BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 59, Number 4, October 2022, Pages 623–627
https://doi.org/10.1090/bull/1780
Article electronically published on August 16, 2022

Continuous semigroups of holomorphic self-maps of the unit disc, by Filippo Bracci, Manuel D. Contreras, and Santiago Díaz-Madrigal, Springer Monographs in Mathematics, Springer, Cham, xxvii+566 pp., ISBN 978-3-030-36781-7

This book is an excellent exposition of the updated achievements in complex geometric function theory, hyperbolic geometry, and related topics. It is an updated self-contained manuscript on the theory of iterates and continuous semigroups which can be used as a reference source for researchers and as an introductory book for graduate students in complex analysis and iteration theory including detailed materials developing Carathéodory prime ends topology and hyperbolicity theory in simply connected domains.

In order to make this review self-contained and for the reader's convenience, we supplement it with some historical remarks and emphasize the relevant references as well.

In recent years important progress has been made in the study of semigroups of linear and nonlinear operators from the viewpoint of the approximation theory. These advances have primarily been achieved by introducing the theory of intermediate spaces. The applications of the theory not only permit integration of a series of diverse questions from many domains of mathematical analysis but also lead to significant new results on classical approximation theory, on the initial and boundary behavior of solutions of partial differential equations (PDEs), and on the theory of singular integrals. It seems that historically a motivation for the study of semigroups of linear and nonlinear operators goes back to the study the equation of heat-conduction for an infinite rod

$$\frac{\partial \omega}{\partial t} = \frac{\partial^2 \omega}{\partial x^2} \qquad (-\infty < x < \infty;\ t > 0) \tag{1}$$

with the given initial temperature distribution

$$\omega(x,0) = p(x). \tag{2}$$

Here $p(x)$ belongs to the space of all bounded, uniformly continuous real-valued functions defined on the real axis.

Denoting a "solution" $\omega(x,t)$ of (1) associated with the given initial value $f(x)$ by $\omega(x,t;f)$, it is essential for the study of existence of a unique solution in the sense for each $t > 0$ and that

$$\lim_{t \to 0_+} \omega(x,t;p) = p(x) \tag{3}$$

uniformly with respect to x.

The above problem is sometimes referred to as *Cauchy's problem* for the particular instance of the heat-conduction equation.

For earlier literature related to the subject of the book under review, among many others, we would like to mention the book of Kazimierz Goebel and Simeon Reich, *Uniform convexity, hyperbolic geometry and nonexpensive mappings*, Marcel Dekker, New York and Basel, 1982, as well as the book of Ian Graham and Gabriela Kohr, *Geometric function theory in one and higher dimensions*, Marcel Dekker, Inc, New York–Basel (2003). Also the paper of Leandro Arosio, Filippo Bracci, Hidetaka Hamada, and Gabriela Kohr, *An abstract approach to Loewner chains*, Journal d'Analyse Mathématique, **119** (2013), 89–114, might be a good introduction to the applications of complex dynamic systems to geometric function theory in their modern context.

In a parallel way, Filippo Bracci, Manuel D. Contreras, Santiago Diaz-Madrigal, and Hidetaka Hamada [7], [6], [9], [8], and [4] have developed geometrical aspects of this theory. They have produced very interesting questions and problems (as well as their solutions) which in our opinion will give a push for further investment the semigroup theory to general complex analysis. In particular, Leandro Arosio, Filippo Bracci, Hidetaka Hamada, and Gabriela Kohr [4] have presented a new geometric construction of Loewner chains in one and several complex variables which holds on a complete hyperbolic complex manifold, and they proved that there is essentially a one-to-one correspondence between evolution families of order d and Loewner chains of the same order. As a consequence they obtained a solution for any Loewner–Kufarev PDE, given by univalent maps.

Returning to the particular content of the book under review, we denote by $\mathrm{Hol}(D,\mathbb{C})$ the family of all holomorphic maps on a domain D in the complex plane $\mathbb{C}$ with values in $\mathbb{C}$. For the special case when $F \in \mathrm{Hol}(D,\mathbb{C})$ is a self-mapping of D, we simply write $F \in \mathrm{Hol}\,(D)$. We recall that a map $f \in \mathrm{Hol}(D,\mathbb{C})$, is said to be a semicomplete vector field on D if the Cauchy problem

$$\begin{cases} \frac{\partial u(t,z)}{\partial t} = f(u(t,z)) \\ u(0,z) = z \end{cases} \tag{4}$$

has a unique solution $u = u(t,z) \in D$ for all $z \in D$ and $t \geq 0$.

Furthermore, one can show (see [12] and [13]) that the function $u(t,z)$ satisfies the following PDE,

$$\frac{\partial u(t,z)}{\partial t} = \frac{\partial u(t,z)}{\partial z} f(z), \quad z \in D.$$

A mapping f is a *complete* vector field if the solution of (4) exists for all $t \in \mathbb{R}$ and $z \in D$. In other words, f is complete if both f and $-f$ are semicomplete.

Note also that f is complete if and only if this solution $\{u(t,\cdot)\}$ of (4) is a group (with respect to the parameter $t \in (-\infty,\infty)$) of automorphisms of D.

The set of semicomplete vector fields on D is denoted by $\mathcal{G}(D)$. Respectively, the set of complete vector fields is denoted by $\mathcal{G}_{\mathrm{aut}}(D)$.

Various criteria and presentations of semicomplete and complete vector fields on the open unit disk D and half-planes can be found in the book under review.

For the open unit ball $\mathcal{B}$ in $\mathbb{C}^n$ and general Hilbert and Banach spaces, see also [1], [3], [2], [13], [12], [10], and [11].

It is well known that in the case where $D = \{z \in \mathbb{C} : |z| < 1\}$, a semicomplete vector field is complete if and only if it admits the representation

$$f\,(z) = a - \overline{a}z^2 + ibz$$

for some complex number a and real b (see, for example, [13] and [10]).

If a family $\{F_t = u(t,\cdot)\}$, $t \geq 0$, $(t \in \mathbb{R})$ forms a semigroup (group) of holomorphic self-maps of the open unit disk D, it follows from the remarkable result of E. Berkson and H. Porta [5] that the limit

$$f\,(z) = \lim_{t\to 0^+} \frac{1}{t}\,(F_t\,(z) - z)$$

exists, and defines a semicomplete (complete) vector field f on D. Clearly, $f \in \mathrm{Hol}(D, \mathbb{C})$ and determines the so-called holomorphic generator of $\{F_t\}$ via the above formula.

In general, if D is a convex domain in $\mathbb{C}$ and the latter limit exists one can identify the set $\mathcal{G}(D)$ of semicomplete vector fields with the set of all holomorphic generators on D. The set $\mathcal{G}(D)$ is a real cone in $\mathrm{Hol}(D, \mathbb{C})$, while the set $\mathcal{G}_{\mathrm{aut}}(D)$ of all group generators on D is a real Banach algebra (see [3] and [12]).

Observe that the authors of the book under review introduce the primary subject of their study: continuous one-parameter semigroups of holomorphic self-maps of the unit disc. They establish the main basic properties of one-parameter semigroups and extend to this context the Denjoy–Wolff theory. Then they characterize groups of automorphisms and more generally of linear fractional self-maps of the unit disc. They also consider continuous semigroups of holomorphic self-maps of C and C_∞, proving that they reduce to groups of Möbius transformations, and they explain why a nontrivial theory of continuous semigroups of holomorphic maps only makes sense for self-maps of the unit disc.

In particular, in the first chapter the authors introduce some basic tools which are necessary for their study. They start recalling the concept of Riemann surfaces, and they focus mainly on the geometry of the unit disc, the complex plane, and the Riemann sphere. Next, from the Schwarz–Pick lemma, they define the hyperbolic metric and hyperbolic distance of the unit disc and extend these concepts to Riemann surfaces. Further the authors switch our attention to the analytic and dynamic properties of holomorphic self-maps of the unit disc. They introduce the notion of horocycles, Stolz's regions, and angular derivatives, and they prove the classical version of the Lindelöf theorem, Julia's lemma, and the Julia–Wolff–Carathéodory theorem, which are proper essential tools in the study of discrete type (iterations) and continuous type semigroups of holomorphic self-maps on the disk.

Observe also, that the Julia–Wolff–Carathéodory theorem can be seen a boundary version of the Schwarz–Pick lemma, which almost automatically brings us to hyperbolic geometry on the disk.

We note in passing that Julia was particularly interested in the case where

$$|f(z)| < |z|,$$

which occurs when $f(0) = 0$, hence $z = 0$ is an attractive fixed point.

Julia also stated a corollary:

> *If $|f(z)| \leq 1$ and ζ is a nonzero fixed point of f in the open unit disk D in C, then*
>
> $$\left|\frac{f(z) - \zeta}{f(z) - \xi}\right| \leq \left|\frac{z - \zeta}{z - \xi}\right|,$$
>
> *where ξ is symmetric to ζ with respect to the unit circle. Equality occurs if and only if*
>
> $$\frac{f(z) - \zeta}{f(z) - \xi} = \exp{(i\vartheta)}\,\frac{z - \zeta}{z - \xi}.$$

Julia followed this with what he called a "New extension of the Schwarz Lemma" (1918).

With those tools, the authors of the book under review consider iteration theory for holomorphic self-maps of the unit disc and prove the Denjoy–Wolff theorem, which says that,

> *except trivial cases, the orbits of a holomorphic self-map of the unit disc converge to the same point on the closed unit disc.*

They discuss also boundary fixed points (and, more generally, boundary contact points) of holomorphic self-maps of the unit disc when no continuous extension to the boundary is assumed.

The aim of this book is probably to present a systematic treatment of semigroups of bounded holomorphic maps on a domain in the complex plane and their connections with approximation theoretical questions in a more classical setting as well as within the setting of the theory of intermediate spaces.

A number of perfect books on various aspects of the latter theory appeared more than half a century ago, for example, Davis (1963), Sard (1963), Meinardus (1964), Rice (1964), Cheney (1966), and Lorents (1966). By contrast, the book under review is primarily concerned with those aspects of semigroup theory that are connected in some way or other with dynamic systems and complex geometry. Special emphasis is placed upon the significance of the relationships between the abstract theory and its various applications.

The book under review is appropriate for graduate students as well as for research mathematicians. It can be read and used by one who is familiar with real variable theory and the elements of complex and functional analysis. To make the exposition self-contained, these foundations are collected in almost all the chapters of the book. Furthermore, certain efforts have been made to make the presentation and proofs of the theorems as clear and detailed as practicable, so that the book will, in fact, be accessible to the student reader.

An essential part of the material is considered here for the first time outside of technical papers, and about half is based upon recent research. Each chapter starts with an introduction and contains detailed sections including references and historical remarks appropriate to the principal results treated, as well as information on important topics related to, but not included among, those given in the body of the text. In this way the book may furnish additional information for research mathematicians.

Finally, we would like to concentrate on the classical notion of the Koenigs function which plays a crucial role in the last chapter of the book in order to construct the models for semigroups in the open unit disk or, more generally, the simply connected Riemann surface. Starting from a given semigroup, the basic idea is to define an abstract space (the space of orbits of a semigroup or abstract basin of attraction) which inherits a complex structure in such a way that the semigroup is conjugated to a continuous group of automorphisms of such a Riemann surface. Moreover, construction implies that all possible (semi-) conjugations of the semigroup factorize through the model. Also, the model respects basic properties of the semigroup, in particular, the divergence rate, which is a measure in the hyperbolic distance of the rate of convergence to the Denjoy–Wolff point. Defining and discussing the concept of models and semimodels, the authors concentrate on studying the canonical models via the Koenigs function of a semigroup, which intertwines the semigroup to a simple group of automorphisms of either the complex plane, a half-plane, or a strip, depending on the properties of the starting semigroup. Those Koenigs functions are univalent functions which are either spiral-like or star-like at infinity. The authors introduce in detail the models for semigroups of linear fractional maps and noncanonical semimodels with the base hyperbolic space through conjugation operation. They show how to translate dynamical properties of a semigroup into geometrical properties of the image of the associated Koenigs function. The final chapter ends by considering topological models for semigroups, showing that from a topological point of view, there are only three possible models: the model of hyperbolic rotations, the model of elliptic semigroups (which are not groups), and the model of hyperbolic semigroups.

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David Shoikhet

Holon Institute of Technology

Holon & Braude Academic College

Karmiel, Israel

Email address: `davs@braude.ac.il`