BOOK REVIEWS

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The Cremona group and its subgroups, by Julie Déserti, Mathematical Surveys and Monographs, Vol. 252, American Mathematical Society, Providence, RI, xii+187 pp., ISBN 978-1-4704-6012-9

An example of a Cremona transformation is familiar to any mathematician through the notion of the inversion transformation: a point x in $\mathbb{R}^n$ is mapped to a point $T(x)$ on the line $\mathbb{R}\cdot x$ such that $|T(x)|\cdot|x| = r$ for a fixed $r > 0$. In the case $n = 2$, the transformation was familiar to Apollonius of Perga in 200 BC. The ancient notion of a stereographic projection that goes back to Ptolemy is a result of applying the inversion transformation in $\mathbb{R}^3$ to a sphere. The distinguishing feature of the inversion transformation is that it may transform spheres to planes and vice versa, but, most importantly, it cannot be extended to an everywhere defined map. To solve the latter problem is possible only in the case $n = 2$ by adding to $\mathbb{R}^2 = \mathbb{C}$ the point at infinity. The introduction of projective coordinates made it possible to identify the extended plane with the complex projective line $\mathbb{CP}^1$. The inversion transformation acquires an analytic formula $(z_0 : z_1) \mapsto (\bar{z}_1 : r\bar{z}_0)$, and hence becomes a composition of a projective transformation and the conjugation. Another way to obtain an analytic formula for the inversion transformation that works for any n is to consider the real projective space $\mathbb{RP}^n$ and define the transformation by the formula $T : (x_0 : \cdots : x_n) \mapsto (x_1^2 + \cdots + x_n^2 : rx_0x_1 : \cdots : rx_0x_n)$. In this way we can define this transformation in the complex projective space $\mathbb{CP}^n$. It is given by homogeneous polynomials of the same degree (called the algebraic degree of the transformation) in projective coordinates. Although it is still not defined on the subset $x_0 = x_1^2 + \cdots + x_n^2 = 0$, it is invertible on the complement of this set. This leads to the definition of a Cremona transformation of $\mathbb{CP}^n$ as a transformation given in projective coordinates by homogeneous polynomials of degree d that may vanish simultaneously on a closed subvariety of codimension ≥ 2 and which is invertible on a some subset complementary to a closed subvariety. Transformations with $d = 2$ are called quadratic Cremona transformations. Until the fundamental works of Luigi Cremona and Ernest de Jonquières in the second half of the 19th century, it was widely believed that all Cremona transformations are quadratic. In attempts to extend a Cremona transformation to an everywhere defined map, many various constructions were introduced in the 19th century, all based on different concepts of infinitely near points. This led to basic tools of modern birational geometry of algebraic varieties, such as σ-processes, blowups and blowdowns, which are some examples of birational transformations of algebraic varieties. Nevertheless, a general Cremona transformation of algebraic degree $d > 1$ cannot be extended to an automorphism of any birationally equivalent complete algebraic variety, so the indeterminacy of the transformation is unavoidable. This makes Cremona transformations very different from automorphisms of other geometric structures, such as homeomorphisms, diffeomorphisms, analytic automorphisms of manifolds.

Cremona transformations of $\mathbb{CP}^n$ form a group, denoted by $\mathrm{Cr}(n)$, and the yet unpopular concept of a group in the beginning of the 19th century was probably the reason why Cremona transformations of higher algebraic degree were overlooked at that time. The fundamental theorem of Max Noether asserts that any Cremona transformation in the complex projective plane is a composition of projective transformations and only one non-projective quadratic transformation that inverts the affine coordinates. Although the theorem may give a false impression that the group $\mathrm{Cr}(2)$ is quite simple, it is far from the truth. For example, until very recently, it was unknown whether the group is simple as an abstract group. The first proof of its simplicity by Serge Cantat and Stéphane Lamy [5] used very technical tools of hyperbolic geometry of infinite-dimensional hyperbolic spaces. This fundamental contribution to algebraic geometry came from experts in completely different fields of mathematics—hyperbolic geometry and complex dynamics—demonstrating once more the unity of mathematics. Incidentally, the author of the book under review originates from that same group of brilliant young mathematicians.

An algebraic equivalent of the Cremona group $\mathrm{Cr}(n)$ is the group of automorphisms of the field $\mathbb{C}(t_1, \dots, t_n)$ of rational functions in n variables that are identical on the coefficients. At this point, we can replace the field $\mathbb{C}$ of coefficients with any field K and introduce the Cremona group $\mathrm{Cr}_K(n)$. The description of the group $\mathrm{Cr}_K(1)$ is easy and, I believe, should be given in any undergraduate course of abstract algebra: it consists of the fractional linear transformation $t \mapsto \frac{at+b}{ct+d}$ with $ad - bc = 1$. By assigning to this transformation the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we obtain an isomorphism between $\mathrm{Cr}_K(1)$ and the projective linear group $\mathrm{PGL}(2, K)$. Over the complex numbers, the transformations are known as Möbius transformations. They come in three kinds: elliptic, parabolic, and loxodromic (or hyperbolic) according to whether $|a + d| < 2$, $= 2$, or > 2, respectively. A group containing only elliptic transformations is a finite group. A group that contains elliptic or parabolic transformations is an extension of an infinite cyclic group by a finite group. A group containing at least two non-commuting hyperbolic transformations is essentially non-abelian infinite group. Among them we encounter discrete subgroups of the Lie group $\mathrm{PSL}(2, \mathbb{C})$ like Kleinian groups, in particular, the modular group $\mathrm{PSL}(2, \mathbb{Z})$ or its subgroups of finite index, the main objects of study in the modern theory of modular forms and three-dimensional hyperbolic geometry.

The classification of finite subgroups of $\mathrm{Cr}_{\mathbb{C}}(1)$ was first given in the work of Felix Klein and Hermann Schwarz in the 1880s. The groups were known from antiquity as symmetry groups of Platonic solids. The subgroups come in two infinite series of cyclic and dihedral subgroups and three exceptional groups: the tetrahedral group $\mathbb{T}$ isomorphic to the alternate group $\mathfrak{A}_4$, the octahedral group $\mathbb{O}$ isomorphic to the symmetric group $\mathfrak{S}_4$, and the icosahedron group $\mathbb{I}$ isomorphic to the alternate group $\mathfrak{A}_5$. Some modern techniques allow us to classify all finite subgroups of $\mathrm{Cr}_K(1)$ for an arbitrary field K [1].

The extension of Klein–Schwarz classification to the rank 2 Cremona group $\mathrm{Cr}_{\mathbb{C}}(2)$ began at the end of the 19th century in the works of Seligmann Kantor [15] and Anders Wiman [21]. A conceptual modern explanation of their work led Yuri Manin to develop the theory of minimal models of rational G-surfaces (X, G), projective algebraic surfaces X together with an action of a finite group G. It turns out that any finite subgroup of $\mathrm{Cr}_{\mathbb{C}}(2)$ can be realized as a finite group of automorphisms of a G-minimal del Pezzo surface or a G-minimal conic bundle. The complete classification of all possible groups G (up to conjugation) that arise in this way was achieved in this century in the work Jéremy Blanc [2] (for abelian groups) and in the work of the reviewer and Vassily Iskovskikh [11] for arbitrary finite groups (with some gaps that have been later fixed by various mathematicians).

The classification of infinite subgroups of the Cremona groups is the subject of modern research. The Cremona group $\mathrm{Cr}_{\mathbb{C}}(n)$ can be made into a topological group in different ways [4]. Although one can define its Lie algebra, which is a simple infinite-dimensional Lie algebra, no structure of an infinite-dimensional Lie or algebraic group can be put on it. However one may ask which algebraic groups besides the obvious one, $\mathrm{PGL}(n+1, \mathbb{C})$ and its subgroups, it may contain.

The first known work on algebraic subgroups of $\mathrm{Cr}_{\mathbb{C}}(n)$ is the 1893 work of Federigo Enriques [12] that classifies all maximal connected algebraic groups contained in $\mathrm{Cr}_{\mathbb{C}}(2)$. All such groups are of rank 2, i.e., they contain a two-dimensional torus, and the classification of such subgroups is equivalent to Segre's classification of minimal rational ruled surfaces. It was shown by Michele Demazure that any maximal connected algebraic subgroup of $\mathrm{Cr}_{\mathbb{C}}(n)$ is of rank $\leq n$ [8]. His work contains a classification of such subgroups of maximal rank n by using the so-called Enriques systems that led to the first time appearance of the concept of a torus embedding [16]. In a joint paper with Gino Fano [13], Enriques extended his work to algebraic subgroups of $\mathrm{Cr}_{\mathbb{C}}(3)$, and they proved that any algebraic subgroup of $\mathrm{Cr}_{\mathbb{C}}(3)$ extends to a group of automorphisms of a rational projective threefold. The complete classification of algebraic subgroups of $\mathrm{Cr}_{\mathbb{C}}(n), n > 2$, is known only in the case $n = 3$ [20].

It is natural to ask which discrete subgroups of Lie groups can be embedded in $\mathrm{Cr}_{\mathbb{C}}(n)$. Let Γ be a lattice (a discrete subgroup of finite covolume) in a connected simple Lie group G of real rank $r \geq 2$. Robert Zimmer and others conjectured that, if Γ acts by diffeomorphisms on a compact n-manifold and the kernel of the action is finite, then the rank of Γ is bounded by n. For example, according to a result of Étienne Ghys (improved by the author of the book under review [9]), a subgroup Γ of finite index of $\mathrm{SL}(n, \mathbb{Z})$ cannot be embedded in the group of diffeomorphisms of the two-dimensional sphere S^2 if $n \geq 4$. In a striking analogy of this result, the same is true for an embedding of Γ in $\mathrm{Cr}_{\mathbb{C}}(2)$ [10].

Although the Cremona group $\mathrm{Cr}_{\mathbb{C}}(2)$ is not a linear group in the sense that it does not admit faithful finite-dimensional linear representations, it admits a natural infinite-dimensional representation in an infinite-dimensional hyperbolic space $\mathbb{H}^{\infty}$ introduced for the first time by Yuri Manin. It is defined as a certain inductive limit of the cohomology spaces of all blowups of $\mathbb{CP}^2$. The study of this action is the main tool for proving many algebraic properties of the Cremona group of rank 2. For example, one may ask whether the group is simple, perfect, Hopfian (every surjective homomorphism is bijective), or obeys the Tits alternative (every finitely generated subgroup either contains a solvable subgroup of finite index or contains a non-abelian free group). Note that examples of groups for which the Tits alternative holds are linear groups, the mapping class groups, or hyperbolic groups in the sense of Gromov. As we have already mentioned, the group $\mathrm{Cr}_{\mathbb{C}}(2)$ is not simple; if one takes a quadratic transformation at random, the smallest normal subgroup that contains some power of it is a proper subgroup. The group is perfect, Hopfian [7], and satisfies the Tits alternative [6].

Iterations of a Cremona transformation ϕ give examples of discrete complex dynamical systems. For example, one can define the notion of the dynamical degree $\lambda(\phi) = \lim_{n\to\infty} \mathrm{alg.deg}(\phi^n)^{1/n}$ for a plane Cremona transformation ϕ. Like Möbius transformations, Cremona transformations satisfy a trichotomy: $\lambda(\phi) = 0, 1$, or larger than 1. The first category consists of transformations of finite order, and the second one consists of transformations that preserve a pencil of rational or elliptic curves. The last category is the most interesting and represents transformations of positive entropy $\log\lambda(\phi)$. The number $\lambda(\phi)$ is a Pisot or Salem number; in the latter case ϕ can be extended to a regular automorphism of a rational surface. It is an interesting geometric problem of realizing Salem numbers by an automorphism of a given rational algebraic surface [18].

As the reader has probably already noticed, most of the discussion so far concerns the study of the complex plane Cremona groups $\mathrm{Cr}_{\mathbb{C}}(2)$. The book contains very little information about other cases. However, it is worth mentioning the recent progress in other cases when $K \neq \mathbb{C}$ or $n > 2$. For example, the real Cremona group $\mathrm{Cr}_{\mathbb{R}}(2)$ is very interesting since some of its subgroups may act without indeterminacy points and hence define subgroups of diffeomorphisms of the real projective space $\mathbb{RP}^n$. Over a finite field $K = \mathbb{F}_q$, it acts on a finite set $\mathbb{P}^n(\mathbb{F}_q)$ and provides interesting subgroups of the symmetric groups. Susanna Zimmermann proved that the real Cremona group $\mathrm{Cr}_{\mathbb{R}}(2)$ is neither simple nor perfect [22], and together with Stéphane Lamy [17], they proved that $\mathrm{Cr}_K(2)$ is not simple for any field K that admits an algebraic extension of degree 8. Many different presentations of $\mathrm{Cr}_K(2)$ by generators and relations are known for all algebraically closed fields K and, somewhat non-explicitly, for any perfect field K. The group $\mathrm{Cr}_K(n)$ is still very mysterious for $n \geq 3$. It was proven by Hilda Hudson (the author gives a wrong reference to her book [14]) that $\mathrm{Cr}_{\mathbb{C}}(3)$ cannot be generated by the subgroup $\mathrm{PGL}(4,\mathbb{C})$ and a finite set of non-projective transformations. Recently, it was proved that the group $\mathrm{Cr}_K(3)$ is not simple, and is not perfect for any subfield K of $\mathbb{C}$ [3]. No analogues of the hyperbolic space $\mathbb{H}^\infty$ where this group can be represented are known, so no methods of hyperbolic geometry can be applied.

The classification of finite subgroups of Cremona groups $\mathrm{Cr}_{\mathbb{C}}(n)$ of higher rank n is still in a rudimentary state. Although there is a theory of minimal G-models of Fano varieties, which are higher-dimensional analogues of del Pezzo surfaces, many of them are not rational, and hence G is not a subgroup of the Cremona group. However, in this way, one can classify, for example, finite simple groups contained in $\mathrm{Cr}_{\mathbb{C}}(3)$ [19].

An important subgroup of $\mathrm{Cr}_{\mathbb{C}}(n)$ is the group of polynomial automorphisms of the affine space $\mathbb{C}^n$, or, equivalently, the group of automorphisms of the polynomial algebra $K[t_1,\ldots,t_n]$. We still have no ideas about possible sets of suitable generators of this group for $n \geq 3$.

Almost everything discussed here, and much more, is elaborated with more details and often with proofs in the book under review. I believe that mathematicians working in almost any field of mathematics may find some of the material from this book relevant to their research. Although it must be warned that the book is not for easy reading—it requires a broad mathematical culture to appreciate the beauty of its subject. It will serve as a very good source of references to some exciting recent progress in the study of the group of Cremona transformations.

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