)$.

The algebraic structure is obtained from *divided difference equations*. For example, the set of (x, x') such that $f(x) = f(x')$ is the zero set of $f(x) - f(x')$, which includes the points where $x = x'$ so we may factor $f(x) - f(x') = (x - x') \cdot g(x, x')$ with $g = 0$ capturing the real double points. This is combined with a consequence for the complex analytic case of a result proven with Andre Galligo that states: for a class of local algebras that include those for the stable germs in the nice dimension, the maximum k such that $\mathcal{D}^k(f_0) \neq \emptyset$ equals $\dim_{\mathbb{C}} Q(f)$. This together with multitransversality provides a handle on basic geometric properties of the $\mathcal{D}^k(f_0)$ for finitely $\mathcal{A}$-determined germs f_0. Specifically, the *double point space* $\mathcal{D}^2(f_0)$, is shown to be Cohen–Macaulay. In the case of stable corank-1 germs (i.e., $\operatorname{rank}(df_0(0)) = n - 1$), the $\mathcal{D}^k(f_0)$ are smooth. Then, by multitransversality, for finitely $\mathcal{A}$-determined germs, the $\mathcal{D}^k(f_0)$ have the appropriate dimension $n - (k-1)(p-n) = p - k(p-n)$ and are shown to be isolated complete intersection singularities. Also, for a stabilization f_t, the $\mathcal{D}^k(f_t)$ are the Milnor fibers, so the earlier described results for $n > p$ can be applied.

For germs of corank ≥ 2, the methods no longer directly apply and the divided difference scheme becomes considerably more complicated. Already for stable germs f_0, $\mathcal{D}^2(f_0)$ is not smooth, and the divided difference scheme is more complicated. A beginning approach to understanding this case is by constructing specific resolutions of the $\mathcal{D}^k(f_0)$ using a method of iterated *blowing ups* from algebraic geometry. In the case of the double point space, this joins with earlier work for the global case by Kleinman and by Ronga to provide a resolution for $\mathcal{D}^2(f_0)$. This extends to the triple point space. However, for the general case, the complete understanding of the structure of the $\mathcal{D}^k(f_0)$ is still a work in progress, and the book points to much work still to be done.

3.6. **Topology of the image via alternating homology.** The insights gained from the local analysis of the $\mathcal{D}^k(f_0)$ for finitely $\mathcal{A}$-determined f_0 do provide the foundations for understanding the topology of the image. For a stabilization $f_t : B_\varepsilon \to \mathbb{C}^p$, the restriction $f_t|\mathcal{D}^k(f_t)$, for $t = 0$ or $t \neq 0$, is a type of branched covering with a natural action of the symmetry group S_k. This leads to an approach developed by Victor Goryunov and David Mond for computing the topology of the image for a finite-to-one surjective algebraic map $f : X \to Y$ between compact semialgebraic sets using *alternating homology*, which we can think of as a partial replacement in the case $n < p$ for de Rham cohomology used when $n > p$. This homology theory restricts to the homology chains c which are alternating, i.e., $\sigma \cdot c = \operatorname{sgn}(\sigma)c$ for $\sigma \in S_k$.

For a coefficient ring this gives a chain complex $C_\bullet^{\mathrm{Alt}} = \{C_n^{\mathrm{Alt}}(\mathcal{D}^k(f), R)\}$ with induced homology groups denoted $AH_n(\mathcal{D}^k(f); R)$ (in fact, this works for any topological space with S_k action). Here principally $R = \mathbb{Z}$ or $\mathbb{Q}$. In the case of $R = \mathbb{Q}$, $AH_n(\mathcal{D}^k(f); \mathbb{Q}) \simeq H_n^{\mathrm{Alt}}(\mathcal{D}^k(f); \mathbb{Q})$, the subspace of ordinary homology on which $\sigma \in S_k$ acts by multiplication by $\operatorname{sgn}(\sigma)$.

These groups were initially modeled by Goryunov based on Vassiliev's work on knot invariants. Now, they can be understood either via using a complex of equivariant sheaves or as presented in the book using simplicial complexes via a triangulation of the maps, using a result of Hardt. Several very instructive examples in the book illustrating this homology are explained in depth.

The map $\varepsilon^{k+1} : \mathcal{D}^{k+1}(f) \to \mathcal{D}^k(f)$ induces a homomorphism

$$\varepsilon_\sharp^{k+1} : C_n^{\mathrm{Alt}}(\mathcal{D}^{k+1}(f), R) \longrightarrow C_n^{\mathrm{Alt}}(\mathcal{D}^k(f), R)$$

by viewing S_k as the subgroup of S_{k+1} fixing the $(k+1)$-st element. This yields a double complex $\{C_p^{\mathrm{Alt}}(\mathcal{D}^q(f), R)\}$ with standard horizontal homology boundary operators and vertical operators $\varepsilon_\sharp^k$. In a standard manner this yields a total complex, and a standard method can be applied to obtain a spectral sequence converging to the homology of the total complex. In another direction, each vertical simplicial complex, when extended by $f_\sharp : C_\ell(X, R) \to C_\ell(Y, R)$ is shown to be acyclic. It follows that the total complex has homology groups $H_\ell(Y; R)$.

3.7. **Image computing spectral sequence.** This leads to the *imaging computing spectral sequence* (ICSS) developed by Goryunov and Mond and further by Kevin Houston. This spectral sequence applies generally for a finite-to-one surjective algebraic maps $f : X \to Y$ between compact semialgebraic sets. We might think of this as the dual of the Serre spectral sequence for fibrations, now for the images of finite mappings. Here, $E^1_{p,q} = AH_q(\mathcal{D}^{p+1}(f); R)$ converging to $H_{p+q}(Y; R)$, with differential $d^1_{q,p} = \varepsilon^{p+1}_*$ on $AH_q(\mathcal{D}^{p+1}(f); R)$.

In the case of a $\mathcal{A}$-finitely determined corank-1 germ $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^{n+\ell}, 0$ with stabilization f_t, the preceding geometric properties imply, for $R = \mathbb{Q}$, that $AH_q(\mathcal{D}^\ell(f_t); \mathbb{Q})$ is zero except in one nonzero dimension $n - (k-1)(\ell - 1)$. The spectral sequence for f_t then collapses at the E^1 term, allowing us to compute the alternating homology for the deformed image $Y_t = f_t(B_\varepsilon)$. First, for $\ell \geq 2$, if $q = n - (k-1)(\ell - 1)$ (for some k) and $q \neq 0$, then

$$H_q(Y_t; \mathbb{Q}) \simeq H^{\mathrm{Alt}}_{n-(k-1)\ell}(\mathcal{D}^k(f_t); \mathbb{Q}),$$

and it is $\mathbb{Q}$ if $q = 0$. While if $\ell = 1$ (which involves the Mond conjecture)

$$H_n(Y_t; \mathbb{Q}) \simeq \bigoplus_{k=2}^{n+1} H^{\mathrm{Alt}}_{n-k+1}(\mathcal{D}^k(f_t); \mathbb{Q})$$

and $H_q(Y_t; \mathbb{Q}) = 0$ if $q \neq 0, n$. Here the ranks of $H^{\mathrm{Alt}}_{n-k+1}(\mathcal{D}^k(f_t); \mathbb{Q})$ are called the *alternating Milnor numbers* and can in principle be computed using the specific ideals defining them.

There are two extensions. For a multigerm, Houston shows that instead the spectral sequence collapses at E_2. In addition for germs f_0 of corank ≥ 2, although the $\mathcal{D}^k(f_0)$ do not have isolated singularities and can have homology in multiple dimensions, Houston shows the alternating homology is concentrated in the one appropriate dimension, and analogous formulas still hold. There are also results using instead cohomology.

Concerning the Mond conjecture, an ongoing challenge now would be to use the alternating homology of multiple point spaces to compute the topology of the deformed image of f_t and relate it to the $\mathcal{A}_e$-codimension of f_0. When $\mathrm{corank}(f_0) = 1$, the preceding gives a formula for the image Milnor number in terms of the alternating Milnor numbers. However, as of now, they do not yield an algebraic formula that would equal the $\mathcal{A}_e$-codimension. Also, the conjecture includes f_0 which have higher corank and at present involve greater complications. Nonetheless, these results supply important steps in understanding the topology of the images as an important class of highly nonisolated singularities, identifying where future advances still need to be made.

4. Placing this book among standard references and textbooks

There are a number of reference and text books which concern various aspects of either smooth or holomorphic singularity theory. Several principal ones include Golubitsky and Guillemin [GG], Jean Martinet [Mt2], Bruce and Giblin [BG1], and Arnold, Gusein-Zade, and Varchenko [AGV, vol I, Part I] for smooth singularity theory, and Looijenga [L1] and Arnold, Gusein-Zade, and Varchenko [AGV, vol II] for the complex case.

The material treated in this book largely complements the results in these books, so we briefly contrast their approaches. The books [GG], [Mt2], [BG1], and [AGV, vol I, Part I] consider only the smooth theory and do not attempt to fully develop the local theory, although Martinet comes closest devoting more attention to the local theory, versal unfoldings and the beginning of the classification. The book [BG1] gives a gentle introduction, concentrating on smooth functions and applications to elementary differential geometry. The first part of [AGV, vol I] gives an exposition of the results in Thom–Mather theory, often without full proofs (moving on in later parts to classification methods for functions and to Lagrange and Legendre singularities). Both [GG] and [Mt2] cover the Thom transversality theorem and Malgrange's theorem. Golubitsky and Guillemin give a full-throated treatment of global smooth stability, including proving the Thom transversality theorem (and its multitransversality version), the Malgrange (and Weierstrass) preparation theorem, and the division theorem from which it follows, and giving their own proof of Mather's theorem, *infinitesimal stability implies stability.* This reviewed book then benefits from the above treatments for the transversality theorem and Malgrange's theorem by referring to, e.g., [GG] for complete proofs, and allowing more attention to the local Thom–Mather theory.

For the complex analytic case, Looijenga [L1] and Arnold, Gusein-Zade, and Varchenko [AGV, vol II] concentrate on the geometry and topology of isolated hypersurface and complete intersection singularities, covering many of the topics listed earlier, but which is complementary to the results in the complex case for $n < p$.

The authors of the reviewed book give a thorough development of local Thom–Mather theory, beginning with the simplest examples for $\mathcal{R}$-equivalence of both smooth and holomorphic functions. Then they gradually extend the results to the more complicated work for $\mathcal{A}$-equivalence, where the algebra becomes more demanding. They have done so with many carefully explained examples and many exercises, making frequent use of computer algebra, based on the software Macaulay 2 (an alternative would be for a reader to use the software Singular). They have leaned somewhat more in the direction of the holomorphic case, with the second part of the book in mind. As a result, the level of treatment demands somewhat more of the reader's algebraic and geometric sophistication beyond what is found in the other books on smooth equivalence. They have compensated by providing a useful collection of five detailed appendices covering jet bundles and the transversality theorem, stratifications, commutative algebra, local analytic geometry, and sheaves. For the second part of the book, the higher level of presentation is roughly that of [L1] and [AGV, vol II].

A reader who does learn the theory from the first part of the book will be well rewarded with a complete treatment of the local theory. Then as a textbook, the first part would be a good choice for an upper level graduate course for students with good backgrounds in algebra (especially commutative algebra) and the fundamentals of algebraic geometry. Continuing on to the second part will open to the reader an area on complex analytic maps rich in explanations of interesting existing results, new methods, and pointing to many remaining problems of interest.

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