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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
IMRE BARANY AND GIL KALAI

MR0109315 (22 #201) 52.00

Birch, B. J.

On $3N$ points in a plane.

Proceedings of the Cambridge Philosophical Society **55** (1959), 289–293.

The following theorem is proved in this note. Theorem 1: Given $3N$ points in a plane, we can divide them into N triads such that, when we form a triangle with the points of each triad the N triangles will all have a common point.

The proof is given on the basis of three lemmas and two corollaries. The first two lemmas are the fixed-point theorem for n-space and Caratheodory's $(n+1)$-point theorem. Lemma 3 is as follows, where E^n is the unit n-ball: Let a mass distribution in E^n be defined by an integrable density function $\rho(x)$; then we can find a point r inside E^n so that every closed half-space with r on its boundary will contain at least $1/(n+1)$ of the total mass. The first corollary states that Lemma 3 holds if the mass-distribution is not continuous, and the second corollary is as follows: Let Y be a finite set consisting of M points in n-space, and suppose that $M > (n+1)R$. Then there is a point common to all the closed half-spaces which contain at least $(M-R)$ points of Y.

{The author states that he believes Lemma 3 is new. Actually the most important new feature concerns the number and dispositions of closed half-spaces containing an "optimal" portion of the mass stated in the proof. For the Lemma 3 itself, it may be interpreted as a corollary of the finite point-mass problem which is a straight-forward generalization to n-space of a theorem stated and proved in Jaglom and Boltjanski [*Konvexe Figuren*, VEB Deutscher Verlag, Berlin, 1955; MR0079789; p. 16], where they also observe that the continuous case is special. To prove Theorem 1, then, the author could have applied his proof directly to the theorem stated by Jaglom and Boltjanski. However, the procedure used and the application of the method to other problems of optimization are of further interest.}

P. C. Hammer

From MathSciNet, August 2022

MR0458437 (56 #16640) 57C15; 13H10, 52A25

Stanley, Richard P.

The upper bound conjecture and Cohen-Macaulay rings.

Studies in Applied Mathematics **54** (1975), *no.* 2, 135–142.

Let Δ be a triangulation of the $(d-1)$-sphere. If Δ has f_j j-dimensional faces $(j=-1,0,\cdots,d-1; f_{-1}=1)$, write $h_i = \sum_{j=-1}^{i-1}(-1)^{i-j-1}\binom{d-j-1}{d-i} f_j$ $(i=0,1,\cdots,d)$. Further, if h and i are positive integers, h can be written uniquely in the form $h = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots + \binom{n_j}{j}$, where $n_i > n_{i-1} > \cdots > n_j \geq j$. Let

$h^{\langle i\rangle} = \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \cdots + \binom{n_j+1}{j+1}$, and $0^{\langle i\rangle} = 0$. The author's main result is then: $0 \le h_{i+1} \le h_i{}^{\langle i\rangle}$, for each $i = 0, 1, \cdots, d-1$. This is proved by using a characterization of Cohen-Macaulay rings of G. A. Reisner ["Cohen-Macaulay quotients of polynomial rings," Ph.D. Thesis, Univ. Minnesota, Minneapolis, 1974], and showing that $A_\Delta R/I$ is a Cohen-Macaulay ring, when $R = \mathbf{Q}[v_1, \cdots, v_n]$, ($v_1, \cdots, v_n$ being the vertices of Δ), and I is the homogeneous ideal generated by the square-free monomials $v_{i(1)} \cdots v_{i(s)}$, where $\{v_{i(1)}, \cdots, v_{i(s)}\} \in \Delta$. The theorem has a similar form to and is probably a useful step towards a conjecture of the reviewer [Israel J. Math. **9** (1971), 559–570; MR0278183], which would characterize all possible f-vectors $(f_0, f_1, \cdots, f_{d-1})$ of simplicial d-polytopes, and, conceivably, also of all triangulations of $(d-1)$-spheres.

P. McMullen
From MathSciNet, August 2022

MR0676720 (84c:52014) 52A35
Bárány, Imre
A generalization of Carathéodory's theorem.
Discrete Mathematics **40** (1982), *no.* 2–3, 141–152.

A well-known theorem of Carathéodory states that, given a set $V \subseteq \mathbf{R}^n$ with $p \in \operatorname{conv} V$, there is subset A of V consisting of $n+1$ or fewer points with $p \in \operatorname{conv} A$. Applications and generalizations appear in Helly's theorem and its relatives [L. W. Danzer et al., *Proceedings of Symposia in Pure Mathematics, VII*, 101–180, Amer. Math. Soc., Providence, R.I., 1963; MR0157289]].

In this paper, the author presents the following interesting generalization of Carathéodory's theorem: If $V_1, \cdots, V_{n+1} \subseteq \mathbf{R}^n$ and $p \in \bigcap \{\operatorname{conv} V_i \colon 1 \le i \le n+1\}$, then there exist elements $v_i \in V_i$, $1 \le i \le n+1$, such that $p \in \operatorname{conv}\{v_1, \cdots, v_{n+1}\}$. Furthermore, the author gives several related results, including this Helly-type theorem: Let $\mathcal{C}_1, \cdots, \mathcal{C}_m$, $m > 1$, be nonempty families of nonempty sets in $\mathbf{R}^n$. Assume that the sets in $\mathcal{C}_1$ and $\mathcal{C}_2$ are compact. If for every choice $C_i \in \mathcal{C}_i$, $1 \le i \le m$, the union $\bigcup\{C_i \colon 1 \le i \le m\}$ is convex, then for some i the intersection $\bigcap \mathcal{C}_i$ is nonempty. Finally, applications are made to many areas, including trees, simple polytopes, and convex functions.

Marilyn Breen
From MathSciNet, August 2022

MR1234493 52A01; 05C12, 06A06, 52-02, 54H12
van de Vel, M. L. J.
Theory of convex structures. (English)
North-Holland Mathematical Library, 50.
North-Holland Publishing Co., *Amsterdam*, 1993, xvi+540 pp., \$157.25,
ISBN 0-444-81505-8

The author has done a wonderful job. He shows how pervasive the idea of convexity is, by collecting notions and results from a large set of mathematical disciplines and by putting all of these in a common perspective. We find here an introduction to convexity aspects of fundamental structures such as median algebras, convexity spaces, ordered sets, normed vector spaces, superextensions, semilattices, matroids, metric spaces, trees, and more. A huge number of results are given together with a rich bibliography, some of them being sketched in paragraphs entitled "Further topics".

Chapter 1 introduces the general concept of a convex structure, i.e. a set provided with a collection of subsets stable for intersections and nested unions. The resulting hull operator is considered, as well as the notion of segment. Although all of these concepts are fairly general, the findings described are very interesting because they show the specificity of the various examples to which they apply and also the interrelationships among these examples.

Investigation of convex structures can be purely combinatorial or have a topological flavour. The central part of the book is accordingly divided into two chapters. Chapter 2 is mainly devoted to invariants inspired from classical theorems (associated with the names of Radon, Helly, Tverberg, ...), but also has a section on infinite combinatorics. It ends with a report on the famous Eckhoff conjecture.

Topological convex structures are obtained by building, or simply assuming, a topology as an additional ingredient. In this framework, continuity of the hull operator or further separation theorems can be studied. Again in this Chapter 3, a wealth of applications in specific fields is given.

Miscellaneous topics are collected in the final chapter, e.g., affine representation of convexity spaces, selection theorems, dimension theory in connection with invariants, fixed point theorems.

My appreciation of this dense handbook is very high. Many of the accomplishments presented are due to the author and/or his close collaborators. Moreover, the book will serve as a road map to (mostly recent) developments in various branches of mathematics.

Jean-Paul Doignon
From MathSciNet, August 2022

MR1921545 (2003g:52004) 52A35; 05D15

Alon, Noga; Kalai, Gil

Transversal numbers for hypergraphs arising in geometry.

Advances in Applied Mathematics **29** (2002), *no.* 1, 79–101.

A set system (hypergraph) $\mathcal{F}$ has the (p, q) property if among any p sets of $\mathcal{F}$ some q have a common point. The (p, q) theorem conjectured by Hadwiger and Debrunner and proved by N. Alon and D. J. Kleitman [Adv. Math. **96** (1992), no. 1, 103–112; MR1185788] extends Helly's theorem on convex sets as follows: For $p \geq q \geq d+1$, if $\mathcal{F}$ is a finite family of convex sets in $\mathbf{R}^d$ and $\mathcal{F}$ has the (p, q) property, then its transversal number is bounded by a function of p, q, and d.

The authors consider analogues and relatives of the (p, q) theorem for other topological and abstract hypergraph classes. The basic tools of the original proof of the (p, q) theorem are the fractional Helly theorem, the fractional transversal number, and weak epsilon-nets for convex sets. The main result of the paper shows that an appropriate fractional Helly property implies the validity of an abstract (p, q) theorem. Consequences of the general result include a topological $(p, d+1)$ theorem involving d-Leray complexes, and a $(p, 2^d)$ theorem for convex lattice sets in $\mathbf{Z}^d$.

The present work can be regarded as a contribution towards understanding of the following questions: For which classes of hypergraphs is the fractional transversal number, $\tau^*(\mathcal{F})$, bounded by some (p, q) property of $\mathcal{F}$ uniformly for all hypergraphs $\mathcal{F}$ in the class; and for which classes $\mathbf{F}$ is the transversal number, $\tau(\mathcal{F})$, bounded by some function of $\tau^*(\mathcal{F})$ uniformly for all $\mathcal{F} \in \mathbf{F}$?

Jenő Lehel
From MathSciNet, August 2022

MR3974609 52A35; 57M99, 90C25, 91A80, 91B32

De Loera, Jesús A.; Goaoc, Xavier; Meunier, Frédéric

The discrete yet ubiquitous theorems of Carathéodory, Helly, Sperner, Tucker, and Tverberg.

American Mathematical Society. Bulletin. New Series **56** (2019), *no.* 3, 415–511.

This paper surveys the theory and applications of the five fundamental theorems of discrete geometry mentioned in the title. In the first part, the authors present some of the many reformulations and variations of these theorems and explore how these results fit together. The second part of the paper is devoted to the multiple applications of the five theorems. The authors work on wide areas and examine examples from game theory and fair division, from graph theory, from optimization, and from geometric data analysis. Some of the examples given are classical (e.g., Nash equilibria, von Neumann's min-max theorem, linear programming), and others are more specialized (e.g., Dol'nikov's colorability defect or the polynomial partitioning technique), but for all these, the five theorems provide elegant and simple proofs. For other examples (for instance for Meshulam's lemma, or for the ham sandwich theorem) the authors present new proofs. The paper is well written, supplying ample background information, and interesting open problems accompany the presentation.

Mircea Balaj
From MathSciNet, August 2022

MR4287348 52A35; 52A40, 52C07

Dillon, Travis; Soberón, Pablo

A mélange of diameter Helly-type theorems.

SIAM Journal on Discrete Mathematics **35** (2021), *no.* 3, 1615–1627.

The first theorem is a combined fractional and colorful version, actually a relaxation, of a long standing quantitative Helly-type conjecture for diameter due to I. Bárány, M. Katchalski and J. Pach [Proc. Amer. Math. Soc. **86** (1982), no. 1, 109–114; MR0663877].

It is proved that there exists a decreasing function $\gamma\ (0, \sqrt{2}) \to (0, 1]$ such that $\gamma(c) \to 1$ as $c \to 0$ and the following holds for every $c \in (0, \sqrt{2})$, $\alpha \in (0, 1]$, and $d \geq 2$: Let $\beta = 1 - (1 - \alpha \cdot \gamma(c))^{1/2d}$; assume that $\mathcal{F}_1, \ldots, \mathcal{F}_{2d}$ are finite families of convex sets in $\mathbb{R}^d$ and set $N = \prod_{i=1}^{2d} |\mathcal{F}_i|$; if $\bigcap_{i=1}^{2d} F_i$ has diameter greater than or equal to 1 for at least αN different $2d$-tuples $(F_1, \ldots, F_{2d})$, where $F_i \in \mathcal{F}_i$, $i = 1, \ldots, 2d$, then for some $k \in \{1, \ldots, 2d\}$, there exists a subfamily $\mathcal{G} \subset \mathcal{F}_k$, $|\mathcal{G}| \geq \beta |\mathcal{F}_k|$, such that the diameter of the set $\bigcap_{F \in \mathcal{G}} F$ is greater than or equal to $cd^{-1/2}$.

A convex set has large 'discrete diameter' if it contains many colinear integer points. As an extension of the Doignon-Bell-Scarf theorem [J.-P. Doignon, J. Geom.

3 (1973), 71–85; MR0387090] an exact Helly-type theorem for discrete diameter is proved here as follows. Let k be a positive integer and let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$; if the intersection of every 4^d or fewer elements of $\mathcal{F}$ contains k colinear integer points, then the set $\bigcap_{F\in\mathcal{F}} F$ contains k colinear integer points.

Further quantitative Helly-type results are obtained when particular Minkowski norms measure the diameter. Three proofs are presented for the next theorem, one of which implies a colorful version as well. Let ρ be a norm in $\mathbb{R}^d$ whose unit ball is a polytope with k facets, and let $\mathcal{F}$ be a finite family of convex sets in $\mathbb{R}^d$; if the intersection of every kd or fewer members of $\mathcal{F}$ has ρ-diameter greater than or equal to 1, then the set $\bigcap_{F\in\mathcal{F}} F$ has ρ-diameter greater than or equal to 1; moreover, this statement is not true if kd is replaced by $kd-1$.

Jenő Lehel

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