THE ALGEBRO-GEOMETRIC METHOD: SOLVING ALGEBRAIC DIFFERENTIAL EQUATIONS BY PARAMETRIZATIONS

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ABSTRACT. We present a survey of the algebro-geometric method for solving algebraic ordinary differential equations by means of parametrizations of the associated algebraic sets. In particular, we deal with equations of order one, and also systems of algebro-geometric dimension one. Various classes of solutions are treated symbolically, such as rational, algebraic, and power series solutions. We also consider classes of algebraic transformations of the associated algebraic sets preserving the solutions of the differential equations. Two Maple packages, implementing some of these solution methods, are presented.

CONTENTS

1. INTRODUCTION

A differential equation is a relation between a function, finitely many of its derivatives, and possibly the variable of differentiation. If the function depends on

more than one variable, the derivatives will be partial. Correspondingly, we speak of ordinary differential equations (ODEs) or partial differential equations (PDEs). In a system of differential equations we simultaneously consider more than one equation. The study and analysis of differential equations is a well-established branch in mathematics. One may distinguish, among others, some main directions of study in this field: the existence of solutions, the analysis of properties of solutions, the actual computation of solutions, and the applications. Moreover, approximate solutions can be computed by means of numerical methods, or exact expressions for solutions can be derived by symbolic computer algebra methods.

In this paper, we describe several methods for the exact computation of solutions of some classes of algebraic ordinary differential equations (AODEs), that is, equations where the relationships between the functions and their derivatives are given by polynomials. This approach we call the algebro-geometric method. The key idea will be to use information derived from the geometric object defined by these polynomials. In order to give a more precise motivation for the ideas and methods presented in this survey, we conduct this introduction with the help of some examples.

Let us consider the differential equation (see Example 2.6 for further details)

$$(1) \qquad\qquad 20\,y^3 + y^2 + 20\,yy' - 25\,y'^2 + y' = 0,$$

where $' = \frac{d}{dx}$ denotes the usual derivative with respect to the independent variable x. This differential equation is algebraic, since it is defined as $F(y, y') = 0$, where $F(u, v)$ is the polynomial $20\,u^3 + u^2 + 20\,uv - 25\,v^2 + v$. The idea now is to use the algebraic curve $\mathcal{C}_F$ defined by the polynomial $F(u, v)$; that is, $\mathcal{C}_F := \{(a, b) \in \mathbb{C}^2 \mid F(a, b) = 0\}$ is the zero-locus of F in $\mathbb{C}^2$ (see Figure 1 for a plot of the real part of the cubic curve $\mathcal{C}_F$). The irreducible curve $\mathcal{C}_F$ has genus zero and admits the rational parametrization

$$\mathcal{P} = (p_1(x), p_2(x)) := \left(\frac{(1 + 6\,x)\,x}{(x + 1)^2}, -\frac{(1 + 11\,x)\,x^2}{(x + 1)^3} \right).$$

We look for a rational function $T(x)$ such that the reparametrization $\mathcal{P}(T(x))$ takes into account the differential aspect of the given equation, i.e., such that $\frac{d}{dx}(p_1(T(x))) = p_2(T(x))$. This is the case exactly for $T(x) = 1/x$. Since $\mathcal{P}(T(x))$ is also a rational parametrization of $\mathcal{C}_F$, we have that

$$F\left(p_1(T(x)), \frac{d}{dx}(p_1(T(x))) \right) = 0$$

and $p_1(T(x)) = (x + 6)/(x + 1)^2$ is a rational solution of the differential equation $F(y, y') = 0$. In Section 2.1 we explain that the general solution $p_1(T(x+c))$ covers all rational solutions of $F(y, y') = 0$.

Let us now consider the AODE

$$(2) \qquad\qquad 20y^3 + 2y^2 - 25y'^2 + y' = 0.$$

Note that (1) and (2) only differ in one term, namely $20yy'$. Since the algebraic curve $\mathcal{C}_F$ corresponding to the polynomial $F(u, v) = 20u^3 + 2u^2 - 25v^2 + v$ has genus one, there is no rational parametrization of $\mathcal{C}_F$. It will be shown that this also means $F(y, y') = 0$ does not have nonconstant rational solutions. Instead we

can work with local parametrizations. A Puiseux expansion of $F(u, v)$ at $u = 0$ is

$$v(u) = \frac{1}{25} + u^2 + 20u^3 - 25u^4 - 1000u^5 + \mathcal{O}(u^6).$$

The corresponding local parametrization $\mathcal{P} = (p_1(x), p_2(x)) = (x, v(x))$ describes the curve $\mathcal{C}_F$ around the curve-point $(0, 1/25)$. Again we look for a reparametrization such that $\frac{d}{dx}(p_1(T(x))) = p_2(T(x))$. Setting $T(0) = 0$, this is exactly the case for

$$T(x) = \frac{x}{25} + \frac{x^3}{1875} + \frac{x^4}{3125} + \mathcal{O}(x^5).$$

Then $p_1(T(x)) = T(x)$ defines a formal power series solution of the differential equation (2).

When trying to solve (2) by using the local parametrization

$$(q_1(x), q_2(x)) = (x, -x^2 - 20x^3 + 25x^4 + 1000x^5 + \mathcal{O}(x^6)),$$

one obtains that the above strategy cannot be followed. The reason is that the corresponding solution is a formal power series expanded around infinity. For this type of solution, and for formal power series with fractional exponents, some additional transformations have to be performed, as will be explained in the second part of Section 2.1.

Let us also remark that if, instead of having one autonomous first-order algebraic ODE, we have a system of autonomous AODEs of maximum order m, we can associate to it a system of algebraic equations that will define a zero-set in $\mathbb{C}^{m+1}$. Now, if this zero-set is a space curve, the previous ideas can be extended; this is explained in Section 3.

In examples (1) and (2), the differential equations were autonomous and of order one. This allowed us to associate a planar curve to them. When working with nonautonomous first-order AODEs $F(x, y, y') = 0$, we can either try to parametrize the surface $\mathcal{S}_F := \{(a, b, c) \in \mathbb{C}^3 \mid F(a, b, c) = 0\}$ or consider $F(x, u, v)$ as a polynomial in the variables $\{u, v\}$, and take the zero-set of F in the algebraic closure of the field $\mathbb{C}(x)$, i.e., using the curve $\{(a, b) \in \overline{\mathbb{C}(x)}^2 \mid F(x, a, b) = 0\}$. Although these associated geometric objects are more complicated, it is possible to develop a theory on the existence of rational general solutions of $F(x, y, y') = 0$ as we explain in Section 2.2.

Sometimes one can transform a given nonautonomous AODE into an autonomous AODE. Following the general idea of working with the algebraic sets associated to the AODEs, in Section 4 we describe the class of transformations that can be used for this purpose.

Historical Background. The problem of finding exact solutions of ODEs and also PDEs has been extensively studied in the literature. The huge majority of these methods, however, make implicit assumptions on the structure of the equations, such as that the system is in normal form where the well-known Cauchy–Kovalevskaya theorem can be applied. We focus here on methods where these assumptions are dropped and note that there are a lot of famous examples which are not of normal form, such as Navier–Stokes equations, Maxwell equations, and many others.

In Zwillinger [Zwi98, Section II.B] several different exact methods are proposed. In Hubert [Hub96] one can find a method for computing the implicit general and singular solutions of AODEs by means of Gröbner bases. Eremenko [Ere98] provides a degree bound for rational solutions of AODEs and hence a method for determining

them. In Ince [Inc26], linear differential equations and generic solutions of first-order AODEs are considered.

The treatment considered in this paper is the so-called *algebro-geometric approach*, which relies on the combination of three fields: differential algebra (see [Rit50] and [Kol73]), computer algebra (see [Win96] and [vzGG13]), and algebraic geometry (see [CLO05]). The main idea, introduced in [FG04] for autonomous first-order AODEs, is as follows: we associate to the given AODE, or the system of AODEs, an algebraic set. Now we have two problem levels: the differential level corresponding to the input AODEs, and the purely algebraic geometric level corresponding to the algebraic set. In this situation, the underling strategy is to analyze whether, and how, the properties and computations on the algebraic geometric level can be translated to the input differential equations. For a wide panoramic vision, we refer to the PhD theses [Ngô11], [Gra15], [Vo16], and [Fal20].

In this paper we focus on AODEs and some special systems of this type of equation. For the case of algebraic partial differential equations we refer, e.g., to Robertz [Rob14] or Grasegger et al. [GLSW18].

Structure. Section 2 is devoted to first-order AODEs. The stage is prepared with some basic definitions. Then we first focus on the autonomous case, dealing with the existence and actual computation of rational general solutions, algebraic general solutions, and finally formal power series solutions. Afterward the nonautonomous case is treated. Here we take two different approaches: the given AODE may be considered geometrically as a surface over the ground field, or it may be considered as a curve with coefficients in the field of rational functions in the independent variable. In a comparison of these approaches, we find that they essentially determine the same rational general solutions. Additionally, algebraic general solutions of nonautonomous AODEs are also treated. In Section 3 systems of autonomous AODEs of dimension one are studied. We distinguish first the case of rational solutions and later the case of formal power series solutions. In Section 4 we treat the problem of detecting those rational transformations, on the algebraic level, that preserve the differential information of the given AODE. This analysis is first performed for linear transformations and later for birational transformations. In Section 5 we briefly describe two Maple packages for dealing with some of the algorithmic ideas introduced in this paper. Finally, in Section 6, we summarize the algebro-geometric approach and we give an outlook to some future research topics. In the appendices, basic notions and results on formal power series (Appendix A) and local parametrizations (Appendix B) are recalled.

Notation. We finish this section by introducing the notation and terminology that will be used throughout this paper.

For every set A containing a zero-element, we use the notation $A^* = A \setminus \{0\}$. In particular, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \{1, 2, \ldots\}$.

Let R be an integral domain, and let $' : R \to R$ be a map which is additive, i.e., $(a+b)' = a' + b'$, and satisfies the product rule or Leibniz rule, i.e., $(ab)' = a'b + ab'$ for $a, b \in R$. Then R is called a differential ring, and in the case where it is a field, we speak of a differential field. We will work with differential rings and fields of characteristic zero. When we extend a differential ring or field algebraically by some α, the derivation extends uniquely according to the minimal polynomial. In a transcendental extension by t we are free to define the derivation of t.

The ring of differential polynomials in the variable y over the differential ring R, written $R\{y\}$, consists of the polynomials in $R[y, y_1, y_2, \ldots]$ in infinitely many variables, where $y_1 = y', y_2 = y''$, etc. For y_i we also write $y^{(i)}$.

We consider $\mathbb{K}$ as a field of constants (i.e., $a' = 0$ for $a \in \mathbb{K}$) and we extend the derivation to $\mathbb{K}[x]$ by letting $x' = 1$. An AODE over $\mathbb{K}$ is an equation of the form

$$F(x, y, y', \ldots, y^{(n)}) = 0,$$

where $F \in \mathbb{K}[x]\{y\} \setminus \mathbb{K}[x]$. By abuse of notation we often do not distinguish between the polynomial F and the associated differential equation $F = 0$. Additionally, in Section 3, we will deal with some particular systems of AODEs. We use the notation $\hat{y}$ to represent a solution of the differential equation.

When working with the geometric counterpart of the differential problem, we will denote by $\mathbb{A}^n(\overline{\mathbb{K}})$ the n-dimensional affine space over $\overline{\mathbb{K}}$, the algebraic closure of $\mathbb{K}$. Sometimes, for a set of polynomials $\mathcal{F}$, we will use the notation $\mathbb{V}(\mathcal{F})$ to represent the algebraic set defined by $\mathcal{F}$.

Let K be a field of characteristic zero, and let R be a differential ring. In Table 1 we introduce the basic algebraic structures with coefficients in K or R, respectively, an independent variable x and indeterminates $y_1, \ldots, y_\ell$. For more detail on formal power series we refer to Appendix A.

TABLE 1. Notation for basic algebraic structures.

$\overline{K}$	algebraic closure of K
$K[x]$	ring of polynomials
$K(x)$	field of rational functions
$K[[x]]$	ring of formal power series with exponents in $\mathbb{N}$
$K((x))$	field of formal Laurent series
$K\{\{x\}\}$	field of algebraic Puiseux series
$K\langle\langle x \rangle\rangle$	field of formal Puiseux series
$R\{y_1, \ldots, y_\ell\}$	ring of differential polynomials

2. AODEs of order one

Let $\mathbb{K}$ be a differential field of characteristic zero. We assume that $\mathbb{K}$ is equipped with a derivation, written as $'$, such that $'$ maps all elements of $\mathbb{K}$ to 0, i.e., $\mathbb{K}$ is its own field of constants. Over $\mathbb{K}$ we can consider various types of functions, such as rational or algebraic functions, and the derivation $'$ can be extended to such classes of functions.

Definition 2.1. Let $F(x, y, y', \ldots, y^{(n)})$ be a differential polynomial in $\mathbb{K}[x]\{y\} \setminus \mathbb{K}[x]$. This polynomial generates an AODE over $\mathbb{K}$ of the form

$$(3) \qquad\qquad F(x, y, y', \ldots, y^{(n)}) = 0.$$

If F does not explicitly depend on x, we have an *autonomous* AODE of the form

$$F(y, y', \ldots, y^{(n)}) = 0.$$

This differential equation is called the AODE *defined by* F, and F is called the *defining polynomial* of the differential equation (3). If the degree of F in $y^{(n)}$ is positive, we call n the *order* of the AODE. If F is a differential polynomial of

positive order n, then the *separant* of F, denoted by S_F, is the partial derivative of F with respect to (w.r.t.) $y^{(n)}$.

Observe that an algebraic equation is simply an AODE of order zero. When we substitute η for the variable y in F, we often simply write $F(\eta)$ instead of $F(x, \eta, \eta', \ldots, \eta^{(n)})$.

Definition 2.2. We extend the differential field $(\mathbb{K}, ')$ to $(\mathbb{K}(x), ')$, so that $x' = 1$. Let $\mathbb{L}$ be a differential extension field of $\mathbb{K}(x)$. An element $\hat{y} \in \mathbb{L}$ is a *zero* of F iff $F(\hat{y}) = 0$. We call a zero of F also a *solution* of the AODE (3) defined by F.

If F can be factored as

$$F = F_1 \cdot \ldots \cdot F_k$$

over $\overline{\mathbb{K}}$, then the set of solutions of F is simply the union of the sets of solutions of the factors F_i. So let us assume that the defining polynomial F is irreducible over $\overline{\mathbb{K}}$.

By $\{F\}$ we denote the radical differential ideal generated by F. According to Ritt [Rit50, Chapter II], this ideal can be decomposed as the intersection of two divisor ideals, namely the ideal of all differential polynomials G, with the property that $S_F G \in \{F\}$, and the radical differential ideal generated by F and S_F. In terms of the solutions this means that a solution is either a solution for which the separant does not vanish, or a solution for which the separant does vanish.

Definition 2.3. Let F be an irreducible differential polynomial in $\mathbb{K}[x]\{y\}$. Then

$$\{F\} = \underbrace{(\{F\} : S_F)}_{\text{general component}} \cap \underbrace{\{F, S_F\}}_{\text{singular component}}.$$

So $\{F\}$ can be decomposed into a so-called *general component* and a *singular component*. By $\{F\} : S_F$ we mean the quotient of the radical generated by F and the ideal generated by S_F.

According to Ritt, the general component is a prime differential ideal. So it has a generic zero $\hat{y}$ in some differential extension field, such that (s.t.) a differential polynomial G is in $\{F\} : S_F$ if and only if $G(\hat{y}) = 0$.

Definition 2.4. Let F and S_F be as above. A generic zero of the prime differential ideal $\{F\} : S_F$ is called a *general solution* of the AODE $F = 0$.

Ritt proves in [Rit50, Chapter II, Section 13] that a differential polynomial G is in the general component of $\{F\}$ if and only if G can be pseudo-reduced to 0 modulo F. So S_F cannot be in the general component, because the pseudo-remainder of S_F modulo F is S_F itself, which is not 0. In other words, if η is a general solution of $F = 0$, then $S_F(\eta) \neq 0$.

From the point of view of analysis (compare [Inc26]) we know that the family of solutions of a differential equation of order one depends on an arbitrary constant c. So a general solution of (3) must be an element of a transcendental extension field of $\mathbb{K}(x)$, expressible in terms of a transcendental element c, s.t. $c' = 0$. Such a c is called an *arbitrary constant*. For the case of rational general solutions of AODEs of order one, this is stated in [NW10, Lemma 3.12].

Example 2.5. Consider the AODE

$$F(x, y, y') = y'^2 + 3y' - 2y - 3x = 0.$$

A general solution of this AODE is $y = \frac{1}{2}((x+c)^2 + 3c)$, where c is an arbitrary constant. The separant of F is $S_F = 2y' + 3$ and the singular solution of F is $\hat{y} = -\frac{3}{2}x - \frac{9}{8}$.

So how can we determine a general solution of an AODE (3)? We can implicitly describe a generic zero of the prime differential ideal $\{F\} : S_F$ by the congruence class of y in the quotient ring $\mathbb{K}(x)\{y\} / (\{F\} : S_F)$, which is actually an integral domain. Or we could describe the solution by computing a Gröbner basis for $\{F\} : S_F$, as Hubert does in her paper [Hub96] for the case that $F = 0$ is of order one. But this is an implicit description of the general solution. In this section we want to describe ways for determining explicit solutions of first-order AODEs.

2.1. **Autonomous AODEs of order one.** We consider an autonomous AODE over $\mathbb{Q}$ of order one, i.e., a differential equation of the form

(4) $$F(y, y') = 0,$$

where $F \in \mathbb{Q}[y, y'] \setminus \mathbb{Q}[y]$ is absolutely irreducible. Once we fix a class of functions in which we want to find solutions, there may or may not be an algorithm for deciding the existence of solutions, and, in the positive case, determining the solutions. The starting point is the work of Feng and Gao [FG04, FG06] and generalizes either to a bigger class of differential equations or to more general functions where the solutions are sought.

Rational general solutions. This case is considered by R. Feng and X.-S. Gao in [FG04, FG06], where an algorithm is given for deciding the existence of a rational general solution and, in the positive case, for determining the rational general solution. We describe their approach to this situation.

If equation (4) has a nonconstant rational solution $\hat{y} \in \overline{\mathbb{Q}}(x)$, then $F(\hat{y}, \hat{y}') = 0$, and therefore $\mathcal{P} = (\hat{y}, \hat{y}')$ is a rational parametrization[1] of the corresponding algebraic curve $\mathcal{C}_F$ defined by $F(u, v) = 0$. For the theory of parametric curves we refer to [SWPD08]. Feng and Gao prove that $\overline{\mathbb{Q}}(\hat{y}, \hat{y}') = \overline{\mathbb{Q}}(x)$ (see [FG06, Theorem 3.7]), which implies (for instance by [SWPD08, Theorem 4.14]) that the parametrization $(\hat{y}, \hat{y}')$ is proper, i.e., it induces a birational map from $\mathbb{A}^1(\overline{\mathbb{Q}})$ onto the curve $\mathcal{C}_F$. And this implies that every other proper parametrization $\tilde{\mathcal{P}}$ can be expressed as $\tilde{\mathcal{P}} = \mathcal{P}(T)$, where T is a Möbius transformation, i.e., a linear birational map of the form $(ax+b)/(cx+d)$ with $ad - bc \neq 0$. Moreover, if (4) has a rational solution, then such a solution can be found with coefficients in $\mathbb{Q}$ (see [FG04, Theorem 6]).

So we have an algorithm for deciding whether (4) has a rational solution:

(a) Compute a proper rational parametrization $\mathcal{P} = (p_1(x), p_2(x))$ of $\mathcal{C}_F$ over $\mathbb{Q}$, e.g., by the algorithm in [SW97]. If such a parametrization cannot be found, then there is no rational solution of the differential equation (4).

(b) Compute the coefficients of a Möbius transformation T s.t. $p_1(T)$ solves (4). The existence of such a transformation can be decided by a Gröbner basis

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[1]A pair of rational functions $\mathcal{P}$ is a rational parametrization of the irreducible curve $\mathcal{C}$ iff $\mathcal{P}$ induces a rational mapping from $\mathbb{A}^1(\overline{\mathbb{Q}})$ onto $\mathcal{C}$. $\mathcal{P}$ is proper iff the mapping is birational.

computation. If such a transformation cannot be found, then there is no rational solution of the differential equation (4).

(c) $\hat{y} = p_1(T)$ is a rational solution.

In fact, the existence of such a Möbius transformation as in (b) can be decided by very basic means. Let $r(x) = p_2(x)/p_1'(x)$. If r is a constant in $\mathbb{Q} \setminus \{0\}$, then the desired transformation is $T : x \mapsto rx$. If $r(x) = a(x - b)^2$ for $a, b \in \mathbb{Q}$ and $a \neq 0$, then the desired transformation is $T : x \mapsto (abx - 1)/ax$. Otherwise, such a Möbius transformation does not exist.

Once we have found a particular rational solution $\hat{y}(x)$, then $\hat{y}(x+c)$ is a rational general solution (see [FG06, Lemma 3.1]). So we have a decision algorithm for the existence of a rational general solution of an autonomous AODE over $\mathbb{Q}$ of order one.[2]

Example 2.6. We consider the differential equation $F(y, y') = 0$, where

$$F(y, y') = 20\, y^3 + y^2 + 20\, yy' - 25\, y'^2 + y'.$$

F defines a cubic curve in $\overline{\mathbb{Q}}^2$ of genus zero (see Figure 1), which can be properly parametrized as

$$\mathcal{P} = (p_1(x), p_2(x)) := \left(\frac{(1 + 6\,x)\,x}{(x+1)^2}, -\frac{(1 + 11\,x)\,x^2}{(x+1)^3} \right).$$

Since

$$r(x) := \frac{p_2(x)}{p_1'(x)} = -x^2,$$

we know that $F = 0$ has a rational solution, and the required Möbius transformation is

$$T = \frac{1}{x}.$$

Applying T, we get

$$\mathcal{P}(T) = \left(\frac{x+6}{(x+1)^2}, \frac{-x-11}{(x+1)^3} \right).$$

Therefore, $\hat{y} = (x+6)/(x+1)^2$ is a rational solution of F, and

$$\hat{y}(x + c) = \frac{x + c + 6}{(x + c + 1)^2}$$

is a rational general solution of $F(y, y') = 0$.

Algebraic general solutions. The work of Feng and Gao on rational solutions was extended to algebraic solutions in [ACFG05] and further examined in [Fal20]. In particular, in [ACFG05] an algorithm for deciding the existence of an algebraic general solution and, in the positive case, for computing its minimal polynomial is presented. It is worth mentioning that rational functions are particular instances of algebraic functions.

Definition 2.7. Let $\hat{y} \in \overline{\mathbb{Q}}\langle\langle x \rangle\rangle$ be a solution of a differential equation $F(y, y') = 0$, and let $Q \in \overline{\mathbb{Q}}[x, y]$ be an irreducible polynomial with $Q(x, \hat{y}) = 0$. Then $\hat{y}$ is called an *algebraic solution* (over $\overline{\mathbb{Q}}$).

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[2] As stated here, the algorithm does not detect the rational general solution $\hat{y} = c$ for the family of AODEs $ky' = 0$, where $k \in \mathbb{Q} \setminus \{0\}$. This is due to the fact that AODEs of this form do not have nontrivial rational solutions. However, this is the only case that requires special treatment.
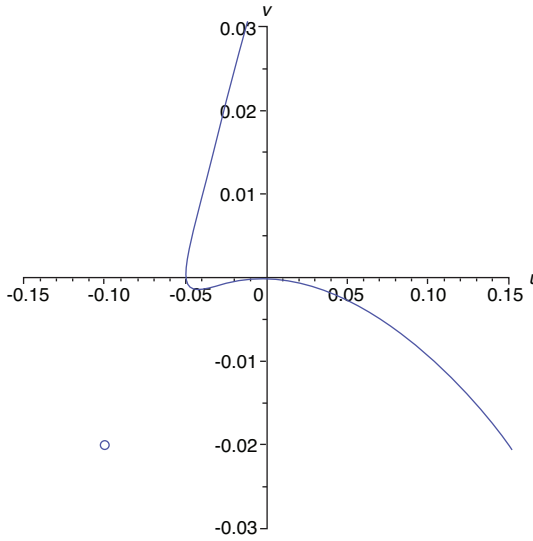
FIGURE 1. The real part of the cubic curve $\mathcal{C}_F$ defined by the AODE of Example 2.6. Note that the point at $(-0.1, -0.02)$ is an isolated singularity of the curve.

If equation (4) has a nonconstant algebraic solution $\hat{y}$ with minimal polynomial $Q(x, y) \in \overline{\mathbb{Q}}[x, y]$, then every conjugate root of $Q(x, y)$ is a solution of (4). Moreover, all algebraic solutions of (4) are found by a shift of the independent variable. Let us summarize this in the following theorem (see [Fal20, Theorem 4.1.22]).

Theorem 2.8. *Let* $F \in \mathbb{Q}[y, y']$ *be as in* (4), *and let* $\hat{y} \in \overline{\mathbb{Q}}\langle\langle x \rangle\rangle$ *be an algebraic solution of* $F = 0$ *with minimal polynomial* $Q(x, y) \in \overline{\mathbb{Q}}[x, y]$. *Then all formal Puiseux series solutions of* $F = 0$ *are algebraic and are given by* $Q(x + c, y)$, *where* $c \in \overline{\mathbb{Q}}$.

Since, by Theorem 2.8, it is equivalent to find any algebraic solution or all of them, we simplify the problem by additionally considering an initial value $(y_0, y_1) \in \mathcal{C}_F$ with $y_1 \neq 0$ and $S_F(y_0, y_1) \neq 0$. There exists a unique formal power series solution $\hat{y} \in \overline{\mathbb{Q}}[[x]]$ of (4) fulfilling the initial values $\hat{y}(0) = y_0$, $\hat{y}'(0) = y_1$. This solution can be easily computed term by term (see [Fal20, Proposition 2.1.7]).

By using elementary properties on places of the algebraic curve $\mathcal{C}_F$ and the Riemann–Hurwitz formula, a degree bound on the minimal polynomial Q can be derived [ACFG05, Theorems 3.4 and 3.8]:

Theorem 2.9. *Let* $F \in \mathbb{Q}[y, y']$ *be as in* (4), *and let* $Q \in \overline{\mathbb{Q}}[x, y]$ *be the minimal polynomial of an algebraic solution of* $F = 0$. *Then*

$$\deg_x(Q) = \deg_{y'}(F) \quad and \quad \deg_y(Q) \leq \deg_y(F) + \deg_{y'}(F).$$

The above theorems lead to the following algorithm for computing all algebraic solutions of a given differential equation (4):

(a) Compute sufficiently many terms of a formal power series solution $\hat{y} \in \overline{\mathbb{Q}}[[x]]$ of $F = 0$.
(b) Make an ansatz of unknown coefficients for $Q(x, y) \in \overline{\mathbb{Q}}[x, y]$ fulfilling the degree bound from above, and perform coefficient comparison for $Q(x, \hat{y}) = 0$ in order to determine a set of candidates.
(c) Check whether there is an irreducible polynomial implicitly defining a solution of $F = 0$ in the set of candidates.
(d) In the negative case, no algebraic solution exists; in the affirmative case, all solutions are given by $Q(x + c, y)$.

Example 2.10. Let us consider the differential equation

$$F(y, y') = y^4 + 3y' = 0.$$

For the initial value $(1, -\frac{1}{3}) \in \mathcal{C}_F$, we obtain the formal power series solution

$$\varphi = 1 - \frac{x}{3} + \frac{2x^2}{9} - \frac{14x^3}{81} + \mathcal{O}(x^4)$$

(see Figure 2). Let $Q(x, y) = \sum_{0 \leq i \leq 4, 0 \leq j \leq 1} a_{i,j} \, x^i \, y^j$. Then $Q(x, \varphi) = 0$ leads to the possible choice $Q(x, y) = x\,y^3 - 1$ and the solutions, namely $\hat{y} = \frac{\zeta}{\sqrt[3]{x+c}}$ for $\zeta^3 = 1$, are determined by $Q(x + c, y)$.
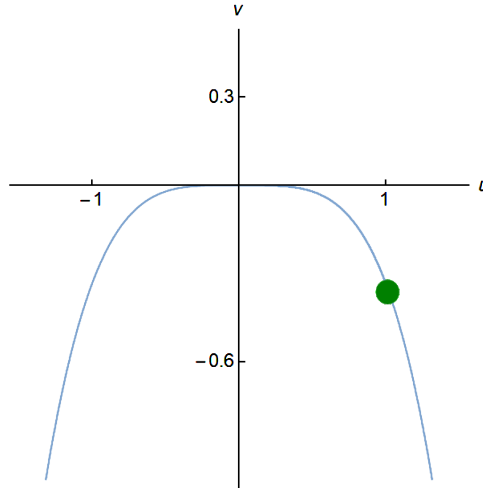

FIGURE 2. The real part of the curve $\mathcal{C}_F$ defined by the AODE of Example 2.10. The highlighted point at $(1, -1/3)$ denotes the initial value of the used formal power series solution.

Power series solutions. In [FS20] formal power series solutions with nonnegative integer exponents of equations of the type (4) are studied. It is shown that every formal power series solution of $F(y, y') = 0$ is convergent. This result is generalized in [CFS20] to formal Puiseux series, i.e., formal power series with rational exponents (see Appendix A). The proof is constructive and provides an algorithm to describe

all Puiseux series solutions. Let us outline it here in a similar manner as shown above for rational and algebraic solutions. Additionally, let us note that algebraic functions can always be expanded as formal Puiseux series. Hence, these results generalize [ACFG05] and the previous two sections.

If equation (4) has a nonconstant formal Puiseux series solution $\hat{y}$ with ramification index equal to n, then $F(\hat{y}, \hat{y}') = 0$, and therefore $\mathcal{P} = (p_1(t), p_2(t)) = (\hat{y}(t^n), \hat{y}'(t^n))$ is a local parametrization of the corresponding algebraic curve $\mathcal{C}_F$ defined by $F(u, v) = 0$, called a *solution parametrization*. For the theory of local parametrizations we refer to [Duv89, Wal50]. By performing order comparison in $\frac{d\,p_1(t)}{dt} = nt^{n-1}\,\hat{y}'(t^n) = nt^{n-1}\,p_2(t)$, the necessary condition

$$(5) \qquad \operatorname{ord}_t(p_1'(t)) + 1 = \operatorname{ord}_t(p_1(t) - p_1(0)) = n + \operatorname{ord}_t(p_2(t))$$

follows. Since reparametrizations are of the same order, a necessary condition on a place is found such that it can contain a solution parametrization. If the place $[\mathcal{P}]$ indeed contains a solution parametrization, we may speak about a *solution place*.

It turns out that condition (5) is already a sufficient condition [CFS20, Theorem 10]. This can be seen by expanding the reparametrization $\frac{d\,p_1(T(t))}{dt} = p_2(T(t))$ in order to obtain the equation

$$(6) \qquad p_1'(T(t))\,T'(t) = nt^{n-1}\,p_2(T(t)),$$

which can be solved for formal power series $T(t) \in \overline{\mathbb{Q}}[[t]]$ of order one, for example by the Newton polygon method for differential equations [Can93]. For positive n there exist exactly n solutions, and after computing the first coefficient of $T(t)$, the next coefficients are uniquely determined. The solutions of $F(y, y') = 0$ *corresponding to* $\mathcal{P}$ are now given by $\hat{y} = p_1(T(x^{1/n}))$.

Theorem 2.11. *Let $F \in \mathbb{Q}[y, y']$ be as in (4), and let $\mathcal{P} \in \overline{\mathbb{Q}}((t))^2$ be an irreducible local parametrization of $\mathcal{C}_F$ centered at (y_0, p_0). Then $[\mathcal{P}]$ is a solution place if and only if equation (5) holds for an $n \in \mathbb{N}^*$. In the affirmative case, there are n many solutions of $F = 0$ corresponding to $\mathcal{P}$ and all of them have ramification index equal to n.*

The above observations address power series solutions expanded around 0. Since the given differential equation is autonomous, the independent variable can be shifted and solutions expanded around any $x_0 \in \mathbb{C}$ can be found. Formal power series expanded around infinity are obtained in a similar way as described above, coming from the solutions of the associated differential equation

$$(7) \qquad p_1'(T(t))\,T'(t) = nt^{-n-1}\,p_2(T(t)).$$

Equation (7) has either no solution or infinitely many. More precisely, solutions of (7) and, therefore, solutions of $F = 0$ expanded around infinity involve an unspecified parameter.

Using results on the convergence of Puiseux expansions (see [Duv89]), we can show convergence of the solutions. The components of local parametrizations $(p_1(t), p_2(t))$ arising from such Puiseux expansions are convergent. Then, also the solutions $T(t)$ of (6) and (7) have a positive radius of convergence. This property remains valid for the composition leading to the following result [CFS20, Theorem 11].

Theorem 2.12. *Let $F \in \mathbb{Q}[y, y']$. Then all formal Puiseux series solutions of $F = 0$, expanded around any finite point or around infinity, are convergent.*

The algorithm for computing all local solutions of (4) with given initial data $y(0) = y_0 \in \mathbb{C}$ follows from the proof of Theorem (2.11) and is given by the following:

(a) Compute for every curve branch of $\mathcal{C}_F$ centered above y_0 a local parametrization $\mathcal{P} = (p_1(t), p_2(t))$, e.g., by the algorithm in [Duv89].
(b) Check whether the necessary condition (5) is fulfilled for positive n.
(c) In the affirmative case, compute all reparametrizations $\mathcal{P}(T(t))$ by solving (6).
(d) $\hat{y} = p_1(T(x^{1/n}))$ are the Puiseux series solutions.

Let us emphasize that in [Duv89] computational bounds are presented such that all local parametrizations centered above a given $y_0 \in \mathbb{C}$ are in one-to-one correspondence to a set of truncations. This enables us to represent the Puiseux series solutions $\hat{y}$ as truncated Puiseux series where existence and uniqueness are ensured.

For almost every $y_0 \in \mathbb{C}$, there exist $\deg_v(F(u,v))$ many values $y_1 \in \mathbb{C}$ such that $(y_0, y_1) \in \mathcal{C}_F$ corresponds to a formal power series solution $\hat{y}$ with ramification index equal to 1 and $(\hat{y}(0), \hat{y}'(0)) = (y_0, y_1)$. Curve points where this may not happen are as follows:

• Singular curve points and points of ramification with respect to the projection onto the v-axis computed by $F(y_0, y_1) = \frac{\partial F}{\partial v}(y_0, y_1) = 0$.
• Curve points lying on the v-axis computed by $F(y_0, 0) = 0$.
• Curve points with second component equal to infinity computed with $\mathrm{lc}_v(F)(y_0) = 0$, where lc_v is the leading coefficient of $F(u,v)$ considered as polynomial in v.

Such points are called *critical curve points*. Curve points with first component equal to infinity correspond to solutions $\hat{y}$ of negative order. They can be found by considering the numerator of $F(1/u, -u/v^2)$ and the initial value $\hat{y}(0) = 0$.

The above reasonings enable the representation of all local solutions of (4) as follows:

(a) Compute the critical curve points of $\mathcal{C}_F$. For every other value $(y_0, y_1) \in \mathcal{C}_F$ the solution is represented by $y_0 + y_1 x$.
(b) Compute for every critical curve point the solution truncations with $y(0) = y_0$ as initial value by the preceding algorithm.
(c) Compute the solutions with negative order by considering the numerator of $F(1/u, -u/v^2)$ and $\hat{y}(0) = 0$ as initial value.
(d) Compute the solutions expanded around infinity.

Example 2.13. Let us continue Example 2.10. For the differential equation $F(y, y') = y^4 + 3y'$, we obtain a generic solution represented by the truncation $y_g = y_0 - \frac{y_0^4}{3} x + \mathcal{O}(x^2)$, where $y(0) = y_0$ is an arbitrary initial value, and the particular solutions are represented as

$$\hat{y}_1 = 0, \quad \hat{y}_2 = \frac{\zeta}{x^{1/3}}, \quad \hat{y} = \frac{\zeta}{x^{1/3}} + \frac{c}{x^{4/3}} + \mathcal{O}(\frac{1}{x^{5/3}}),$$

where $\zeta^3 = 1$. The solution $\hat{y}$ corresponds to the expansion of y_g at infinity, where $c\, y_0^3 = 1$, and also to the expansion of the general solution $\hat{y} = \frac{\zeta}{\sqrt[3]{x+c}}$. The solution $\hat{y}_2$ corresponds to the particular choice $c = 0$, $y_0 = \infty$.
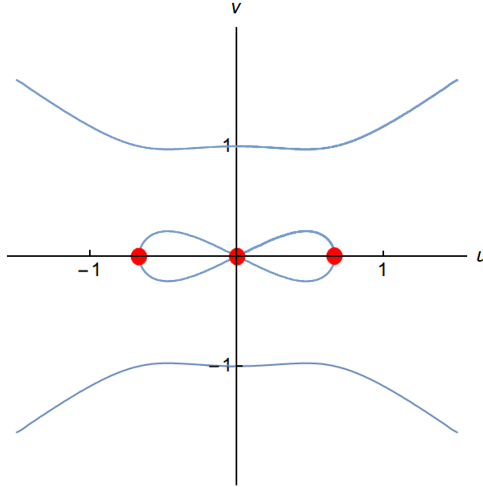
FIGURE 3. The real part of the Devil's curve from Example 2.14. The highlighted points denote the critical curve points.

Example 2.14. Let us consider

$$F(y, y') = y'^4 - y'^2 - y^4 + \frac{4}{9} y^2 = 0.$$

The corresponding algebraic set given by $F(u, v) = 0$ is called the Devil's curve (see Figure 3) and has the critical curve points $\{(0, 0), (\pm\frac{2}{3}, 0), (\alpha, \beta)\}$, where $36\alpha^4 - 16\alpha^2 + 9 = 0$ and $2\beta^2 - 1 = 0$.

The local parametrizations centered at $(0, 0)$ are $(t, \pm\frac{2}{3} t + \mathcal{O}(t^2))$, which do not provide any solution (see equation (5)) and the constant 0 is the only solution with $(0, 0)$ as an initial tuple.

A local parametrization at $(\frac{2}{3}, 0)$ is, for example,

$$(p_1(t), p_2(t)) = (\frac{2}{3} - \frac{16}{27} t^2, -\frac{16}{27} t + \frac{11 \cdot 2^5 \cdot 31}{3^9} t^3 + \mathcal{O}(t^5)),$$

and (5) is fulfilled with $n = 1 > 0$. Then equation (6) corresponding to $(p_1(t), p_2(t))$ is $T'(t) = p_2(T(t))$ and has the unique solution

$$T(t) = \frac{2}{3} t - \frac{11 \cdot 31}{2^2 \cdot 3^7} t^3 + \mathcal{O}(t^5).$$

Therefore,

$$p_1(T(x)) = \frac{2}{3} - \frac{4}{27} x^2 + \frac{2^2 \cdot 11 \cdot 31}{3^{10}} x^4 + \mathcal{O}(x^5)$$

is a solution truncation of $F = 0$. Similarly, we can find solution truncations coming from the other local parametrizations, such as

$$(q_1(t), q_2(t)) = (\alpha + (2\alpha^3 - \frac{4\alpha}{9}) t^2, \beta + (2\alpha^3 - \frac{4\alpha}{9}) t + \mathcal{O}(t^2)),$$

centered at (α, β). Equation (5) is fulfilled with $n = 2$ and leads to the Puiseux series solutions

$$\hat{y} = \alpha + \beta x + \frac{\gamma(36\alpha^3 - 8\alpha)}{27} x^{3/2},$$

where γ are the roots of $65\gamma^2 = 98\alpha\beta - 144\alpha^3\beta$.

For the numerator of $F(1/y, -y'/y^2)$, the local parametrizations at the origin, given by $(t, \pm\frac{13i}{36} t^2 + \mathcal{O}(t^3))$, do not fulfill (5) and no solutions with negative order exist.

2.2. **Nonautonomous AODEs of order one.** Now let us consider AODEs of order one in which the variable x might appear explicitly. In particular, we consider an algebraic differential equation over $\overline{\mathbb{Q}}$ of the form

$$(8) \qquad\qquad\qquad F(x, y, y') = 0.$$

If we apply the same approach as for autonomous equations, we might take two paths. First, we could associate to the differential equation (8) the surface $\mathcal{S}_F$ in $\mathbb{A}^3(\overline{\mathbb{Q}})$ defined by $F(x, u, v) = 0$. Every rational solution $\hat{y}$ of the AODE gives rise to a rational curve on $\mathcal{S}_F$ parametrized by $(x, \hat{y}(x), \hat{y}'(x))$. Alternatively, we might associate to the AODE (8) the curve $\mathcal{C}_F \subseteq \mathbb{A}^2(\overline{\mathbb{Q}(x)})$ defined by $F(x, u, v) = 0$, where we consider this as an equation in u and v with coefficients in $\overline{\mathbb{Q}}(x)$. In this case, a rational solution $\hat{y}$ of the AODE corresponds to a point on $\mathcal{C}_F$. If we can parametrize either the surface $\mathcal{S}_F$ or the curve $\mathcal{C}_F$ by suitable rational functions, then it is possible to find a rational general solution of the differential equation (8) by a reparametrization of these rational functions, provided such a solution exists.

In what follows we investigate both approaches and compare them w.r.t. their applicability.

Rational general solutions via surface parametrization. This approach has been pursued by Ngô and Winkler in [NW10, NW11b]. Consider the surface $\mathcal{S}_F \subseteq \mathbb{A}^3(\overline{\mathbb{Q}})$ defined by $F(x, u, v) = 0$. Assume that $\mathcal{S}_F$ is parametrizable by rational functions, and let

$$\mathcal{P}(s, t) = (p_1(s, t), p_2(s, t), p_3(s, t)) \in \overline{\mathbb{Q}}(s, t)^3$$

be a proper rational parametrization.[3] We call AODEs whose surface $\mathcal{S}_F$ is parametrizable in this way *surface parametrizable*. Observe that by Castelnuovo's theorem (see, e.g., [Zar58]), any unirational algebraic surface over $\overline{\mathbb{Q}}$ is rational. In other words, if $\mathcal{S}_F$ has a rational parametrization, then we can find a proper rational parametrization as well. Any rational solution $\hat{y} = f(x)$ of the AODE (8) gives rise to a rational solution curve parametrizable by $\mathcal{P}$ if and only if the space curve $\mathcal{C}_f = \{(x, u, v) \mid x \in \overline{\mathbb{Q}}, u = f(x), v = f'(x)\}$ is almost contained in $\operatorname{im}(\mathcal{P}) \cap \operatorname{dom}(\mathcal{P}^{-1})$, viz. except for finitely many points of $\mathcal{C}_f$. By [NW10, Proposition 3.4 and the subsequent remark], we can turn the search for a rational solution of (8), and in particular for a rational general solution, into the search for a rational solution of a planar rational system. Namely, a surface parametrizable AODE (8) has a rational solution whose solution curve is parametrizable by $\mathcal{P}$ if and only if the autonomous system

$$(9) \qquad\qquad \begin{cases} s' = \dfrac{f_1(s, t)}{g(s, t)} \\[2mm] t' = \dfrac{f_2(s, t)}{g(s, t)} \end{cases}$$

---

[3]Similarly to the case of algebraic curves, a triple of rational functions $\mathcal{P}$ is a rational parametrization of the irreducible surface $\mathcal{S}$ iff $\mathcal{P}$ induces a rational mapping from $\mathbb{A}^2(\overline{\mathbb{Q}})$ onto $\mathcal{S}$ such that the Jacobian has rank two. $\mathcal{P}$ is proper iff the mapping is birational.

has a rational solution, where $f_1(s,t)$, $f_2(s,t)$, $g(s,t) \in \overline{\mathbb{Q}}(s,t)$ are given by

$$f_1(s,t) = \frac{\partial p_2(s,t)}{\partial t} - p_3(s,t)\,\frac{\partial p_1(s,t)}{\partial t},$$

$$f_2(s,t) = p_3(s,t)\,\frac{\partial p_1(s,t)}{\partial s} - \frac{\partial p_2(s,t)}{\partial s},$$

$$g(s,t) = \frac{\partial p_1(s,t)}{\partial s}\frac{\partial p_2(s,t)}{\partial t} - \frac{\partial p_1(s,t)}{\partial t}\frac{\partial p_2(s,t)}{\partial s}.$$

We call the system (9) the *associated system* of the differential equation (8) w.r.t. the parametrization $\mathcal{P}$. By construction, the associated system is an autonomous system of quasi-linear differential equations of order one. Solution methods for systems of this particular shape have been studied in [NW11b], for example. Assume that we have a method for finding rational general solutions of system (9). In this case, [NW10, Theorem 3.15] asserts that we can construct a solution of the differential equation (8) via a reparametrization of $\mathcal{P}$. In particular, if $(\hat{s}, \hat{t})$ is a rational general solution of the associated system (9) w.r.t. $\mathcal{P}(s,t) = (p_1(s,t), p_2(s,t), p_3(s,t))$, then

$$\hat{y} = p_2(\hat{s}(x+k), \hat{t}(x+k))$$

is a rational general solution of the AODE (8), where

$$k = x - p_1(\hat{s}, \hat{t})$$

is a constant. In other words, we translate the solution $(\hat{s}, \hat{t})$ of the associated system by a constant and substitute this quantity into the parametrization $\mathcal{P}$. By construction, $\mathcal{P}(\hat{s}(x+k), \hat{t}(x+k)) = (x, f(x), f'(x))$ for some rational function $f(x)$, cf. [NW10, Proposition 3.4]. Every parametrization of $\mathcal{S}_F$ annihilates the polynomial $F(x, u, v)$ and so does the reparametrization $(x, f(x), f'(x))$, i.e., $F(x, f(x), f'(x)) = 0$, which means that $f(x)$ is a rational solution of the differential equation (8). Furthermore, if $f(x) = p_2(\hat{s}(x+k), \hat{t}(x+k))$ is constructed from a general solution of the associated system, then it is a rational general solution of the original AODE.

Note that proper rational parametrizations of algebraic surfaces are far from being unique, and system (9) depends on the parametrization being used. However, the existence of a rational general solution is an invariant among the systems obtained in this way. In particular, the success of the method described above does not depend on the proper rational parametrization being used. Furthermore, in the case of rational general solutions, we can assume without loss of generality that the corresponding solution curve is parametrizable by $\mathcal{P}$, cf. [NW11a, the remarks preceding Theorem 2.1]. Consequently, if the differential equation (8) is surface parametrizable, then it has a rational general solution if and only if its associated system w.r.t. any proper rational parametrization of $\mathcal{S}_F$ has such a solution.

We summarize these ideas in the subsequent algorithm for computing rational general solutions of surface parametrizable first-order AODEs:

(a) Given $F(x, y, y') = 0$, a surface parametrizable AODE of order one. Compute a proper rational parametrization

$$\mathcal{P}(s,t) = (p_1(s,t), p_2(s,t), p_3(s,t)) \in \overline{\mathbb{Q}}(s,t)^3$$

of the associated surface $\mathcal{S}_F$, e.g., by using the algorithm in [Sch98].

(b) Construct the associated system (9) w.r.t. $\mathcal{P}$ and compute a rational general solution $(\hat{s}, \hat{t})$ of this system. An algorithm for finding such solutions is given in [NW11b]. If no such solution exists, then the input AODE does not have a rational general solution.

(c) Compute the constant $k = x - p_1(\hat{s}, \hat{t})$ and return

$$\hat{y} = p_2(\hat{s}(x + k), \hat{t}(x + k)),$$

a rational general solution of the input AODE.

Example 2.15. Let us consider a first-order AODE $F(x, y, y') = 0$, where

$$F(x, y, y') = x^6 y' - 2x^5 y + x^3 y'^3 - 3x^2 y y'^2 + 3x y^2 y' - y^3.$$

The zero-locus of $F(x, u, v)$ defines a rational surface in $\mathbb{A}^3(\overline{\mathbb{Q}})$ (see Figure 4) which has the proper rational parametrization

$$\mathcal{P}(s, t) = (p_1(s, t), p_2(s, t), p_3(s, t)) = \left(\frac{t - s^3}{2s}, \frac{t^2 - s^6}{4s}, t \right).$$
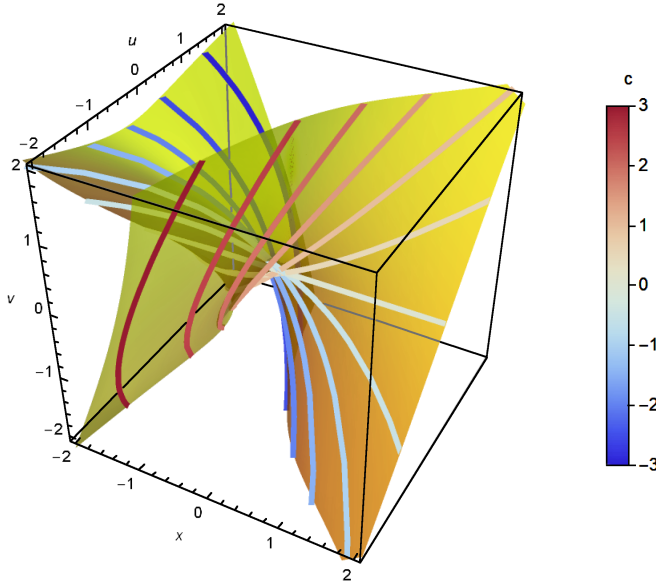

FIGURE 4. The real part of the surface $\mathcal{S}_F$ defined by the AODE of Example 2.15. The highlighted curves on the surface correspond to the space curves generated by particular rational solutions. These particular solutions are obtained from the general solution $\hat{y} = cx(4x + c^2)/8$ by setting the arbitrary constant to values from the real interval $[-3, 3]$.

With the aid of this parametization, we are able to reduce the differential equation to the following associated system:

$$\begin{cases} s' = \dfrac{f_1(s,t)}{g(s,t)} = \dfrac{\frac{\partial p_2(s,t)}{\partial t} - p_3(s,t)\frac{\partial p_1(s,t)}{\partial t}}{\frac{\partial p_1(s,t)}{\partial s}\frac{\partial p_2(s,t)}{\partial t} - \frac{\partial p_1(s,t)}{\partial t}\frac{\partial p_2(s,t)}{\partial s}} = 0 \\[2em] t' = \dfrac{f_2(s,t)}{g(s,t)} = \dfrac{p_3(s,t)\frac{\partial p_1(s,t)}{\partial s} - \frac{\partial p_2(s,t)}{\partial s}}{\frac{\partial p_1(s,t)}{\partial s}\frac{\partial p_2(s,t)}{\partial t} - \frac{\partial p_1(s,t)}{\partial t}\frac{\partial p_2(s,t)}{\partial s}} = 2s. \end{cases}$$

Due to the simplicity of this system, it is clear that a rational general solution is of the form $(\bar{s}, \bar{t}) = (c_1/2, c_1 x + c_2)$, where c_1 and c_2 are arbitrary constants. Since the system is autonomous, we can eliminate one of these constants by a suitable translation[4] and use the simpler solution $(\hat{s}, \hat{t}) = (c/2, cx)$ with arbitrary constant c. Finally, we compute the constant $k = x - p_1(\hat{s}, \hat{t}) = c^2/8$ and obtain $(\hat{s}(x + k), \hat{t}(x + k)) = (c/2, cx + c^3/8)$. This translated solution of the associated system yields the following rational general solution of the differential equation $F(x, y, y') = 0$:

$$\hat{y} = p_2(\hat{s}(x + k), \hat{t}(x + k)) = \frac{cx(4x + c^2)}{8}.$$

Rational general solutions via curve parametrization. Another approach for computing rational general solutions of first-order AODEs has been investigated by Vo, Grasegger, and Winkler in [VGW18]. Instead of viewing $F(x, u, v) = 0$ as the defining equation of a surface in three-dimensional affine space over $\overline{\mathbb{Q}}$, we consider the curve $\mathcal{C}_F \subseteq \mathbb{A}^2(\overline{\mathbb{Q}(x)})$ defined by the zero-locus of $F(x, u, v)$, where the latter is considered as a polynomial in u and v with coefficients from the rational function field $\overline{\mathbb{Q}}(x)$. We assume that the curve $\mathcal{C}_F$ is parametrizable by rational functions, analogous to the previous approach, but with coefficients from this rational function field. Let

$$\mathcal{P} = (p_1(t), p_2(t)) \in \overline{\mathbb{Q}}(x)(t)^2$$

be a proper rational parametrization of $\mathcal{C}_F$. A first-order AODE is called *curve parametrizable* if its curve $\mathcal{C}_F$ has a parametrization of the antecedent form. Notice the requirement on the coefficients of the parametrization: If we would allow coefficients in an algebraic extension of $\overline{\mathbb{Q}}(x)$—as is typically required for parametrizing curves defined over a nonalgebraically closed field—then we might not get a rational solution after a reparametrization of $\mathcal{P}$. The requirements on the parametrization, however, do not impose restrictions in addition to the unirationality of $\mathcal{C}_F$. By Lüroth's theorem and [VGW18, Theorem 4.3], unirationality of $\mathcal{C}_F$ implies the existence of a proper rational parametrization of this particular form.

Given a rational solution $\hat{y}$ of the differential equation (8), we see that the pair $(\hat{y}, \hat{y}')$ describes a point on the curve $\mathcal{C}_F$. Since $\mathcal{P}$ is a rational parametrization, at most finitely many points of $\mathcal{C}_F$ cannot be covered by $\mathcal{P}$, i.e., $\mathcal{C}_F \setminus \operatorname{im}(\mathcal{P})$ is finite. However, we cannot miss a rational general solution as such can be seen as an infinite family of rational solutions. By [VGW18, Lemma 5.2], the search for a rational general solution of (8) can be reduced to the search for a rational

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[4]The class of solutions of an autonomous system of differential equations is invariant under translation by a constant. In the present case, the shift $(\bar{s}(x - c_2/c_1), \bar{t}(x - c_2/c_1))$ is used to eliminate the second arbitrary constant from the solution.

general solution of a single quasi-linear ODE of order one. Namely, the curve parametrizable AODE (8) has a rational general solution if and only if

$$(10) \qquad t' = \frac{p_2(t) - \frac{\partial p_1(t)}{x}}{\frac{\partial p_1(t)}{t}}$$

has a rational general solution. We call the quasi-linear ODE (10) the *associated differential equation* of the AODE (8) w.r.t. the parametrization $\mathcal{P}$. Any rational general solution $\hat{t}$ of the associated differential equation gives rise to

$$\hat{y} = p_1(\hat{t}),$$

a rational general solution of the AODE (8) by [VGW18, Theorem 5.3].

We see that the computation of a rational general solution is reduced to a reparametrization of $\mathcal{P}$. By construction, $\mathcal{P}(\hat{t}) = (f(x), f'(x))$ for some rational function $f(x)$ and, by a similar reasoning as in the preceding section, $f(x) = p_1(\hat{t})$ yields a rational general solution of the differential equation (8). Notice that the associated differential equation depends again on the chosen parametrization—which is not unique.

It is known that equation (10) cannot have a rational general solution if it is neither a linear differential equation nor a Riccati equation. Computing rational general solutions of linear differential equations is easily done via integration. For Riccati equations, Kovacic [Kov86] describes an algorithm for finding all rational solutions of such an equation. Chen and Ma [CM05] later modified this method to look for rational general solutions only. Therefore, the existence of a rational general solution of the associated differential equation (10) can be decided, and such solutions are computable.

We summarize these steps in the subsequent algorithm for computing rational general solutions of curve parametrizable AODEs:

(a) Given $F(x, y, y') = 0$, a curve parametrizable AODE of order one, compute a proper rational parametrization

$$\mathcal{P} = (p_1(t), p_2(t)) \in \overline{\mathbb{Q}}(x)(t)^2$$

of the associated curve $\mathcal{C}_F$, e.g., by using [VGW18, Algorithm 1].

(b) Construct the associated differential equation (10) w.r.t. $\mathcal{P}$. If this is neither a linear differential equation nor a Riccati equation, then the input AODE does not have a rational general solution.

(c) Find a rational general solution $\hat{t}$ of the associated differential equation. If no such solution exists, then the input AODE does not have a rational general solution.

(d) Return $\hat{y} = p_1(\hat{t})$, a rational general solution of the input AODE.

Example 2.16. Consider again the first-order AODE from Example 2.15. The curve $\mathcal{C}_F$ defined by the zero-locus of $F(x, u, v)$, considered as a polynomial in u and v with coefficients in $\overline{\mathbb{Q}}(x)$, has the proper rational parametrization

$$\mathcal{P}(p_1(t), p_2(t))$$
$$= \left(\frac{(x - t)(x^5 t^2 + t^2 - 2tx + x^2)}{x^5 t^3}, \frac{(x - t)(2x^5 t^2 + t^2 - 2tx + x^2)}{x^6 t^3} \right).$$

By utilizing this parametrization, the differential equation can be reduced to the following associated differential equation:

$$t' = \frac{p_2(t) - \frac{\partial p_1(t)}{x}}{\frac{\partial p_1(t)}{t}} = \frac{2}{x^2}t^2 - \frac{1}{x}t.$$

This associated differential equation is a Riccati equation and has the rational general solution $\hat{t} = x/(cx^2 + 1)$, where c is an arbitrary constant. From the solution $\hat{t}$ we obtain that

$$\hat{y} = p_1(\hat{t}) = cx(x + c^2)$$

is a rational general solution of the differential equation $F(x, y, y') = 0$. Substitution of c by $c/2$ in $\hat{y}$ results in the same solution as in Example 2.15.

Comparison of the previous approaches. In the case of autonomous first-order AODEs, a rational general solution is constructed by translating a nontrivial rational solution by an arbitrary constant; cf. Section 2.1. As it turns out, adjoining a single arbitrary constant to the coefficient field is the only extension needed for constructing a rational general solution of an autonomous first-order AODEs, granted that such a quantity exists. Solutions of this particularly simple form are of major interest in the rest of this section and warrant a special name. Let $\hat{y}$ be a general solution of a (not necessarily autonomous) first-order AODE. If $\hat{y} \in \overline{\mathbb{Q}}(x, c) \setminus \overline{\mathbb{Q}}(x)$, where c is an arbitrary constant, then we call $\hat{y}$ a *strong rational general solution* [VGW18].

A simple extension by an arbitrary constant is, in general, no longer sufficient in the general (nonautonomous) case: there exist first-order AODEs which have a rational general solution, but not a strong rational general solution.

Example 2.17. The subsequent AODE can be found in [MW22, Example 4.4]:

$$x^2y'^2 - 2xyy' - y'^3 + y^2 - 2 = 0.$$

A rational general solution is given by $\hat{y} = cx + \sqrt{c^3 + 2}$, which is not strong since the arbitrary constant does not appear purely rationally. Furthermore, it can be shown that there does not exist a strong rational general solution of the above AODE.

The previously introduced algorithms for computing rational general solutions of AODEs of order one require that the associated curve and surface are parametrizable. However, not all first-order AODEs are parametrizable as a curve or as a surface. The antecedent example describes such an AODE which is not solvable by either approach since it cannot be parametrized. Naturally, this leads to the question of whether there is a specific class of AODEs which can be solved by either of the algorithms introduced. For this we introduce the following notation: Let $\mathcal{AODE}^{(1)}$ denote the class of all first-order AODEs, and let $\mathcal{AODE}^{(1)}_{CP}$ and $\mathcal{AODE}^{(1)}_{SP}$ be the subclass of curve parametrizable and surface parametrizable AODEs, respectively. By $\mathcal{AODE}^{(1)}_{RGS}$ we denote the class of AODEs of order one which have a rational general solution and $\mathcal{AODE}^{(1)}_{SRGS}$ stands for the subclass which have a strong rational general solution.

It is known that every first-order AODE which has a strong rational general solution is curve parametrizable. Conversely, if an AODE of order one has a rational general solution and is curve parametrizable, then it has a strong rational general solution [VGW18, Theorem 5.4(i) and Corollary 5.5]. In other words, $\mathcal{AODE}_{SRGS}^{(1)} = \mathcal{AODE}_{CP}^{(1)} \cap \mathcal{AODE}_{RGS}^{(1)}$.

Every curve parametrizable first-order AODE is surface parametrizable in a natural way: Let $\mathcal{P} = (p_1(x,t), p_2(x,t)) \in \overline{\mathbb{Q}}(x)(t)^2$ be a proper rational parametrization of the associated curve, where we indicate the dependency on the independent variable x in the components of $\mathcal{P}$. By interpreting x as an additional parameter, $\mathcal{P}$ can be turned into the surface parametrization $\tilde{\mathcal{P}}(s,t) = (s, p_1(s,t), p_2(s,t)) \in \overline{\mathbb{Q}}(s,t)^3$. Furthermore, properness of $\mathcal{P}$ implies that $\tilde{\mathcal{P}}$ is proper as a surface parametrization [MW22, Theorem 4.1(I)]. On the other hand, there exist surface parametrizable AODEs which cannot be parametrized as a curve.

Example 2.18. The first-order AODE $y'^2 - y^3 - x = 0$ can be parametrized as a surface by $\mathcal{P}(s,t) = (t^2 - s^3, s, t)$. However, the associated curve of this AODE is not rational and cannot be parametrized. This AODE does not have a rational general solution; cf. [MW22, Example 4.1].

Recall that the algorithm via curve parametrization cannot solve AODEs outside the class $\mathcal{AODE}_{SRGS}^{(1)}$. One might expect that, given that $\mathcal{AODE}_{SP}^{(1)}$ is a strict superclass of $\mathcal{AODE}_{CP}^{(1)}$, the algorithm via surface parametrization can solve AODEs beyond those that possess a strong rational general solution. As it turns out, this is not the case. If an AODE is surface parametrizable and has a rational general solution, then it has a strong rational general solution [MW22, Theorem 4.3(II)]. Therefore, $\mathcal{AODE}_{SRGS}^{(1)} = \mathcal{AODE}_{SP}^{(1)} \cap \mathcal{AODE}_{RGS}^{(1)}$, and we see that first-order AODEs which are surface parametrizable but not curve parametrizable cannot have a rational general solution.

Finally, let $\mathcal{P}$ be a proper rational parametrization of the associated curve of a first-order AODE, and let $\tilde{\mathcal{P}}$ be the corresponding surface parametrization. In this case, it can be shown that the associated system w.r.t. $\tilde{\mathcal{P}}$ actually reduces to the associated differential equation w.r.t. $\mathcal{P}$; cf. [MW22, Section 4.2]. In other words, for a common rational parametrization the methods via curve parametrization and surface parametrization actually have to solve the same associated equation(s). We illustrate this by an example.

Example 2.19. Consider the subsequent AODE:

$$(11) \qquad xy'^2 - 2yy' - x = 0.$$

Its associated curve is rational and has the proper parametrization $\mathcal{P} = ((xt^2 - x)/(2t), t)$. The associated differential equation w.r.t. $\mathcal{P}$ is

$$(12) \qquad t' = \frac{t}{x}$$

and has the rational general solution $\hat{t} = cx$. From this we obtain the rational general solution $\hat{y} = (c^2 x^2 - 1)/(2c)$ of the AODE (11). Alternatively, from $\mathcal{P}$ we obtain the proper rational parametrization $\tilde{\mathcal{P}}(s,t) = (s, (st^2 - s)/(2t), t)$ of the associated surface. The associated system w.r.t. $\tilde{\mathcal{P}}$ is

$$\begin{cases} s' = 1 \\ t' = \frac{t}{s}. \end{cases}$$

The first equation has the trivial solution $\hat{s} = x$, and substituting this into the second equation shows that the associated system reduces to the associated differential equation (12). A rational general solution of the associated system is given by $(\hat{s}, \hat{t}) = (x, cx)$ which yields the solution $\hat{y} = (c^2 x^2 - 1)/(2c)$ of the AODE (11).

Since both methods compute solutions in the same solution class, it is natural to ask if one approach is to be preferred over the other. In general, the method via curve parametrization provides a complete decision algorithm, while the surface parametization method can do so only in the generic case. If the corresponding surface of the differential equation is of a special geometric shape, however, then it is worthwhile to follow the method via surface parametrization. In this case, the parametrization can be deduced easily and some results on the solvability of the associated planar system exist; cf. Ngô, Sendra, and Winkler [NSW12b, Sections 4 and 5].

Algebraic general solutions. Aroca et al. [ACFG05] presented a polynomial time algorithm for computing algebraic general solutions of autonomous AODEs $F(y, y') = 0$. Vo and Winkler (in [VW15]) adapted the algebro-geometric method in order to compute algebraic general solutions of nonautonomous parametrizable AODEs $F(x, y, y') = 0$. But in the absence of a degree bound for such algebraic solutions, a bound for the algebraic extension degree must be specified.

As in the previous development, via a proper rational parametrization
$$\mathcal{P}(s, t) = (p_1(s, t), p_2(s, t), p_3(s, t))$$
of the corresponding surface $\mathcal{S}_F$, one reduces the problem of solving $F(x, y, y') = 0$ to an associated planar rational system of the form

(13)
$$\begin{cases} s' = M(s, t) \\ t' = N(s, t), \end{cases}$$

where M and N are rational functions in s and t; compare (9). If the surface parametrizable AODE $F(x, y, y') = 0$ has an algebraic general solution, then its associated system w.r.t. a proper rational parametrization has a *rational first integral*; i.e., a nonconstant rational function $W(s, t)$ such that $M\frac{\partial W}{\partial s} + N\frac{\partial W}{\partial t} = 0$; cf. [VW15, Proposition 3]. Furthermore, by [VW15, Corollary 1], if $W = P/Q$ is a rational first integral in reduced form of the associated system and $(\hat{s}, \hat{t})$ is an algebraic general solution, then $\hat{s}$ is an algebraic general solution of the autonomous first-order AODE $F_1(s, s') = 0$, where
$$F_1(s, r) := \mathrm{res}_t(P(s, t) - cQ(s, t), rM_2(s, t) - M_1(s, t));$$
analogously for $\hat{t}$. Theorem 2 in [VW15] states that if $(\hat{s}, \hat{t})$ is an algebraic general solution of the associated system, then
$$\hat{y} = p_2(\hat{s}(2x - p_1(\hat{s}, \hat{t})), \ \hat{t}(2x - p_1(\hat{s}, \hat{t})))$$
is an algebraic general solution of the original AODE.

As in [ACFG05], we need to bound the degree of the desired algebraic solution. Theorem 5 in [VW15] states that if $F(x, y, y') = 0$ has an algebraic solution with minimal polynomial of degree less than or equal to n, then the associated system has a rational first integral of degree $m(n)$; an explicit formula for $m(n)$ is given. So we can decide the existence of an algebraic solution having extension degree less than or equal to n.

Example 2.20 (from [VW15])**.** Consider the differential equation

$$4x(x-y)y'^2 + 2xyy' - 5x^2 + 4xy - y^2 = 0.$$

The solution surface of the differential equation is rational and has the proper rational parametrization

$$\mathcal{P}(s,t) = \left(s, -\frac{t^2 - 5ts + 5s^2}{s}, \frac{t^2 - 4st + 5s^2}{2s(t-2s)} \right).$$

The associated system with respect to $\mathcal{P}$ is

$$\begin{cases} s' = 1 \\ t' = \frac{t^2 - 3s^2}{2s(t-2s)}. \end{cases}$$

If we look for an algebraic general solution $\hat{y}$ of degree at most $n = 2$, we need to find a rational first integral of degree at most $m(n) = 32$ of the associated system. In this case, the associated system has the rational first integral $W = (t^2 - 4st + 3s^2)/s$. Thus it has an algebraic solution $(\hat{s}, \hat{t}) = (x, \bar{t}(x,c))$, where $\bar{t}(x,c)$ is a root of the algebraic equation $t^2 - 4xt + 3x^2 - cx = 0$. So, by [VW15, Theorem 2], we see that

$$\hat{y} = \frac{\sqrt{cx(cx+1)} - 1}{c}$$

is an algebraic general solution of the differential equation.

3. Systems of autonomous AODEs of dimension one

In Section 2 we have seen how to deal with the rational solutions and the formal Puiseux series solutions of an autonomous AODE by analyzing the associated curve of the given differential equation. In this section we show how to extend these results to the case of systems of AODEs in one differential indeterminate. For this propose, the key property is that the dimension of the associated geometric object has dimension one. In Section 3.1 we treat the case of rational solutions, and in Section 3.2 we focus on the case of formal Puiseux series solutions. The results and ideas presented in this section are essentially based on [LSNW15] for the case of rational solutions and on [CFS21] for the case of formal Puiseux series solutions. Recently, these results were extended to systems involving several differential indeterminates [CFRS21].

Throughout this section, we will work with the ground field $\mathbb{Q}$ of rational numbers and with its algebraic closure $\overline{\mathbb{Q}}$. In addition, we consider finitely many polynomials

$$(14) \qquad \mathcal{F} = \{F_j(w_0, w_1, \ldots, w_n) \mid j \in J\} \subset \mathbb{Q}[w_0, w_1, \ldots, w_n],$$

where $J \subset \mathbb{N}$ is a finite subset. Associated to $\mathcal{F}$, we consider the autonomous algebraic system $\mathcal{S}$ of ordinary differential equations

$$(15) \qquad \mathcal{S} = \{F_j(y, y', \ldots, y^{(n)}) = 0 \mid j \in J\},$$

where y is an indeterminate over a differential extension field of $\overline{\mathbb{Q}}(x)$, and $'$ denotes the differentiation w.r.t. x. We impose on $\mathcal{S}$ the following two hypotheses:

(1) The algebraic variety defined by $\mathcal{F}$, in the affine space $\mathbb{A}^{n+1}(\overline{\mathbb{Q}})$, has dimension one, not necessarily pure; note that this dimension is the algebro-geometric one and not the usual concept of dimension in differential algebra. This hypothesis will allow us to connect to the theory in Section 2. We denote by $\mathcal{C}_{\mathcal{S}}$ this one-dimensional variety.

(2) For all $j \in J$, there exists $i \in \{1, \ldots, n\}$ such that $\deg_{w_i}(F_j) > 0$ or, equivalently, none of the equations in $\mathcal{S}$ is purely algebraic.

For finding solutions of $\mathcal{S}$, we will present two different approaches:

- project the system, compute solution candidates, and check whether they are indeed solutions;
- derive a simplified system of differential equations with the same solution set as $\mathcal{S}$, where solutions can be found directly.

The first approach is used for rational solutions. The projection can be applied either to the parametrization, as is done here, or to the rational solution, see [LSNW15, Section 5]. This approach works if the parametrizations and solutions are described in a closed form. Thus, for finding formal Puiseux series solutions, we have to simplify $\mathcal{S}$ first by (differential) elimination methods. As we will show in the examples, for computing rational solutions, this approach could be used as well.

3.1. **Rational solutions.** In this section, we analyze the existence, and actual computation, of rational solutions of $\mathcal{S}$. We start by observing that if $\hat{y}(x)$ is a nonconstant rational solution of $\mathcal{S}$, then $\hat{y}(x + c)$, where c is transcendental over $\overline{\mathbb{Q}}$, is a rational general solution (see [LSNW15, Theorem 2.3]). Therefore, for our purposes, it is enough to find one nonconstant rational solution. But first, we deduce some necessary conditions for the existence of such solutions. If there exists a nonconstant rational solution $\hat{y}$ of the system $\mathcal{S}$, then $\mathcal{P}_{\hat{y}} = (\hat{y}(x), \ldots, \hat{y}^{(n)}(x))$ is a parametrization of $\mathcal{C}_{\mathcal{S}}$. Thus, $\mathcal{C}_{\mathcal{S}}$ has to be a rational curve. Furthermore, $\mathcal{P}_{\hat{y}}$ is proper (see [LSNW15, Corollary 2.2]). Since all proper parametrizations of a curve are related by the composition with a Möbius transformation, the properness of $\mathcal{P}_{\hat{y}}$ implies that all proper parametrizations of $\mathcal{C}_{\mathcal{S}}$ must have, as first component, a nonconstant rational function. See also [LSNW15, Corollary 2.4] for the analysis of polynomial solutions. In addition to the above conditions, in [LSNW15, Section 3] the notion of partial degree of a space curve is introduced, and from there new sufficient conditions are derived (see [LSNW15, Theorem 4.1]).

Now, let us say that $\mathcal{C}_{\mathcal{F}}$ satisfies the *minimal* sufficient conditions mentioned above: let $\mathcal{C}_{\mathcal{F}}$ be rational, and let

$$(16) \qquad \mathcal{P} = (p_1(x), \ldots, p_{n+1}(x))$$

be a proper parametrization of the space curve $\mathcal{C}_{\mathcal{S}}$, where $p_1(x)$ is not constant. Then, the following theorem shows how to proceed in order to decide, and compute, the nonconstant rational solutions of $\mathcal{S}$.

Theorem 3.1 ([LSNW15, Theorem 4.2]). *Let $\mathcal{P}$ be as in* (16). *The following assertions are equivalent:*

(1) *$\mathcal{S}$ admits a nonconstant rational solution.*
(2) *There exist $a, b \in \overline{\mathbb{Q}}$, $a \neq 0$ such that either*
   (2.1) *$a\, p_j'(x) = p_{j+1}(x)$ for every $j = 1, \ldots, n$, or*
   (2.2) *$a(x - b)^2 p_j'(x) = p_{j+1}(x)$ for every $j = 1, \ldots, n$.*

Moreover, if one of these equivalent statements holds, then $\hat{y} = p_1(ax)$ (if (2.1) *holds) and $\hat{y} = p_1(\frac{abx-1}{ax})$ (if* (2.2) *holds) is a nonconstant rational solution of $\mathcal{S}$.*

Clearly, Theorem 3.1 provides an algorithm for our purposes. In [LSNW15] a second computational approach, based on a suitable projection of $\mathcal{C}_{\mathcal{F}}$ over a plane

curve, is discussed. Moreover, in [LSNW15] one can also find a detailed analysis on the optimality of the required field to express the coefficients of the rational general solution of $\mathcal{S}$. In the following, we illustrate these ideas by an example.

Example 3.2. We consider the set $\mathcal{F} = \{F_1, \ldots, F_7\} \subset \mathbb{Q}[w_0, w_1, w_2, w_3]$, where

$$
\left\{
\begin{array}{rcl}
F_1 & = & 27\,w_2{}^4 + 2\,w_3{}^3 \\
F_2 & = & 2\,w_1 w_3 - 3\,w_2{}^2 - 2\,w_3 \\
F_3 & = & 9\,w_1 w_2{}^2 - 9\,w_2{}^2 + w_3{}^2 \\
F_4 & = & 6\,w_1{}^2 - 12\,w_1 + w_3 + 6 \\
F_5 & = & w_0 w_3 - 3\,w_1 w_2 + 6\,w_2 + 2\,w_3 \\
F_6 & = & 3\,w_0 w_2 + 6\,w_1 + 6\,w_2 + w_3 - 6 \\
F_7 & = & 6\,w_0{}^2 w_1 - 6\,w_0{}^2 + 24\,w_0 w_1 - 24\,w_0 + 12\,w_1 - w_3 - 6
\end{array}
\right\}.
$$

The autonomous system $\mathcal{S}$ is obtained by the substitution $w_i = y^{(i)}$ for $i = 0, \ldots, 3$. The genus of the associated curve $\mathcal{C}_{\mathcal{F}}$ is zero, and $\mathcal{C}_{\mathcal{F}}$ can be properly parametrized as

$$
\mathcal{P} = \left(\frac{(x+1)^2}{3\,(2\,x-1)\,t}, \frac{(5\,x-1)\,(x+1)}{9\,x^2}, \frac{2\,(2\,x-1)^3}{27\,x^3}, -\frac{2\,(2\,x-1)^4}{27\,x^4} \right).
$$

Condition (2.1) in Theorem 3.1 does not hold. Nevertheless, condition (2.2) holds with

$$
a(x - b)^2 = -\frac{(2\,x-1)^2}{3}.
$$

So taking

$$
T(x) = \frac{2\,x+3}{4\,x},
$$

we get that

$$
\hat{y} = p_1(T(x)) = \frac{(2\,x+1)^2}{4\,x+6}
$$

is a nonconstant rational solution of $\mathcal{S}$. Therefore, the rational general solution of $\mathcal{S}$ is

$$
\hat{y}(x + c) = \frac{(2\,x+2\,c+1)^2}{4\,x+4\,c+6}.
$$

3.2. Formal power series solutions. In this section, we generalize the previous section to formal Puiseux series. More precisely, existence, uniqueness and convergence of formal Puiseux series solutions of $\mathcal{S}$ as in (15) are covered. In contrast to Section 3.1, we first simplify the given system and then apply the results from Section 2.1.

By using algebraic decomposition methods such as regular chains [Kal93], $\mathcal{S}$ can be decomposed into a finite union of subsystems $\mathcal{S}_k$ with the same set of *regular zeros*. In our case it turns out that regular zeros are the nonconstant Puiseux series solutions. The $\mathcal{S}_k$ are of the form

$$
(17) \qquad \mathcal{S}_k =
\left\{
\begin{array}{l}
G_1(y, y') = \sum_{j=0}^{r_1} G_{1,j}(y) \cdot (y')^j = 0 \\
G_2(y, y', y'') = \sum_{j=0}^{r_2} G_{2,j}(y, y') \cdot (y'')^j = 0 \\
\quad \vdots \\
G_m(y, \ldots, y^{(m)}) = \sum_{j=0}^{r_m} G_{m,j}(y, \ldots, y^{(m-1)}) \cdot (y^{(m)})^j = 0
\end{array}
\right\}
$$

with $r_j \geq 1$ and $G_{j,r_j} \neq 0$ for every $1 \leq j \leq m$. By using differential elimination methods (see [Rit50]), we can further simplify $\mathcal{S}_k$ to obtain a single differential equation of order one, namely an equation of the type

(18) $$H_k(y, y') = 0$$

such that $H_k \in \mathbb{Q}[y, y'] \setminus \{0\}$. We call H_k the *reduced differential equation* of $\mathcal{S}_k$.

Theorem 3.3 ([CFS21, Theorem 3]). *Let $\mathcal{S}$ be a differential system as in* (15). *Then there are reduced differential equations $H_1, \ldots, H_m \in \mathbb{Q}[y, y']$ such that the union of the nonconstant Puiseux series solutions is equal to the solution set of $\mathcal{S}$.*

Note that the cases $H_k = 1$ and $H_k \in \mathbb{Q}[y]$ are not excluded and are indeed possible. If this happens, there are no or only constant solutions, respectively, and the subsystem $\mathcal{S}_k$ and its reduced differential equation H_k can be neglected.

Hence, in order to find all nonconstant solutions (rational, algebraic, or formal Puiseux series solutions) of $\mathcal{S}$, we can analyze the reduced differential equations of the subsystems of $\mathcal{S}$. In particular, the convergence of the formal Puiseux series solutions and the computational bounds from Section 2.1 remain true [CFS21, Theorems 4 and 6]. Let us highlight the result on convergence:

Theorem 3.4. *Let $\mathcal{S}$ be a differential system as in* (15). *Every formal Puiseux series solution of $\mathcal{S}$, expanded around any finite point or around infinity, is convergent.*

From the reduced differential equations, all Puiseux series solutions and algebraic solutions can be computed. For the Puiseux series the representation of the solutions is done by a set of truncations such that they are in one-to-one correspondence with the series. Algebraic solutions are represented by their minimal polynomials.

Example 3.5. Let us continue Example 3.2 and apply the reduction process to the system $\mathcal{S}$. The regular chain decomposition of $\mathcal{S}$ consists of the single subsystem

$$\mathcal{S}_1 = \left\{ \begin{array}{l} G_1 = y^2 y' - y^2 + 4yy' + y'^2 - 4y = 0 \\ G_2 = (y+2)y'' + (2y^2 + 8y + 6)y' - 2y^2 - 8y - 4 = 0 \\ G_3 = y^{(3)} - (6y^2 + 24y + 12)y' + 6y^2 + 24y + 6 = 0 \end{array} \right\}.$$

By differential elimination, we obtain that G_2, G_3 are consequences of G_1 and the reduced differential equation is

$$H(y, y') = y^2 y' - y^2 + 4yy' + y'^2 - 4y = 0$$

(see Figure 5). Using the results from Section 2.1 on rational general solutions,

$$\hat{y} = \frac{(2x + 2c + 1)^2}{4x + 4c + 6}$$

can be deduced. The local solutions are given by the generic solution

$$y_g = y_0 + y_1 x + \mathcal{O}(x^2), \qquad y_1^2 + (y_0^2 + 4y_0)y_1 - y_0^2 - 4y_0 = 0, \qquad y_0 \notin \{0, -2, -4\}$$

and the solutions at the critical points by

$$\left\{ \begin{array}{c} -4, \ 0, \ \hat{y}_1 = x^2 - x^3 + \mathcal{O}(x^5), \ -2 + 2x \pm ix^2 + \mathcal{O}(x^3), \ -2 + \frac{1}{x} + x, \\ -4 - x^2 - x^3 + \mathcal{O}(x^3), \ x + \frac{1}{x} + \frac{c-2}{x^2} + \frac{c^2 - 4c + 4}{x^3} + \mathcal{O}(\frac{1}{x^4}) \end{array} \right\}.$$

By plugging the constant solutions of H into $\mathcal{S}$, it can be checked that they are not solutions. The generic solution y_g corresponds to the expansion of the rational
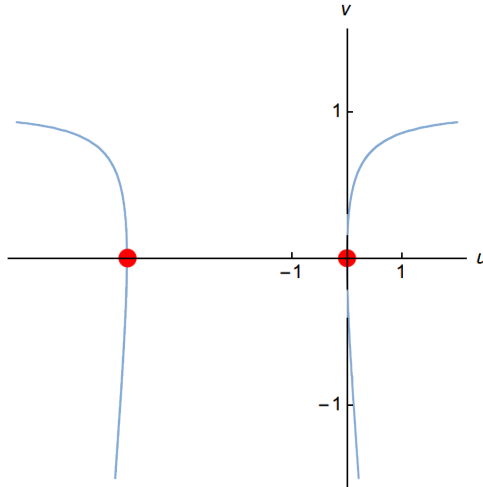
FIGURE 5. The real part of the plane curve $H(u,v) = 0$, obtained from the reduced differential equation of Example 3.5. The highlighted points denote the critical curve points.

general solution $\hat{y}$ with $y_0 = \frac{(2c+1)^2}{4c+6}$, $y_1 = \frac{4c^2+12c+5}{(2c+3)^2}$. The solution $\hat{y}_1$ is covered by $\hat{y}$ for $c = -1/2$. The other particular solutions at the critical points are not covered by $\hat{y}$ and, hence, do not correspond to rational solutions.

4. TRANSFORMATIONS OF AODES

As we have seen in the previous sections, in order to solve an AODE or a system of AODEs, we associate an algebraic variety, and from its properties we derive, if possible, the solutions belonging to some class of functions as the rational functions. On the other hand, in the study of algebraic varieties, birational transformations play an important role since they preserve the essential information of the geometric object. So, the natural question arises: how do these transformations affect the differential equation or a system of differential equations? In this section we study the subset, indeed a group, of these transformations that preserve the rational solvability of the differential equation establishing a direct connection on the sets of rational solutions of the given AODE and its transformation. The ideas and results presented here have been elaborated in [NSW12b], [NSW15]; see also [NH20].

In this situation, we introduce the set

$$(19) \quad \mathcal{AODE}^{(n)} = \{F(x, y, \ldots, y^{(n)}) = 0 \mid F \in \mathbb{K}[x, u_0, \ldots, u_n], \ \deg_{u_n}(F) > 0\},$$

where $\mathbb{K}$ is an algebraically closed field of characteristic zero. Moreover, for $F \in \mathcal{AODE}^{(n)}$, we denote by $\mathbb{V}(F)$ the hypersuface defined by F in $\mathbb{A}^{n+2}(\mathbb{K})$. Note that in Section 2.2, where F is of order one, i.e., $F \in \mathcal{AODE}^{(1)}$, we have introduced the surface $\mathcal{S}_F$ that, with the notation here, is $\mathcal{S}_F = \mathbb{V}(F) \subset \mathbb{A}^3(\mathbb{K})$. Similarly, in Section 2.1, if $F \in \mathcal{AODE}^{(1)}$ is autonomous, we have associated to the differential equation a plane algebraic curve that, in fact, can be seen as a cylinder in $\mathbb{A}^3(\mathbb{K})$.

Additionally, we also introduce the set

$$(20) \quad \mathcal{PODE}^{(n)} = \{F \in \mathcal{AODE}^{(n)} \mid \mathbb{V}(F) \text{ is birationally parametrizable}\}.$$

4.1. **Affine linear transformations.** We start by analyzing the case of affine linear transformations. We describe the theory for first-order AODEs, but it can be easily extended to other orders, as we show in Section 4.2. We consider the group $\mathcal{A}$ of all affine linear transformations of $\mathbb{A}^3(\mathbb{K})$

$$(21) \qquad \begin{aligned} L: \quad \mathbb{A}^3(\mathbb{K}) &\rightarrow \quad \mathbb{A}^3(\mathbb{K}) \\ v &\mapsto \quad A\,v + B, \end{aligned}$$

where A is a nonsingular 3×3 matrix over $\mathbb{K}$ and B is a 3×1 matrix over $\mathbb{K}$. By abuse of notation, we will also denote by L the natural extension of the map L to $\mathbb{A}^3(\mathbb{K}(x))$.

We represent L, given as in (21), as the pair of matrices $L = [A, B]$. In this situation, we introduce the set

$$(22) \qquad \mathcal{G} = \left\{ \left[\begin{pmatrix} 1 & 0 & 0 \\ b & a & 0 \\ 0 & 0 & a \end{pmatrix}, \begin{pmatrix} 0 \\ c \\ b \end{pmatrix} \right] \in \mathcal{A} \text{ where } a, b, c \in \mathbb{K} \text{ and } a \neq 0 \right\}.$$

Note that $\mathcal{G}$ is a subgroup of $\mathcal{A}$.

$\mathcal{G}$ defines a left group action on each of the sets $\mathcal{AODE}^{(1)}$ and $\mathcal{PODE}^{(1)}$ and, in consequence, the corresponding orbits induce a partition of $\mathcal{AODE}^{(1)}$ and $\mathcal{PODE}^{(1)}$, respectively. More precisely, given $F, G \in \mathcal{AODE}^{(1)}$ (similarly for $F, G \in \mathcal{PODE}^{(1)}$), the equivalence relation is defined as $F \sim G$ if and only there exists $L \in \mathcal{G}$ such that $F \circ L^{-1} = G$.

The important fact is that the solvability (in particular, the rational solvability) is invariant within the equivalent classes: if $G = F \circ L^{-1}$, with $L \in \mathcal{G}$ and $F, G \in \mathcal{AODE}^{(1)}$, and $y(x)$ is a solution of $G(x, y, y') = 0$ and $(x, \hat{y}, \hat{y}') := L^{-1}(x, y(x), y'(x))$, then $\hat{y}$ is a solution of $F(x, y, y') = 0$.

Moreover, in [NSW12b, Theorem 3.1] it is shown that, in the classes induced in $\mathcal{PODE}^{(1)}$, the associated system (see (9)) is preserved. We refer the reader to [NSW12b] for a detailed analysis of associated systems for different types of elements in $\mathcal{PODE}^{(1)}$.

Let us illustrate these ideas by means of an example.

Example 4.1. We consider the differential equation $F(x, y, y') = 0$, where

$$\begin{aligned} F(x, y, y') = {}& 20\,y^3 + 120\,y^2 x + 240\,yx^2 + 160\,x^3 + 181\,y^2 + 724\,yx \\ &+ 20\,yy' + 724\,x^2 + 40\,xy' - 25\,y'^2 + 586\,y + 1172\,x \\ &- 39\,y' + 571. \end{aligned}$$

First we check whether there exists an autonomous AODE in the equivalence class of F. For this purpose, we compute $F \circ L^{-1}$ for $L \in \mathcal{G}$ generic, we collect the coefficients of $F \circ L^{-1}$ in $\{y, y'\}$, and we analyze the existence of solutions of the algebraic system of equations, in the parameters that define L, corresponding to x. In this case, one gets that by taking

$$[A, B] = \left[\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \right],$$

$G := F \circ L^{-1}$ is autonomous. Indeed

$$G(x, y, y') = 20\,y^3 + y^2 + 20\,yy' - 25\,y'^2 + y'.$$

Applying the ideas in Section 2.1 (see Example 2.6), one gets that

$$\hat{y} = \frac{x + c + 6}{(x + c + 1)^2}$$

is a rational general solution of $G(x, y, y') = 0$. Now,

$$L^{-1}(x, \hat{y}_g(x), \hat{y}_g'(x)) = \left(x, -2\,x + \frac{x + c + 6}{(x + c + 1)^2} - 3, -2 + \frac{-x - c - 11}{(x + c + 1)^3} \right).$$

Therefore,

$$-2\,x + \frac{x + c + 6}{(x + c + 1)^2} - 3$$

is a rational general solution of the initial equation $F(x, y, y') = 0$.

4.2. **Birational transformations.** In this section we treat a more general case. On one hand, we consider AODEs of order n, and, on the other, we extend the analysis to birational transformations. For this purpose, let $\mathcal{T}$ denote the set of all birational transformations of $\mathbb{A}^{n+2}(\mathbb{K})$. In the following we introduce a subgroup of $\mathcal{T}$ that will preserve the information of the differential equation. First, we consider the Möbius transformations of $\mathbb{A}^1(\mathbb{K}(x))$, that is, rational functions of the form

$$\frac{a(x)u_0 + b(x)}{c(x)u_0 + d(x)},$$

where $a, b, c, d \in \mathbb{K}[x]$ and $ad - bc \neq 0$. Let $\mathcal{M}$ consist in the set of all Möbius transformations of $\mathbb{K}(x)$. Then, for every $L \in \mathcal{M}$, we consider the rational map defined as

(23) $\qquad \Phi_L = (\Phi_1(x), \Phi_2(x, u_0), \Phi_3(x, u_0, u_1), \ldots, \Phi_{n+2}(x, u_0, \ldots, u_n)),$

where

(24) $\qquad \begin{cases} \Phi_1 = x \\ \Phi_2 = L(x, u_0) \\ \Phi_3 = \dfrac{\partial \Phi_2}{\partial x} + u_1 \dfrac{\partial \Phi_2}{\partial u_0} \\ \Phi_r = \dfrac{\partial \Phi_{r-1}}{\partial x} + u_1 \dfrac{\partial \Phi_{r-1}}{\partial u_0} + \cdots + u_{r-2} \dfrac{\partial \Phi_{r-1}}{\partial u_{r-3}} \,, 3 < r \leq n+2, \end{cases}$

and we define the set of transformations as

(25) $\qquad\qquad\qquad \mathcal{G}^{(n)} = \{\Phi_L \mid L \in \mathcal{M}\}.$

In [NSW15, Proposition 2.1] it is proved that $\mathcal{G}^{(n)}$ is a subgroup of $\mathcal{T}$. Similarly, as in Section 4.1, this group generates a left group action on $\mathcal{AODE}^{(n)}$, and on $\mathcal{PODE}^{(n)}$, that induces an equivalence relation. Namely, if $F, G \in \mathcal{AODE}^{(n)}$, similarly if $F, G \in \mathcal{PODE}^{(n)}$, then F and G are related if there exists $\Phi \in \mathcal{G}^{(n)}$ such that $F \circ \Phi^{-1} = G$. In this situation, in [NSW15, Theorem 3.1], it is stated that the existence of rational solutions is an invariant property within each equivalence class. Moreover, in [NSW15, Theorem 3.2], it is proved that the associated system for all elements in a class, given by an equation in $\mathcal{PODE}^{(n)}$, are related. Also, let us mention that [NSW15, Theorem 3.3] shows how to determine the transformed equation, via an element in $\mathcal{G}^{(n)}$, of an element in $\mathcal{AODE}^{(n)}$.

Similarly as in Section 4.1, the equivalence class of a nonautonomous equation could contain an autonomous equation, and hence it might be possible to simplify

the process of solving the first given equation. Nevertheless, differently from what happened in Section 4.1, here it is still an open problem to derive a complete algorithm for this task. Some special situation can be treated as explained in [NSW12a]. Let us briefly give some hints here. First of all, we say that an AODE $G(x, y, y', \ldots, y^{(n)}) = 0$ is *normal* if the leading coefficient of G w.r.t. $y^{(n)}$ belongs to $\mathbb{K}[x, y]$; note that all first-order AODEs are normal. In this situation, the following result holds (see [NSW12a, Theorem 4.1]).

Theorem 4.2. *Let $G \in \mathcal{AODE}^{(n)}$ be an nth-order normal differential equation, and let $M(x, u_1)$ be the leading coefficient of $G(x, u_0, \ldots, u_n)$ w.r.t. u_n.*

  (1) *If M has a nonlinear irreducible factor over $\mathbb{K}[x]$, then no element in the equivalence class of G is autonomous.*
  (2) *If $a_1 u_0 + b_1$ and $a_2 u_0 + b_2$ are two different linear factors of M over $\mathbb{K}[x]$, then the possible transformations $\Phi_L \in \mathcal{G}^{(n)}$, such that $\Phi_L \cdot G$ is autonomous, are defined by*

$$\text{either} \quad L^{-1} = \frac{b_1 u_0 - b_2}{-a_1 u_0 + a_2} \quad \text{or} \quad L^{-1} = \frac{(b_2 - b_1) u_0 + b_1}{-(a_2 - a_1) u_0 - a_1}.$$

In this case, if none of these functions, for every pair of linear factors, transforms G into an autonomous AODE, then there is no autonomous AODE in the equivalence class of G.

Let us finish this section illustrating the previous ideas by an example.

Example 4.3. We consider the second-order AODE $F(x, y, y', y'') = 0$, where

$$
\begin{aligned}
F(x, y, y', y'') = {}& x^5 y'' + 4\, x^4 y y'' - 2\, x^4 y'^2 + 6\, x^3 y^2 y'' - 6\, x^3 y y'^2 \\
& - 4\, x^3 y'^3 + 4\, x^2 y^3 y'' - 6\, x^2 y^2 y'^2 + x y^4 y'' - 2\, x y^3 y'^2 \\
& - 2\, x^4 y' - 4\, x^3 y y' + 12\, x^2 y y'^2 + 4\, x y^3 y' + 2\, y^4 y' \\
& + 2\, x^3 y + 6\, x^2 y^2 + 6\, x y^3 - 12\, x y^2 y' + 2\, y^4 + 4\, y^3.
\end{aligned}
$$

Its defining polynomial $F(x, u_0, u_1, u_2)$ has leading coefficient (w.r.t. u_2)

$$x^5 + 4\, x^4 u_0 + 6\, x^3 u_0^2 + 4\, x^2 u_0^3 + x u_0^4,$$

that factors as

$$x (u_0 + x)^4.$$

Therefore, according to Theorem 4.2, the possible $\Phi_L \in \mathcal{G}^{(n)}$ are those such that

$$L^{-1} \in \left\{ -\frac{u_0 x + x}{u_0}, \frac{-2\, u_0 x + x}{u_0 - 1}, -u_0 x - x, -\frac{2\, u_0 x - x}{u_0} \right\}.$$

Taking the first option, i.e., $L = -\frac{x}{u_0 + x}$, we get that (see [NSW15, Theorem 3.3])

$$G := \Phi_L \cdot F = F(\Phi_{L^{-1}}).$$

The primitive part w.r.t. $\{u_0, u_1, u_2\}$ of the numerator of the above rational function is

$$G(x, u_0, u_1, u_2) = -4 u_1^3 + u_2.$$

Hence, the autonomous AODE

$$G(x, y, y', y'') = -4 y'^3 + y'' = 0$$

belongs to the equivalence class of the nonautonomous AODE $F(x, y, y', y'') = 0$. The equation $G(x, y, , y', y'') = 0$ has the solution

$$\hat{y} = -\frac{c_1 + x}{\sqrt{-2\,c_1 - 2\,x}} + c_2.$$

Thus, taking into account [NSW15, Remark 2.1],

$$L(x, \hat{y}) := -\frac{x\left(-c_2\,\sqrt{-2\,c_1 - 2\,x} - \sqrt{-2\,c_1 - 2\,x} + c_1 + x\right)}{-c_2\,\sqrt{-2\,c_1 - 2\,x} + c_1 + x}$$

is a solution of the original AODE $F(x, y, y', y'') = 0$.

5. SOFTWARE

In this section, we present the structure and content of the software packages `AGADE` and `FirstOrderSolve`. These packages are developed for the popular computer algebra system Maple and consist of the algorithmic methods introduced in Section 2.

5.1. The Maple package `AGADE`. The package `AGADE` (Algebro-Geometric methods for solving Algebraic Differential Equations) implements several algebro-geometric methods for computing rational general solutions of first-order AODEs. The package can be obtained via the website `https://risc.jku.at/sw/agade/`. Download the library `AGADE.mla` from the online repository and make sure that Maple can find this file in its library path. Afterward, the package is loaded via:

```
> with(AGADE);
[RGSautonomousFOAODE, RGScurveParametrizableFOAODE,
    ↪ RGSplanarRationalSystem, RGSsurfaceParametrizableFOAODE]
```

where the abbreviations `RGS` and `FOAODE` stand for rational general solution and first-order AODE, respectively. These four commands perform the following tasks:

- `RGSautonomousFOAODE`: an implementation of the algorithm from Section 2.1 for computing rational general solutions of autonomous first-order AODEs.
- `RGScurveParametrizableFOAODE`: implements the algorithm for computing rational general solutions of first-order AODEs via curve parametrization, cf. Section 2.2.
- `RGSsurfaceParametrizableFOAODE`: an implementation of the algorithm from Section 2.2 for computing rational general solutions of first-order AODEs via surface parametrization.
- `RGSplanarRationalSystem`: computes a rational general solution of a planar rational system. This is the preferred method for finding solutions of the associated planar system for the surface parametrization approach (see [NW11b, Algorithm RATSOLVE]).

All methods, except for the last,[5] take as input a first-order AODE $F(x, y, y') = 0$, where $F \in \overline{\mathbb{Q}}[x, y, y']$,[6] and two symbols y and x denoting the dependent and the independent variable of the differential equation, respectively. For example,

---

[5]The method `RGSplanarRationalSystem` takes as input the planar rational system in the form of a list of equations, symbols denoting the two dependent variables and the independent variable, and a degree bound for the rational first integrals of the system.

[6]Note that F must be an element of $\overline{\mathbb{Q}}[y, y']$ if the first method is used.

consider the autonomous first-order AODE $20y^3 + y^2 + 20yy' - 25y'^2 + y' = 0$ from Section 2.1. We wish to compute a rational general solution with the method for autonomous AODEs. This can be accomplished using the subsequent commands:

```
> F := 20*y(x)^3 + y(x)^2 + 20*y(x)*diff(y(x), x) - 25*diff(y(x), x)^2
    ↪  + diff(y(x), x);
3 2 /d \ /d \2 /d \
F := 20 y(x) + y(x) + 20 y(x) |-- y(x)| - 25 |-- y(x)| + |-- y(x)|
\dx / \dx / \dx /

> RGSautonomousFOAODE(F=0, y, x);
25 _C1 + 25 x + 120
-------------------
2
(5 x + 5 _C1 - 1)
```

All commands support additional arguments which can be used to pass specific parametrizations to the algorithms or to determine information about the used parametrization and the constructed associated equations. A complete specification can be found in Mitteramskogler [Mit21].

Finally, consider the AODE $x^2y'^2 - 2xyy' - 5y'^2 - x^2 = 0$. We want to solve this differential equation with the method via curve parametrization and determine the used parametrization[7] and the constructed associated differential equation. For this, we pass the additional argument `extendedOutput=true`:

```
> F := x^2*diff(y(x), x)^2 - 2*x*y(x)*diff(y(x), x) - 5*diff(y(x), x)
    ↪  ^2 - x^2;
2 /d \2 /d \ /d \2 2
F := x |-- y(x)| - 2 x y(x) |-- y(x)| - 5 |-- y(x)| - x
\dx / \dx / \dx /

> output := RGScurveParametrizableFOAODE(F=0, y, x, extendedOutput=
    ↪ true):
> output:-Solution;
2 2
-1 + (x - 5) _C1
-----------------
2 _C1

> output:-Parametrization;
2 2 2 2
T x - 5 T - x
[-----------------, T, T]
2 x T

> output:-AssocQuasilinEquation;
d T(x)
-- T(x) = ----
dx x
```

Additional examples of how to use the package `AGADE` can be found in the online demo file `AGADE_Demo.mw`.

----

[7]The third argument of the output parametrization is a symbol, denoting the name of the parameter.

5.2. **The Maple package `FirstOrderSolve`.** The Maple package `FirstOrder-Solve` contains several procedures that implement algorithms for computing algebraic and (formal) Puiseux series solutions of first-order autonomous AODEs (cf. Section 2.1). The package is available at the online repository `https://risc.jku.at/sw/firstordersolve/`. Detailed information on the commands can be found in the included help or in [BCFS21].

After downloading the library, the package is initialized by the command

```
> with(FirstOrderSolve);
[AlgebraicSolution, GenericSolutionTruncation,
    ↪ ProlongSolutionTruncation, SolutionTruncations]
```

These main procedures perform the following tasks:

- `SolutionTruncations`: for computing all formal Puiseux series solutions, expanded around zero and around infinity;
- `AlgebraicSolution`: for computing the minimal polynomials of the algebraic solutions;
- `GenericSolutionTruncation`: for computing a truncation of the solutions with noncritical initial values;
- `ProlongSolutionTruncation`: for prolonging the solution truncations up to a higher degree.

For every command a differential polynomial $F \in \mathbb{Q}[y, y']$ is required as input. In case a specific precision of the Puiseux series solutions (the degree of its truncations) is desired, this number has to be given as input. Otherwise, this value will be chosen minimally such that the output truncations are distinct and the truncations are in one-to-one correspondence with the solutions. A specific initial value for the solutions is optional. There are several more options available such as avoiding factorization of F and omitting solutions of a specific type, for example Puiseux series solutions expanded around infinity.

Let us demonstrate the package for Examples 2.10 and 2.14.

```
> F := y(x)^4 + 3*diff(y(x), x);
F := y(x)^4 + 3*diff(y(x), x)

> AlgebraicSolution(F);
{x*y^3 + y^3 - 1}

> SolutionTruncations(F);
[{[_CC - 1/3*_CC^4*x, {}]}, {0, RootOf(_Z^3 - 1)^2/x^(1/3)}, {RootOf(
    ↪ _Z^3 - 1)/x^(1/3) + _CC/x^(4/3)}]
```

The first output covers the algebraic solutions given by the shift in the output minimal polynomial. The generic solution and the two additional power series solutions correspond to particular initial values and expansion points (cf. Example 2.13).

For the second example we first compute the generic solution. Then, we use the optional argument for computing the Puiseux series solutions at two exceptional initial values, namely $y(0) = 0$ and $y(0) = \alpha$, where $36\alpha^4 - 16\alpha^2 + 9 = 0$.

```
> G := diff(y(x), x)^4 - diff(y(x), x)^2 - y(x)^4 + 4*y(x)^2/9;
G := diff(y(x), x)^4 - diff(y(x), x)^2 - y(x)^4 + (4*y(x)^2)/9

> GenericSolutionTruncation(G, 1);
{[_CC + RootOf(-9*_CC^4 + 9*_Z^4 + 4*_CC^2 - 9*_Z^2)*x, {0, -2/3,
    ↪ RootOf(36*_Z^4 - 16*_Z^2 + 9), 2/3}]}
```

```
> SolutionTruncations(G, iv=0);
{0, x, -x}

> SolutionTruncations(G, iv=RootOf(36*_Z^4 - 16*_Z^2 + 9));
{RootOf(36*_Z^4 - 16*_Z^2 + 9) + RootOf(2*_Z^2 - 1)*x + ((4*RootOf(65*
    ↪  _Z^2 + 144*RootOf(2*_Z^2 - 1)*RootOf(36*_Z^4 - 16*_Z^2 + 9)^3 +
    ↪  98*RootOf(2*_Z^2 - 1)*RootOf(36*_Z^4 - 16*_Z^2 + 9))*RootOf
    ↪  (36*_Z^4 - 16*_Z^2 + 9)^3)/3 - (8*RootOf(65*_Z^2 + 144*RootOf
    ↪  (2*_Z^2 - 1)*RootOf(36*_Z^4 - 16*_Z^2 + 9)^3 + 98*RootOf(2*_Z^2
    ↪  - 1)*RootOf(36*_Z^4 - 16*_Z^2 + 9))*RootOf(36*_Z^4 - 16*_Z^2 +
    ↪  9))/27)*x^(3/2)}
```

6. Conclusion and outlook

We have presented the algebro-geometric method for explicitly solving algebraic ordinary differential equations of order one. The central feature of this method consists of associating to the given differential equation an algebraic set, parametrizing this algebraic set, and then transferring this parametrization—if possible—to a solution of the differential equation. Different classes of solution formulas have been considered, such as rational, algebraic, and formal Puiseux series solutions. For some of these solution classes we have given decision procedures. Algebro-geometric transformations may lead to formulations which are advantageous for solving; we have dealt with affine and birational transformations. And finally we have briefly described two Maple packages implementing some of these approaches.

This new symbolic approach to solving algebraic differential equations might open a door to a wide landscape of differential problems. Higher-order equations need a closer analysis of the associated algebraic set. The case of partial differential equations needs to be investigated in the future.

Another possibility for further research is to study necessary field extensions in every step of the computations and restrict to real solutions only. In this way, also the presented algorithms might be adapted and optimized.

Appendix A. Formal power series

Here we recall some classical terminology from [Wal50]. Let $\mathbb{K}$ be a field of characteristic zero. The main algebraic structures used in this paper are listed in Table 1. Formal power series are infinite sums that can be manipulated with the usual algebraic operations on series (addition, subtraction, multiplication, division, differentiation, etc.). Formal Puiseux series are a special type of formal power series. When expanded around zero, formal Puiseux series with coefficients in $\mathbb{K}$ are of the form

$$\varphi = \sum_{i \geq k} c_i\, x^{i/n}.$$

Formal Puiseux series expanded around infinity are of the form

$$\varphi = \sum_{i \geq k} c_i\, x^{-i/n}.$$

Here $c_i \in \mathbb{K}$, $k \in \mathbb{Z}$, and $n \in \mathbb{N}^*$. In case that $n = 1$, we may also speak about formal Laurent series.

With the change of variables $\tilde{x} = x - x_0$, where $x_0 \in \mathbb{K}$, a formal Puiseux series can be expanded around any (finite) point x_0 instead of 0. With the change of variables $\tilde{x} = 1/x$, the series can be expanded around infinity.

Let us recall the relations $\mathbb{K}[x] \subset \mathbb{K}(x) \subset \mathbb{K}\{\{x\}\}$, $\mathbb{K}[x] \subset \mathbb{K}[[x]]$, $\mathbb{K}(x) \subset \mathbb{K}((x))$, $\mathbb{K}\{\{x\}\} \subset \mathbb{K}\langle\langle x \rangle\rangle$, and

$$\mathbb{K}[[x]] \subset \mathbb{K}((x)) \subset \mathbb{K}\langle\langle x \rangle\rangle.$$

It is well known that $\mathbb{K}((x))$ is the fraction field of $\mathbb{K}[[x]]$ and that

$$\mathbb{K}\langle\langle x \rangle\rangle = \bigcup_{n \geq 1} \mathbb{K}((x^{1/n})).$$

For algebraically closed $\mathbb{K}$, $\mathbb{K}\langle\langle x \rangle\rangle$ is the algebraic closure of $\mathbb{K}((x))$. The minimal natural number n such that $\varphi \in \mathbb{K}((x^{1/n}))$ is called the *ramification index* of φ. Note that the ramification index of formal Puiseux series with integer exponents and formal Laurent series is equal to one. The *order* of the series φ is defined as the least index k/n with nonzero coefficient c_k.

Algebraic Puiseux series $\varphi \in \mathbb{K}\{\{x\}\}$ are formal Puiseux series which are the root of a nontrivial polynomial $Q(x,y) \in \mathbb{K}(x)[y]$, i.e., $Q(x,\varphi) = 0$. Algebraic Puiseux series with coefficients in $\mathbb{C}$ have a positive radius of convergence. Since not every formal power series with nonnegative integer exponents is algebraic, the inclusion $\mathbb{K}\{\{x\}\} \subset \mathbb{K}\langle\langle x \rangle\rangle$ is strict.

The composition of formal Puiseux series $f \circ g$ is well defined and again a formal Puiseux series as long as $\mathrm{ord}(g) > 0$. If both series f and g have positive radius of convergence, then also the composition is convergent as Puiseux series.

Appendix B. Local parametrizations

Let $F \in \mathbb{K}[x,y]$ define a plane algebraic curve $\mathcal{C}_F$ in $\mathbb{A}^2(\mathbb{K})$, where $\mathbb{K}$ is algebraically closed. A *local parametrization* centered at $(a_0, b_0) \in \mathcal{C}_F$ is a pair of formal Laurent series $A(t) \in \mathbb{K}((t))^2 \setminus \mathbb{K}^2$ such that $A(0) = (a_0, b_0)$ and $F(A(t)) = 0$. In the set of all local parametrizations of $\mathcal{C}_F$, we introduce the equivalence relation $\sim$ by defining $A(t) \sim B(t)$ if and only if there exists a formal power series $T(t) \in \mathbb{K}[[t]]$ of order one such that $A(T(t)) = B(t)$. A local parametrization is said to be *reducible* if it is equivalent to another one in $\mathbb{K}((t^m))^2$ for some $m > 1$. Otherwise, it is called irreducible. An equivalence class of an irreducible local parametrization $(a(t), b(t))$ is called a *place* of $\mathcal{C}_F$ centered at the common center point (a_0, b_0) and is denoted by $[(a(t), b(t))]$. Observe that the components of the local parametrizations in the same place have the same order.

In the case of finite a_0, the local parametrizations are obtained from the formal Puiseux series φ, expanded around a_0, by solving

$$\tilde{F}(x,y) = F(x - a_0, y) = 0.$$

Since $\mathbb{K}\langle\langle x \rangle\rangle$ is algebraically closed, there are $\deg_y(F)$ many such Puiseux expansions. If $n \in \mathbb{N}^*$ is the ramification index of φ, then $(t^n, \varphi(t^n))$ are local parametrizations of $\mathcal{C}_{\tilde{F}}$ and $(a_0 + t^n, \varphi(t^n - a_0))$ are local parametrizations of the original curve $\mathcal{C}_F$ centered at the (a_0, b_0) with possibly distinct $b_0 = \varphi(0 - a_0)$. Such local parametrizations are called *classical Puiseux parametrizations* centered above a_0 and they are unique up to the substitution $t = \sqrt[n]{1}\, t$. The case $a_0 = \infty$ can be treated by suitable change of coordinates. In algebraic geometry one would consider $F(1/x, y)$. However, in the algebro-geometric approach presented in this paper, the

polynomial F also defines a differential equation such that $F(1/x, -x/y^2)$ has to be considered in order to keep the differential relation.

Acknowledgments

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References

[ACFG05] J. M. Aroca, J. Cano, R. Feng, and X. S. Gao, *Algebraic general solutions of algebraic ordinary differential equations*, ISSAC'05, ACM, New York, 2005, pp. 29–36, DOI 10.1145/1073884.1073891. MR2280526

[BCFS21] F. Boulier, J. Cano, S. Falkensteiner and J.R. Sendra, *Puiseux series and algebraic solutions of first order autonomous AODEs—a MAPLE package*, in Maple in Mathematics Education and Research (MC 2020), pp. 89–103. Springer International Publishing, 2021.

[Can93]   J. Cano, *An extension of the Newton-Puiseux polygon construction to give solutions of Pfaffian forms* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **43** (1993), no. 1, 125–142. MR1209698

[CFRS21] J. Cano, S. Falkensteiner, and J. R. Sendra, *Algebraic, rational and Puiseux series solutions of systems of autonomous algebraic ODEs of dimension one*, Math. Comput. Sci. **15** (2021), no. 2, 189–198, DOI 10.1007/s11786-020-00478-w. MR4260681

[CFS20]   J. Cano, S. Falkensteiner, and J. R. Sendra, *Existence and convergence of Puiseux series solutions for autonomous first order differential equations*, J. Symbolic Comput. **108** (2022), 137–151, DOI 10.1016/j.jsc.2020.06.010. MR4292378

[CFS21]   J. Cano, S. Falkensteiner, and J. R. Sendra, *Algebraic, rational and Puiseux series solutions of systems of autonomous algebraic ODEs of dimension one*, Math. Comput. Sci. **15** (2021), no. 2, 189–198, DOI 10.1007/s11786-020-00478-w. MR4260681

[CLO05]   D. A. Cox, J. Little, and D. O'Shea, *Using algebraic geometry*, 2nd ed., Graduate Texts in Mathematics, vol. 185, Springer, New York, 2005. MR2122859

[CM05]   G. Chen and Y. Ma, *Algorithmic reduction and rational general solutions of first order algebraic differential equations*, Differential equations with symbolic computation, Trends Math., Birkhäuser, Basel, 2005, pp. 201–212, DOI 10.1007/3-7643-7429-2_12. MR2187381

[Duv89]   D. Duval, *Rational Puiseux expansions*, Compositio Math. **70** (1989), no. 2, 119–154. MR996324

[Ere98]   A. Eremenko, *Rational solutions of first-order differential equations*, Ann. Acad. Sci. Fenn. Math. **23** (1998), no. 1, 181–190. MR1601867

[Fal20]   S. Falkensteiner, *Power series solutions of AODEs—Existence, uniqueness, convergence and computation*, PhD Thesis, available online at `www.risc.jku.at/publications/download/risc_6202/20-13.pdf`, 2020.

[FG04]   R. Feng and X.-S. Gao, *Rational general solutions of algebraic ordinary differential equations*, ISSAC 2004, ACM, New York, 2004, pp. 155–162, DOI 10.1145/1005285.1005309. MR2126938

[FG06]   R. Feng and X.-S. Gao, *A polynomial time algorithm for finding rational general solutions of first order autonomous ODEs*, J. Symbolic Comput. **41** (2006), no. 7, 739–762, DOI 10.1016/j.jsc.2006.02.002. MR2232199

[FS20]   S. Falkensteiner and J. R. Sendra, *Solving first order autonomous algebraic ordinary differential equations by places*, Math. Comput. Sci. **14** (2020), no. 2, 327–337, DOI 10.1007/s11786-019-00431-6. MR4096370

[GLSW18] G. Grasegger, A. Lastra, J. R. Sendra, and F. Winkler, *Rational general solutions of systems of first-order algebraic partial differential equations*, J. Comput. Appl. Math. **331** (2018), 88–103, DOI 10.1016/j.cam.2017.10.010. MR3720262

[Gra15]   G. Grasegger, *Symbolic solutions of first-order algebraic differential equations*, PhD Thesis, available online at `https://epub.jku.at/obvulihs/content/titleinfo/753082?lang=en`, 2015.

[Hub96]   E. Hubert, *The general solution of an ordinary differential equation*, in Proceedings of the 1996 international symposium on symbolic and algebraic computation, pp. 189–195, 1996.

[Inc26]   E. L. Ince, *Ordinary Differential Equations*, Dover Publications, New York, 1944. MR0010757

[Kal93]   M. Kalkbrener, *A generalized Euclidean algorithm for computing triangular representations of algebraic varieties*, J. Symbolic Comput. **15** (1993), no. 2, 143–167, DOI 10.1006/jsco.1993.1011. MR1218756

[Kol73]   E. R. Kolchin, *Differential algebra and algebraic groups*, Pure and Applied Mathematics, Vol. 54, Academic Press, New York–London, 1973. MR0568864

[Kov86]   J. J. Kovacic, *An algorithm for solving second order linear homogeneous differential equations*, J. Symbolic Comput. **2** (1986), no. 1, 3–43, DOI 10.1016/S0747-7171(86)80010-4. MR839134

[LSNW15] A. Lastra, J. R. Sendra, L. X. C. Ngô, and F. Winkler, *Rational general solutions of systems of autonomous ordinary differential equations of algebro-geometric dimension one*, Publ. Math. Debrecen **86** (2015), no. 1-2, 49–69, DOI 10.5486/PMD.2015.6032. MR3300577

[Mit21]   J.J. Mitteramskogler, *AGADE—a Maple package for computing rational general solutions of parametrizable first-order algebraic ODEs*, in Maple in Mathematics Education and Research (MC 2020), pp. 268–287. Springer International Publishing, 2021.

[MW22]   J. J. Mitteramskogler and F. Winkler, *Symbolic solutions of algebraic ODEs: a comparison of methods*, Publ. Math. Debrecen **100** (2022), no. 1-2, 143–166, DOI 10.5486/pmd.2022.9100. MR4389251

[Ngô11]   L. X. C. Ngô, *Rational general solutions of first-order algebraic ODEs*, PhD Thesis, available online at `www.risc.jku.at/publications/download/risc_4390/Dissertation-final.pdf`, 2011.

[NH20]   L. X. C. Ngô and T. T. Ha, *Möbius transformations on algebraic ODEs of order one and algebraic general solutions of the autonomous equivalence classes*, J. Comput. Appl. Math. **380** (2020), 112999, 9, DOI 10.1016/j.cam.2020.112999. MR4102762

[NSW12a] L. X. C. Ngô, J.R. Sendra, and F. Winkler, *Birational transformations on algebraic ordinary differential equations*, RISC Report Series 12–18, Research Institute for Symbolic Computation (RISC), Johannes Kepler University Linz, 2012.

[NSW12b] L. X. C. Ngô, J. R. Sendra, and F. Winkler, *Classification of algebraic ODEs with respect to rational solvability*, Computational algebraic and analytic geometry, Contemp. Math., vol. 572, Amer. Math. Soc., Providence, RI, 2012, pp. 193–210, DOI 10.1090/conm/572/11361. MR2953830

[NSW15] L. X. C. Ngô, J. R. Sendra, and F. Winkler, *Birational transformations preserving rational solutions of algebraic ordinary differential equations*, J. Comput. Appl. Math. **286** (2015), 114–127, DOI 10.1016/j.cam.2015.03.007. MR3336231

[NW10] L. X. C. Ngô and F. Winkler, *Rational general solutions of first order non-autonomous parametrizable ODEs*, J. Symbolic Comput. **45** (2010), no. 12, 1426–1441, DOI 10.1016/j.jsc.2010.06.018. MR2733387

[NW11a] L. X. C. Ngô and F. Winkler, *Rational general solutions of parametrizable AODEs*, Publ. Math. Debrecen **79** (2011), no. 3-4, 573–587, DOI 10.5486/pmd.2011.5121. MR2907990

[NW11b] L. X. C. Ngô and F. Winkler, *Rational general solutions of planar rational systems of autonomous ODEs*, J. Symbolic Comput. **46** (2011), no. 10, 1173–1186, DOI 10.1016/j.jsc.2011.06.002. MR2831479

[Rit50] J. F. Ritt, *Differential Algebra*, American Mathematical Society Colloquium Publications, Vol. XXXIII, American Mathematical Society, New York, N. Y., 1950. MR0035763

[Rob14] D. Robertz, *Formal algorithmic elimination for PDEs*, Lecture Notes in Mathematics, vol. 2121, Springer, Cham, 2014, DOI 10.1007/978-3-319-11445-3. MR3307172

[Sch98] J. Schicho, *Rational parametrization of surfaces*, J. Symbolic Comput. **26** (1998), no. 1, 1–29, DOI 10.1006/jsco.1997.0199. MR1633577

[SW97] J. R. Sendra and F. Winkler, *Parametrization of algebraic curves over optimal field extensions*, J. Symbolic Comput. **23** (1997), no. 2-3, 191–207, DOI 10.1006/jsco.1996.0083. Parametric algebraic curves and applications (Albuquerque, NM, 1995). MR1448694

[SWPD08] J. R. Sendra, F. Winkler, and S. Pérez-Díaz, *Rational algebraic curves*: *A computer algebra approach*, Algorithms and Computation in Mathematics, vol. 22, Springer, Berlin, 2008, DOI 10.1007/978-3-540-73725-4. MR2361646

[VGW18] N. T. Vo, G. Grasegger, and F. Winkler, *Deciding the existence of rational general solutions for first-order algebraic ODEs*, J. Symbolic Comput. **87** (2018), 127–139, DOI 10.1016/j.jsc.2017.06.003. MR3744343

[Vo16] N. T. Vo, *Rational and algebraic solutions of first-order algebraic ODEs*, PhD Thesis, available online at `www.risc.jku.at/publications/download/risc_5387/PhDThesisThieu.pdf`, 2016.

[vzGG13] J. von zur Gathen and J. Gerhard, *Modern computer algebra*, 3rd ed., Cambridge University Press, Cambridge, 2013, DOI 10.1017/CBO9781139856065. MR3087522

[VW15] N. T. Vo and F. Winkler, *Algebraic general solutions of first order algebraic ODEs*, Computer algebra in scientific computing, Lecture Notes in Comput. Sci., vol. 9301, Springer, Cham, 2015, pp. 479–492, DOI 10.1007/978-3-319-24021-3_35. MR3447539

[Wal50] R. J. Walker, *Algebraic curves*, Springer-Verlag, New York-Heidelberg, 1978. Reprint of the 1950 edition. MR513824

[Win96] F. Winkler, *Polynomial algorithms in computer algebra*, Texts and Monographs in Symbolic Computation, Springer-Verlag, Vienna, 1996, DOI 10.1007/BF01191382. MR1408683

[Zar58] O. Zariski, *On Castelnuovo's criterion of rationality $p_a = P_2 = 0$ of an algebraic surface*, Illinois J. Math. **2** (1958), 303–315. MR99990

[Zwi98] D. Zwillinger, *Handbook of differential equations*, 2nd ed., Academic Press, Inc., Boston, MA, 1992. MR1149384

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