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Potential theory and geometry on Lie groups, by N. Th. Varopoulos, New Mathematical Monographs, Vol. 38, Cambridge University Press, Cambridge, 2021, xxvii+596 pp., ISBN 978-1-107-03649-9

This book is concerned with the large n behavior of the iterated convolution powers $\mu^{(n)}$ of (nice) probability measures μ. The aim is to understand how the behavior of such convolution powers relates to the algebraic structure of underlying group G, as well as its geometry and topology. The group G is assumed to be a real connected Lie group but no further assumptions are made. The measure μ is assumed to be symmetric (i.e., $\mu(A) = \mu(A^{-1})$ for any Borel set A) and to have a continuous compactly supported density, which is positive at the identity element e of G. Connected Lie groups are algebraic objects whose algebraic structure is largely captured by their Lie algebra. They are also geometric objects when equipped with a left-invariant Riemannian structure (i.e., the choice of a linear basis for their Lie algebra). Importantly, the different possible choices of a left-invariant Riemannian structure all lead to the same *large scale geometry*, which we can think of as intrinsically attached to G (a proper definition entails the notion of quasi-isometry).

Fix a relatively compact symmetric neighborhood $\Omega = \Omega^{-1}$ of the identity element e in G. Is it possible to relate the behavior of the function

$$n \mapsto \mu^{(n)}(\Omega)$$

to the algebraic structure of G? To the large scale geometry/topology of G? What types of behavior should we expect to see? From the perspective of probability theory, $\mu^{(n)}(\Omega)$ is the probability that the random walk on G, started at the identity element and, driven by the measure μ, finds itself in Ω at time n.

One of the simplest *geometric quantities* associated to G is the function $n \mapsto |\Omega^n|$ equal to the Haar volume of Ω^n. In the context of connected Lie groups, it is a fact that this function behaves either like r^d for some integer d or grows exponentially fast (this is not the case for finitely generated groups!). How are the behaviors of the functions $n \to \mu^{(n)}(\Omega)$ and $n \to |\Omega^n|$ related?

At this level, there is no reason to restrict our attention to real connected Lie groups, and one can ask this question for all compactly generated locally compact groups, in particular, all finitely generated groups. The book focuses on connected Lie groups because the author has discovered a rich, sophisticated answer in that context. Extensions of the results presented in the book to finitely generated groups (and other groups) are not easy to formulate, and they appear to be well beyond our present understanding.

In a nutshell, "potential theory" in the title refers to the problem of estimating $\mu^{(n)}(\Omega)$ and to the techniques employed to do so. Although it may not be obvious at first sight, estimating $\mu^{(n)}(\Omega)$ is analogous to studying the heat equation on the Lie group G. This should not be surprising because $\mu^{(n)}$ is the distribution at time n of a random walk on G and, at least in simpler contexts, there are well-established connections between random walks, Brownian motion, and the heat equation. The "geometry" of Lie groups in the title refers to the large scale geometry alluded to above. The importance and value of thinking about groups in such terms (coarse geometry) has been championed prominently and very successfully by M. Gromov over several decades. The geometric concepts considered in the book go well beyond the volume growth function $n \to |\Omega^n|$ and include certain filling invariants we will briefly describe below (see Gromov's book-long article [8]).

Examples and the taxonomy of groups

As we shall see, the results discussed in the book under review provide a complete basic picture of the behavior of $\mu^{(n)}(\Omega)$ in the entire class of real connected Lie groups (note that complex Lie groups are, of course, real Lie groups, and that by real Lie groups we mean finite-dimensional real Lie groups).

For background information, we provide two diagrams. Figure 1 concerns finitely generated groups; Figure 2, real connected Lie groups. Note that a non-trivial finitely generated group is not a connected real Lie group. However, consideration

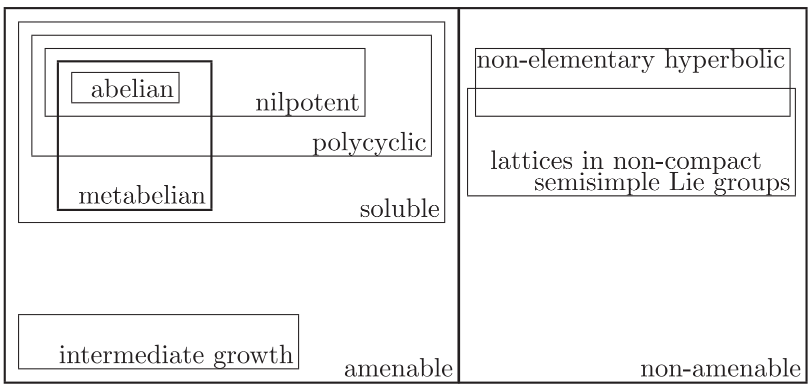

Figure 1. The inclusion relations between various classes of finitely generated groups

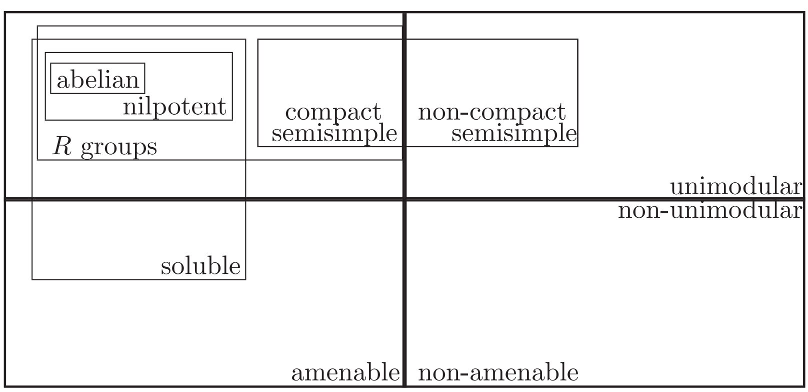

Figure 2. The inclusion relations between various classes of connected Lie groups

of such groups is useful to put into perspective both the content of the book and potential future directions of research. It is hard to appreciate the content of this book without a bird's eye view of the general taxonomy of group structures. In addition to the basic algebraic classes (abelian, nilpotent, polycyclic, soluble,...), two classic dichotomies are important to us: unimodular versus non-unimodular, and amenable versus non-amenable. The following comments should help the reader understand the huge differences that exist between finitely generated groups and connected Lie groups for our present purpose.

A locally compact group admits left-invariant Haar measures and right-invariant Haar measures. Two left-invariant (resp., right-invariant) Haar measures differ by a positive constant multiple, and the map $x \mapsto x^{-1}$ turns one type of Haar measure into the other. Denote by $d^r x$ and $d^l x$ a chosen right-invariant Haar measure and the companion left-invariant Haar measure obtained via the map $x \to x^{-1}$. These two measures are mutually absolutely continuous and the relation $d^l x = m(x) d^r x$ defines the modular function m of G. It satisfies $m(xy) = m(x)m(y)$, $x, y \in G$. A group is unimodular if $m \equiv 1$ (left-invariant and right-invariant measures coincide). Countable groups are unimodular and so are compact groups, abelian groups (!), and nilpotent groups. The simplest non-unimodular group is the group of all orientation-preserving affine bijections of the real line, under composition. In matrix form, it is the group of 2-by-2 matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, $(a, b) \in (0, +\infty) \times (-\infty, +\infty)$. The left-invariant Haar measure is $a^{-2} dadb$, whereas the right-invariant Haar measure is $a^{-1} dadb$.

A locally compact group is amenable if it admits an invariant mean, that is, a linear functional on $L^\infty(G)$ which has norm 1 and is positivity preserving and right-invariant. Compact groups and soluble groups are amenable. Free groups on at least two generators and non-compact semisimple Lie groups[1] are non-amenable.

In Figure 1, the complement of the union of the polycyclic, metabelian, and hyperbolic boxes can be viewed as a region where our understanding is limited (we allow here for some exceptions) and in which much remains to be explored. It is not just that we do not know what the behavior of $n \to \mu^{(n)}(\Omega)$ is, we do not know how to approach the question. In this context, there are groups for which we do not know how to determine if they are amenable or not, and uncountably many essentially different behaviors for $n \mapsto \mu^{(n)}(\Omega)$ can be observed in the soluble box alone. We do not know how to properly describe the behavior of $n \to \mu^{(n)}(\Omega)$ on $\mathrm{SL}_3(\mathbb{Z})$. This is in sharp contrast with the results presented in the book for real connected Lie groups.

In, Figure 2 Lie theory allows us to decompose any connected Lie group into better understood pieces. Namely, a general (finite-dimensional, real) Lie algebra can be decomposed as a semidirect product of its radical $\mathfrak{q}$ (largest soluble ideal) and a Levi subalgebra $\mathfrak{s}$, which is a semisimple Lie algebra acting on $\mathfrak{q}$. Moreover, $\mathfrak{s} = \mathfrak{s}_n \oplus \mathfrak{s}_c$ is the direct sum of one or two semisimple Lie algebras, one of compact type ($\mathfrak{s}_c$) and the other of non-compact type ($\mathfrak{s}_n$). In particular, a Lie group G is amenable if and only if $\mathfrak{s} = \mathfrak{s}_c$ is of compact type. This important result captures a fundamental difference between connected Lie groups and finitely generated

[1] A connected Lie group is semisimple if it contains no non-trivial connected normal solvable (equivalently, abelian) subgroups.

groups: for real connected Lie groups, the amenable/non-amenable dichotomy has no mystery.

The maps $ad(x) : y \mapsto [x, y]$ play a key role in the classification of Lie algebras. A Lie group with Lie algebra $\mathfrak{g}$ is unimodular if and only if $\det(ad(x)) = 1$ for all $x \in \mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is of type R if the eigenvalues of $ad(x)$ are pure imaginary for any $x \in \mathfrak{g}$. A semisimple Lie algebra is of type R if and only if it is of compact type. Nilpotent Lie algebras are of type R. In fact, Lie groups whose Lie algebra is of type R are exactly the groups whose volume growth $n \mapsto |\Omega^n|$ has a polynomial upper bound (see [9, 10]).

Basic illustrative examples used below include the following.

- The semisimple groups $\mathrm{SL}_2(\mathbb{R})$ and its universal cover, $\widetilde{\mathrm{SL}_2(\mathbb{R})}$; $\mathrm{SL}_n(\mathbb{R})$, $n > 2$.
- The soluble groups that are the semidirect products of the form

$$\mathbb{R} \ltimes_A \mathbb{R}^2, A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \left(x, \begin{pmatrix} y \\ z \end{pmatrix}\right)\left(x', \begin{pmatrix} y' \\ z' \end{pmatrix}\right) = \left(x + x', \begin{pmatrix} y \\ z \end{pmatrix} + A^x \begin{pmatrix} y' \\ z' \end{pmatrix}\right),$$

where A is as follows.[2]

(E1) *Heisenberg*: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; this is also the group of matrices $\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$.

(E2) *Sol*: $A = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}$; this is also the group of matrices $\begin{pmatrix} e^{-x} & 0 & y \\ 0 & e^x & z \\ 0 & 0 & 1 \end{pmatrix}$.

(E3) *Solvable, non-unimodular*: $A = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$ with $\lambda_1 + \lambda_2 \neq 0$; this is also the group of matrices $\begin{pmatrix} e^{\lambda_1 x} & 0 & y \\ 0 & e^{\lambda_2 x} & z \\ 0 & 0 & 1 \end{pmatrix}$.

(E4) *Example of an R group that is not nilpotent*: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Examples (E1), (E2), and (E4) are unimodular, whereas (E3) is non-unimodular because of the assumption that $\lambda_1 + \lambda_2 \neq 0$.

Convolution operators

On a locally compact, compactly generated group G, the convolution of an ordered pair of measures (μ, ν), each of finite total mass, is the measure $\mu * \nu$ defined by

$$\mu * \nu(f) = \int_{G \times G} f(xy)\mu(dx)\nu(dy)$$

when f is a continuous, compactly supported function on G (no reference to a Haar measure is needed here). The left-invariant convolution operator associated with a measure μ of finite total mass, $\|\mu\| = |\mu|(G)$, is defined on the same f by

$$f \mapsto f * \mu, \quad f * \mu(x) = \int_G f(xy^{-1})\mu(dy).$$

[2]Each of these A is of the form $A = \exp B$ for some B so that A^x is well defined.

If $\tau_g f(x) = f(gx)$ (left-translation by g), then $\tau_g(f * \mu) = (\tau_g f) * \mu$, that is, the operator $f \mapsto f * \mu$ is left-invariant. This operator extends as a bounded operator on $L^2(G, d^r x)$ with norm

$$\|\mu\|_{\mathrm{op}} = \sup_{\|f\|_2 \leq 1} \{\|f * \mu\|_2\} \leq \|\mu\|.$$

When the group is non-unimodular, this simple fundamental inequality,

$$\|\mu\|_{\mathrm{op}} \leq \|\mu\|,$$

does not hold if one replaces $L^2(G, d^r x)$ by $L^2(G, d^l x)$. The adjoint of this convolution operator (on $L^2(G, d^r x)$) is $f \mapsto f * \check{\mu}$, where $\check{\mu}(A) = \mu(A^{-1})$ for any Borel set A. Assuming that the measure μ admits a continuous density ϕ with respect to the (right-invariant) Haar measure $d^r x$, we let ϕ_n be the continuous density of the iterated convolution $\mu^{(n)}$ with respect to $d^r x$. Following the book, in the non-unimodular case we avoid purposely expressing the density ϕ_n of $\mu^{(n)}$ explicitly in terms of ϕ. Fix a compact generating set Ω of G, which is a symmetric neighborhood of e, and set

$$V(n) = |\Omega^n| = \int_{\Omega^n} d^r x$$

(the symmetry of Ω makes the choice between the measures d^r and d^l irrelevant here). We call V the volume growth function of G because the rough behavior of this function as n tends to infinity does not depend on the choice of the compact generating set Ω. The motivated reader can go through our list of examples (E1)–(E4) above, and decide in which ones $n \to |\Omega^n|$ grows exponentially fast (two have this property) and in which they do not ((E1) and (E4)).

Informally, the basic question addressed in the book is the following.

Question 1. On a locally compact, compactly generated group G with identity element e, let $\mu(dx) = \phi(x) d^r x$ be a symmetric probability measure with compact generating support and continuous density ϕ with $\phi(e) > 0$. What is the behavior of the function $(n, x) \mapsto \phi_n(x)$? In particular, what is the behavior of $\phi_n(e)$ as n tends to infinity? Can we relate that behavior to the algebraic/geometric structure of the group?

Sub-Laplacians and their heat equations

On a (connected, real) Lie group G with Lie algebra $\mathfrak{g}$ (viewed here as the space of left-invariant vector fields on G), let $\{X_1, \ldots, X_k\}$ be a family of left-invariant vector fields which generates $\mathfrak{g}$ as a Lie algebra. Each X_i is a partial differential operator of order 1 acting on smooth functions on G and, being left-invariant, it is entirely determined by its value at the identity element of G. The associated sub-Laplacian $\Delta = \sum_1^k X_i^2$ (this is a second-order differential operator acting on smooth functions on G) is the infinitesimal generator of the heat semigroup $H_t = e^{t\Delta}$. This semigroup solves the heat equation $(\partial_t - \Delta)u = 0$ with initial value $f \in L^2(G, d^r x)$ in the form $u(t, x) = H_t f(x)$. It is easily constructed using spectral theory because Δ gives rise to a self-adjoint operator on $L^2(G, d^r x)$. It is a Markovian semigroup associated with a diffusion process on G, and it commutes with left translations. Moreover, the parabolic partial differential operator $\partial_t - \Delta$ is hypoelliptic and $H_t f = f * \mu_t$, where μ_t is a probability measure admitting a smooth positive density $\mu_t(dx) = h_t(x)d^r x$. It has the property that $\mu_t(A) = \mu_t(A^{-1})$ for any Borel set A.

When G is unimodular, $h : (0, +\infty) \times G$, $(t, x) \mapsto h_t(x)$, is the fundamental solution of the heat equation $\partial_t u = \sum_1^k X_i^2 u$ with initial value δ_e (Dirac mass at the identity element, e of the group G), and the convolution kernel of H_t, that is, $H_t f(x) = \int_G h_t(y^{-1}x)f(y)dy$.

To give a concrete idea of what all this means, consider the Heisenberg group given in its matrix form, $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$. Partial derivatives in the x and y direction at the identity element (i.e., at $x = y = z = 0$) generate two left-invariant vector fields, X and Y, given in coordinates by $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$. An immediate computation gives that the left-invariant vector field Z equal to $\frac{\partial}{\partial z}$ at the identity element is $Z = \frac{\partial}{\partial z} = XY - YX = [X, Y]$. Hence, $\{X, Y\}$ generates the entire Lie algebra, and the sub-Laplacian

$$\Delta = X^2 + Y^2 = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} + x^2\frac{\partial^2}{\partial^2 z} + 2x\frac{\partial^2}{\partial z \partial y}$$

generates a self-adjoint convolution semigroup whose action on a function f is given by $e^{t\Delta}f(x) = H_t f(x) = \int_G h_t(y^{-1}x)f(y)dy$.

Question 2. On a Lie group G with identity element e, what is the behavior of $(t, x) \mapsto h_t(x)$? In particular, what is the behavior of $h_t(e)$ as t tends to infinity? Can we relate that behavior to the algebraic/geometric structure of the group?

To a large extent, Question 2 is nothing but a somewhat sophisticated version of Question 1, set in the context of connected Lie groups. Indeed, for any $t > 0$ and integer $n > 0$, $\mu_t = \mu_{t/n}^{(n)}$ so, for instance, $\mu_n = \mu_1^{(n)}$ (the measure μ_1 is not compactly supported but its density tends to 0 very fast at infinity). In this review, we will focus on convolution operators and Question 1.

Rapid historical review of Question 1 (iterated convolutions)

Coin tossing and de Moivre's central limit theorem in the third edition (1756) of his *The Doctrine of Chances* can serve as our starting point: the group is $\mathbb{Z}$ and de Moivre and Laplace show that $\phi_n(0) \sim cn^{-1/2}$. The second edition of Poincaré's *Calcul des Probabilités* contains a discussion of the problem in the context of card shuffling (the group is the symmetric group, finite but definitely non-abelian). In 1921, Pólya introduced the dichotomy *recurrence versus transience* for a simple random walk on the square lattices $\mathbb{Z}^d$. He proved that a simple random walk is recurrent[3] in dimensions $d = 1, 2$, and transient when $d \geq 3$. Kawada and Ito discussed the behavior of ϕ_n in the context of compact groups in 1940, and one can also mention important works by Gnedenko, Kolmogorov, Khinchine, and Lévy (these efforts concern only basic abelian groups). None of these works really addresses Question 1 in its generality as they focus on very particular groups. It is worth noting that by the mid-1950s, the relation between random walks and Brownian motion (in the context of Euclidean spaces) was well established, providing the source of the natural connection between Question 1 and Question 2.

[3] A random walk is recurrent if, with probability 1, it returns infinitely often to its starting point. It is transient otherwise.

It is Kesten's PhD thesis (1958) that provided the first consideration of Question 1 for general countable groups with what was then a completely new viewpoint—directly relevant to the book under review—emphasizing the relationship between the behavior of $n \mapsto \phi_n(e)$ and the algebraic structure of the underlying group. A later version of Kesten's (amenability) theorem, valid for all locally compact groups including non-unimodular groups, describes the famous dichotomy between amenable and non-amenable groups exactly in terms of the behavior of $n \mapsto \phi_n(e)$. The group G in Question 1 is amenable if and only if $\limsup_{n\to\infty} \phi_n(e)^{1/n} = 1$, equivalently, if and only if $\|\mu\|_{\mathrm{op}} = 1$.

In the early 1960s, Kesten suggested that, perhaps, the only finitely generated groups that carry a (non-degenerate) recurrent random walk are the finite extensions of $\{e\}$, $\mathbb{Z}$, and $\mathbb{Z}^2$, a problem that certainly relates again to Question 1. Kesten's problem provided the impetus for works by Guivarc'h, Keane, Roynette (e.g., [5]), and others during the 1970s (the Lie group version of Kesten's problem was resolved then), but the case of finitely generated groups remained open until Varopoulos solved it in the affirmative in 1985 (Varopoulos's solution uses Gromov's theorem on a group of polynomial volume growth). What Varopoulos proved is Theorem 1 below, which relates volume growth to the behavior of $\phi_n(e)$. Originally, this was proved for finitely generated groups and for connected Lie groups; see [16, Chapter VII]. The version given here takes advantage of Losert's description of groups with polynomial volume growth in the context of general locally compact, compactly generated groups [12–15].

Theorem 1. *Assume G is a locally compact, compactly generated, and unimodular group.*

- Analytic dichotomy: *Either there exists an $a = a(G) \geq 0$ such that, for any μ as in Question 1, $\liminf_{n\to\infty} n^a \phi_n(e) > 0$; or for any $a \geq 1$ and any μ as in Question 1, $\limsup_{n\to\infty} n^a \phi_n(e) < +\infty$.*
- Geometric dichotomy: *The first case arises exactly when there exists b such that $\liminf_{n\to\infty} n^{-b} V(n) < +\infty$.*
- Algebraic dichotomy: *The first case arises exactly when there exist a normal compact group $K_0 \subset G$, an auxiliary compact group K, and a connected simply connected nilpotent Lie group $\widetilde{N}$ (on which K acts) such that $\widetilde{G} = \widetilde{N} \rtimes K$ contains G/K_0 as a closed subgroup with compact quotient.*
- *In the first case there is an integer d, such that both $V(n) \asymp n^d$ and $\phi_n(e) \asymp n^{-d/2}$.*

The somewhat complicated condition spelled out in the third bullet (algebraic dichotomy) should be understood as saying that, at large scales, G looks like the connected simply connected nilpotent Lie group $\widetilde{N}$. When G is finitely generated, the condition is equivalent to the fact that G contains a nilpotent subgroup with finite index and the equivalence between the geometric and algebraic dichotomies is a version of Gromov's theorem on groups of polynomial volume growth.

Note how the first two bullets each state what is, essentially, a trivial (yes/no) dichotomy. However, the information provided in the last two bullets turn these into very informative dichotomies. Among our examples (E1)–(E4), (E3) is excluded (these groups are not unimodular), (E1) and (E4) fall in the first case of the analytic dichotomy with the integer d being $d = 4$ for the Heisenberg group in (E1) and $d = 3$ for (E4), and (E2) falls in the second case (it has exponential volume growth and the return probability decays faster than any inverse power function). Examples (E1) and (E4) also illustrate the algebraic dichotomy: in (E1), $\widetilde{N} = G$, $K_0 = K = \{e\}$; in (E4), $\widetilde{N} = \mathbb{R}^3$, $K = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ (the circle group) acting by rotation on the last two coordinates of $\mathbb{R}^3$, $K_0 = \{e\}$ (the only compact subgroup of G), and G can be identified with the subgroup of $\widetilde{N} \rtimes K$ equal to $\{(\theta, x, y, z) : \theta \in \mathbb{S}^1, x, y, z \in \mathbb{R}, x = \theta \mod \mathbb{Z}\}$. The quotient $\widetilde{G}/G$ is compact.

Kesten's question about finitely generated groups that carry a non-degenerate recurrent random walk is resolved using Theorem 1 because recurrence is characterized by the condition $\sum_n \phi_n(e) = +\infty$. It follows that these groups must satisfy $V(n) \asymp n^d$, equivalently, $\phi_n(e) \asymp n^{-d/2}$, with $d = 0$, 1, or 2, and then it is not hard to show that such groups are finite extensions of $\{e\}$, $\mathbb{Z}$, or $\mathbb{Z}^2$.

If we assume that the group G is a real connected Lie group, then we can describe the algebraic dichotomy purely in terms of the Lie algebra $\mathfrak{g}$ of G: the first case arises exactly when $\mathfrak{g}$ is an R-algebra (i.e., G is an R-group).

The content of the book

The limitations of Theorem 1 and the analytic dichotomy. Theorem 1 is incomplete in two obvious ways:

(a) it is restricted to unimodular groups; and
(b) it gives little information about non-amenable groups, simply putting them into the large bag of groups, where $\phi_n(e)$ decays rapidly (faster than any inverse power function).

In short, one of the aims of the book under review is to alleviate these limitations in the context of real connected Lie groups. This takes us on a journey along which we discover much of great interest. Early results that must have guided the discoveries presented in the book under review include limit theorems for free groups (for this, see the book-length treatment in [17]) and the work of Bougerol [1–3] on local limit theorems for semisimple Lie groups (with finite center) and a class of soluble groups (including non-unimodular examples). For a semisimple real Lie group G with finite center, Bougerol proved that

$$\lim_{n\to+\infty} n^{a(G)} \|\mu\|_{\mathrm{op}}^{-n} \phi_n(e) = c_\mu > 0,$$

where $a(G)$ depend only on G (in fact, $a(G) = (r + 2p)/2$, where r is the real rank and p is the number of indivisible positive roots,[4] that is, $a(G)$ is a number that one reads off the Lie structure of G).

We are ready to state the most basic version of one of the main theorems proved in the book.

Theorem 2 (Analytic dichotomy)**.** *Let G be a connected real Lie group. Either, for any μ as in Question 1, there is a real $a(\mu) \geq 0$ such that*

$$\liminf_{n\to+\infty} n^{a(\mu)} \|\mu\|_{\mathrm{op}}^{-n} \phi_n(e) > 0 \tag{1}$$

or, for any μ as in Question 1, there exists $c > 0$ such that

$$\limsup_{n\to+\infty} e^{cn^{1/3}} \|\mu\|_{\mathrm{op}}^{-n} \phi_n(e) < +\infty. \tag{2}$$

[4] A positive root is indivisible if half of it is not a root.

The statements of both sides of this dichotomy can be sharpened considerably by proving matching upper and lower bounds for $\|\mu\|_{\text{op}}^{-n}\phi_n(e)$ (see the beautiful theorem stated in Section 1.3.1, page 4, of the Introduction). In the first case, one can find $a \geq 0$ and $0 < c \leq C < +\infty$ (all possibly depending on μ) such that, for all n,

$$cn^{-a} \leq \|\mu\|_{\text{op}}^{-n}\phi_n(e) \leq Cn^{-a}.$$

The reader should be warned that the book does not contain a proof of this fact. We are told (on page 103) that a proof can be built on the methods of the book but that the details would be very long to write down in full. In the second case of the dichotomy, there are positive constants c_1, c_2, c_3, c_4 (depending on μ) such that, for all n,

$$c_1 e^{-c_2 n^{1/3}} \leq \|\mu\|_{\text{op}}^{-n}\phi_n(e) \leq c_3 e^{-c_4 n^{1/3}}.$$

A proof of this fact is presented in the book. To illustrate this result, we consider our running examples. Non-compact semisimple Lie groups, the Heisenberg group (E1), and the R-group (E4), all satisfy the first alternative (1). The group Sol (i.e., (E2)) satisfies the second alternative. Groups in the family of examples (E3) satisfy the first alternative if $\lambda_1\lambda_2 > 0$, and the second if $\lambda_1\lambda_2 < 0$. None of these facts are particularly easy to see at this moment, and (E3) illustrates how Theorem 2 applies to non-unimodular groups.

Theorem 2 is one of the main results presented in the book and, with or without the refinements just mentioned, it gives a compelling answer to Question 1 in the case of connected real Lie groups. But the real thrust and goal of the book is to present a series of dichotomies set in very different terms, each of which is also proved to be equivalent to the dichotomy of Theorem 2. The equivalence between these different dichotomies is a significant and impressive result. It is also stimulating because it sheds light on many natural questions:

- What is the natural setting for such results? (The book discusses extensions to algebraic groups.)
- Is it possible to state versions of these results for interesting subclasses of finitely generated groups (e.g., lattices in connected Lie groups)?

This is a question that is completely open. To wit, we do not know what the behavior of $\|\mu\|_{\text{op}}^{-n}\phi_n(e)$ is on the group $SL_3(\mathbb{Z})$! (Thanks to work by S. Gouëzel [6], we know that finitely generated hyperbolic groups behave as in case (1).) To understand one of many reasons why this book is mostly focused on connected Lie groups, note that it is known that $\phi_n(e)$ can exhibit a great variety of different behaviors (uncountably many) among amenable (even soluble) finitely generated groups; see, e.g., [4]. The discussion above shows that this is not at all the case for real connected Lie groups.

The algebraic dichotomy. Let $\mathfrak{q}$ be a soluble Lie algebra, and let $\mathfrak{n}$ be its largest nilpotent ideal (nilradical). Recall that we have the map $\operatorname{ad} x : y \mapsto [x, y]$. For each $v \in V = \mathfrak{q}/\mathfrak{n}$, this map induces a map $\operatorname{ad} v : W \to W$, $W = \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. These maps form a commutative algebra which admits a so-called root decomposition, essentially, a common diagonalization after complexification (there are, of course, finitely many roots). The roots are complex-valued linear functionals on V associated with root spaces (subspaces of the complexification of W) and the action of

$\operatorname{ad} v$, $v \in V$. On a root space and up to the addition of a nilpotent transformation, $\operatorname{ad} v$ has the form $\lambda(v)I$ with the *root*

$$\lambda(v) = \operatorname{Re}\lambda(v) + i\operatorname{Im}\lambda(v) \ \text{(complex linear functional on } V).$$

Let $\{L_1, \ldots, L_p\}$ be the distinct values taken by the real parts $\operatorname{Re}\lambda$ of the roots. By construction, the direct sum W_i of the root spaces associated with a given L_i satisfies $\overline{W}_i = W_i$ (in the complexification of W), and this gives a *real root* decomposition $W = \widetilde{W}_1 \oplus \cdots \oplus \widetilde{W}_p$. Here, "real root" means "the real part of a root". These real roots L_i are distinct linear forms on V. They are the key objects in defining the following algebraic dichotomy.

Definition 1 ((C) versus (NC) for real finite-dimensional Lie algebra)**.**

- A soluble Lie algebra $\mathfrak{q}$ is a (C)-algebra if there exists $(\alpha_i)_1^p \in [0, +\infty)^p$ such that $\sum_1^p \alpha_i L_i = 0$ and $\alpha_i L_i \neq 0$ for at least one $i \in \{1, \ldots, p\}$.
- A soluble Lie algebra $\mathfrak{q}$ is an (NC)-algebra if, for any $(\alpha_i)_1^p \in [0, +\infty)^p$, $\sum_1^p \alpha_i L_i = 0$ implies $\alpha_i L_i = 0$ for all $i \in \{1, \ldots, p\}$.
- A general Lie algebra $\mathfrak{g}$ is a (C)-algebra (resp., (NC)-algebra) if and only if its radical (largest soluble ideal) is a (C)-algebra (resp., (NC)-algebra).

The dichotomy (C)-(NC) is suitable to capture the analytic dichotomy of Theorem 2 in the context of amenable connected Lie groups. Namely, for an amenable connected Lie group G, the Lie algebra $\mathfrak{g}$ of G is (NC) if and only if for any μ as in Question 1, there is a real $a(\mu) \geq 0$ such that (1) holds true (because G is amenable, $\|\mu\|_{\mathrm{op}} = 1$). In the book, this is stated in Theorems 2.3 (C-theorem) and 3.2 (NC-theorem).

To be able to deal with non-amenable groups, one has to use some basic structure theory. A general (finite-dimensional, real) Lie algebra can be decomposed as a semidirect product of its radical $\mathfrak{q}$ (largest soluble ideal) and a Levi subalgebra $\mathfrak{s}$ which is a semisimple Lie algebra. Moreover, $\mathfrak{s} = \mathfrak{s}_n \oplus \mathfrak{s}_c$ is the direct sum of two semisimple Lie algebras of compact type ($\mathfrak{s}_c$) and non-compact type ($\mathfrak{s}_n$). We already mentioned that a Lie group G is amenable if and only if $\mathfrak{s} = \mathfrak{s}_c$ is of compact type.

Now, a semisimple Lie algebra of non-compact type, $\mathfrak{s}_n$, can be decomposed as a direct sum of vector spaces that are subalgebras, $\mathfrak{s}_n = \mathfrak{n}_s + \mathfrak{a} + \mathfrak{k}_n$ ($\mathfrak{n}_s$ is nilpotent, $\mathfrak{a}$ is abelian and $[\mathfrak{n}_s, \mathfrak{a}] \subseteq \mathfrak{n}_s$, and $\mathfrak{k}_n$ is the Lie algebra of a compact group). This is called the Iwasawa decomposition of $\mathfrak{s}_n$, and it provides a decomposition of $\mathfrak{g}$ in the form

$$\mathfrak{g} = \mathfrak{r} + \mathfrak{t}, \quad \mathfrak{r} = \mathfrak{q} + \mathfrak{n}_s + \mathfrak{a}, \quad \mathfrak{t} = \mathfrak{k}_n + \mathfrak{s}_c. \tag{3}$$

Ignoring the difficulties coming from the fact that such decomposition is not unique (which can be shown to be unimportant), we arrive at the key (B)-(NB) dichotomy.

Definition 2 ((B) versus (NB) for real finite-dimensional Lie algebras)**.**

- A Lie algebra $\mathfrak{g}$ is a (B)-algebra if $\mathfrak{r} = \mathfrak{q} + \mathfrak{n}_s + \mathfrak{a}$ is a (C)-algebra.
- A Lie algebra $\mathfrak{g}$ is an (NB)-algebra if $\mathfrak{r} = \mathfrak{q} + \mathfrak{n}_s + \mathfrak{a}$ is an (NC)-alebgra.

Theorem 3 (Equivalence of the analytic and (B)-(NB) dichotomies)**.** *For a connected Lie group G with Lie algebra $\mathfrak{g}$ and referring to* Theorem 2, (1) *holds if and only if $\mathfrak{g}$ is* (NB), *and* (2) *holds if and only if $\mathfrak{g}$ is* (B).

This equivalence is stated in Theorems 4.6 (B-theorem) and 4.7 (NB-theorem) and the entirety of Chapter 4 is devoted to this classification. Note that R-groups are groups of polynomial volume growth, and one can characterize unimodular (NB)-groups as those groups whose Lie algebra is the direct sum of an R-algebra and a semisimple algebra (Part I. B.5, page 202).

The geometric dichotomy for models. Filling invariants have a long history as they address variations on classical isoperimetric problems. In the classical isoperimetric problem, in dimension d, one asks for a bound on the volume of a compact set with smooth boundary in terms of the $(d-1)$-volume of its boundary. One can also ask about the minimal surface area of a surface filling up a close curve of a given length. This leads to the notion of Dehn function. In both cases, one is working in a d-dimensional Riemannian manifold. In these two cases, one uses volume/area to measure the quality of the filling but one may also focus on different quantities; see, e.g., [7,8]. One possibility mentioned in [8, page 74] is what Gromov call the *filling span*. In a fixed dimension n, this involves the Lipschitz constant of Lipschitz extensions to the n-ball of a given Lipschitz embedding of the boundary of that ball. In the book under review, unit cubes are used instead of balls (Section 7.5.1). Namely, let M be a Riemannian manifold diffeomorphic to $\mathbb{R}^d$. For each integer n, let f be a Lipschitz map from the boundary of the unit cube $[0,1]^n \subset \mathbb{R}^n$ to M. Considering all possible Lipschitz extensions $\hat{f} : [0,1]^n \to M$ of f (such extensions always exist), let $A(f)$ be the infimum of the Lipschitz constants $\mathrm{Lip}(\hat{f}) = \sup\{d(\hat{f}(x), \hat{f}(y))/\|x-y\|\}$ of such $\hat{f}$. For fixed $R \geq 1$, define $\varphi_n(R)$ the supremum of $A(f)$ over all f with $\mathrm{Lip}(f) \leq R$.

Definition 3. For a given n, say that M has the polynomial filling property in dimension n if there is a constant C such that, for all $R \geq 1$, $\varphi_n(R) \leq CR^C$.

Definition 4. Say that the pointed space (M, o) is polynomially retractable if there exists a locally Lipschitz map $F : M \times [0,1] \to M$ such that, for all $m \in M$, $F(m,1) = m$, $F(M,0) = o \in M$, and there exists C for which, for all $(m,\lambda) \in M \times [0,1]$, $|dF(m,\lambda)| \leq C(1 + d(o,m))^C$.

Here, dF is the differential of F and $d(o,m)$ is the distance from m to o. In words, one can retract M to a point with at most polynomial distortion along the way. When M is polynomially retractable (to any of its points) it follows that it has the polynomial filling property in any dimension n.

Call a Lie group U a *model* if it is diffeomorphic to some $\mathbb{R}^d$. It turns out that such models U are exactly the simply connected connected soluble Lie groups; see Section 1.5.1.

Theorem 4 (The geometric (C)-(NC) dichotomy for models)**.**

- *If the Lie algebra of a model U is* (NC)*, then U is polynomially retractable.*
- *If the Lie algebra of a model U is* (C)*, then there is an n so that the polynomial filling property fails in dimension n in U.*

There are two ways to describe the (B)-(NB) version of this theorem for connected Lie groups. One way is to use coarse quasi-isometries (here we follow the book in calling "coarse quasi-isometry" what is now most often called a "quasi-isometry"). Given a connected Lie group G, find a model U which is coarse quasi-isometric to G. This can always be done as explained in Section 11.1 of the book (Theorem 11.16). Then G is (B) (resp., (NB)) if and only if U is (C) (resp., (NC)). The second, more elegant way (but perhaps somewhat less directly informative) is to use the notion of polynomial retract to a compact set (instead of a point). It is known that any connected Lie group G always retracts to a compact subgroup K. The group G is (NB) if and only if there exists such a retract, $H(g,t) : G \times (0,1) \mapsto G$, whose gradient is polynomially bounded, i.e., there exists C such that $|dH(g,t)| \leq C(1+d(e,g))^C$ for all $(g,t) \in G \times (0,1)$. This statement is found in Theorem 12.9.

The homological dichotomy. The discussion above is phrased in homotopy terms (continuous deformations, retracts), and the final piece we need to mention is concerned with homology. Consider a model U as above and a closed differential form ω without constant term and which has polynomial growth, i.e., there exists C such that , for all $g \in U$, $|\omega(g)| \leq C(1+d(e,g))^C$. We surely can solve $d\theta = \omega$ on U. The question is whether we can do it with a polynomial bound on θ. The answer is: yes, always, if U is (NB), not always if U is (B). This can be expressed more formally and more generally in terms of the finiteness of the Betti numbers of the complex of (polynomially bounded) differential forms on a connected Lie group G. If these Betti numbers are finite, say that G has *finite polynomial homology*. Then, a connected Lie group G has finite polynomial homology if and only if it is (NB) (see Theorem 12.21).

The structure of the book

The key components of the book are presented in three distinct parts, each coming with a set of appendices and each filling about 200 pages. Part I is concerned with the analytic/probabilistic classification (convolution powers) and the equivalence with its algebraic counterpart (using roots). Parts II and III are entitled "Geometric Theory" and "Homology Theory", respectively. Whereas Part I stands by itself, Parts II and III are interrelated and make use of the algebraic considerations developed in Part I. In fact, the only direct link between Part I and Parts II and III is the algebraic version of the dichotomy. Indeed, it is an open question to formulate and prove any direct link and relation between, say, the behavior of convolution powers and the behavior of filling invariants in this general context. This is in contrast to the fact that such links are well established in the context of amenable unimodular groups when classical isoperimetry directly relates, at least in a coarse way, to the decay of convolution powers.

Each of the three parts involves distinct (often completely unrelated) tools and techniques, each relies on elements of sophisticated existing theories and requires the development of variations on classical arguments as well as new ideas.

Conclusion

The motivated reader will find this book fascinating. It presents, in a somewhat idiosyncratic but readable way, a personal, substantial, and interesting mathematical journey. Because of the variety of concepts and ideas involved, most readers will face some difficulties when reading the book: irritations at the peculiar presentation of concepts they understand, and befuddlement at some of those concepts they are not familiar with. This is despite the author's very genuine attempts to introduce and explain in a self-contained manner the key ideas and constructs. It is perhaps this attempt at being self-contained that explains the absence of mention of existing related works. Certainly the author cites all works that have influenced his journey in developing the presented material, but the four-page reference list is strictly utilitarian, and no visible efforts are made to provide the reader with pointers to works that could facilitate and enhance their understanding of the text by providing additional background information, different but similar developments regarding certain concepts, or special examples treated by different methods. It is not possible to remedy this here but one can point out the obvious: there are many published works on random walks (e.g., [11]) and on filling invariants (e.g., [18, 19]) that have some relation to problems discussed in the book, and the reader would have benefited tremendously if carefully selected pointers to the literature had been given. The reader will, nevertheless, find plenty to think about, either by exploring what is said but which the reader does not completely understand, or by considering the many open directions of research indicated and discussed in the text. In both cases, the reader will have an opportunity to make significant new contributions to our understanding of the original questions raised by the author.

References

[1] Philippe Bougerol, *Un théorème central limite local sur les groupes de Lie semi-simples* (French, with English summary), C. R. Acad. Sci. Paris Sér. A-B **288** (1979), no. 15, 741–744. MR532404

[2] Philippe Bougerol, *Théorème central limite local sur certains groupes de Lie* (French), Ann. Sci. École Norm. Sup. (4) **14** (1981), no. 4, 403–432 (1982). MR654204

[3] Philippe Bougerol, *Exemples de théorèmes locaux sur les groupes résolubles* (French, with English summary), Ann. Inst. H. Poincaré Sect. B (N.S.) **19** (1983), no. 4, 369–391. MR730116

[4] Jérémie Brieussel and Tianyi Zheng, *Speed of random walks, isoperimetry and compression of finitely generated groups*, Ann. of Math. (2) **193** (2021), no. 1, 1–105, DOI 10.4007/annals.2021.193.1.1. MR4199729

[5] A. Brunel (ed.), *Journées sur les marches aléatoires (Nancy 1979)*, Astérisque **74**, Société Mathématique de France, 1980.

[6] Sébastien Gouëzel, *Local limit theorem for symmetric random walks in Gromov-hyperbolic groups*, J. Amer. Math. Soc. **27** (2014), no. 3, 893–928, DOI 10.1090/S0894-0347-2014-00788-8. MR3194496

[7] Mikhael Gromov, *Filling Riemannian manifolds*, J. Differential Geom. **18** (1983), no. 1, 1–147. MR697984

[8] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295. MR1253544

[9] Yves Guivarc'h, *Croissance polynomiale et périodes des fonctions harmoniques* (French), Bull. Soc. Math. France **101** (1973), 333–379. MR369608

[10] J. W. Jenkins, *Growth of connected locally compact groups*, J. Functional Analysis **12** (1973), 113–127, DOI 10.1016/0022-1236(73)90092-x. MR0349895

[11] Emile Le Page and Marc Peigné, *Local limit theorems on some non-unimodular groups* (English, with English and French summaries), Rev. Mat. Iberoamericana **15** (1999), no. 1, 117–141, DOI 10.4171/RMI/252. MR1681639

[12] Viktor Losert, *On the structure of groups with polynomial growth*, Math. Z. **195** (1987), no. 1, 109–117, DOI 10.1007/BF01161604. MR888132

[13] Viktor Losert, *On the structure of groups with polynomial growth. II*, J. London Math. Soc. (2) **63** (2001), no. 3, 640–654. MR1825980

[14] Viktor Losert, *On the structure of groups with polynomial growth III*, J. Algebra **554** (2020), 1–40, DOI 10.1016/j.jalgebra.2020.02.012. MR4079363

[15] Viktor Losert, *On the structure of groups with polynomial growth IV*, J. Algebra **581** (2021), 92–121, DOI 10.1016/j.jalgebra.2021.04.006. MR4249757

[16] N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics, vol. 100, Cambridge University Press, Cambridge, 1992. MR1218884

[17] Wolfgang Woess, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics, vol. 138, Cambridge University Press, Cambridge, 2000, DOI 10.1017/CBO9780511470967. MR1743100

[18] Robert Young, *Homological and homotopical higher-order filling functions*, Groups Geom. Dyn. **5** (2011), no. 3, 683–690, DOI 10.4171/GGD/144. MR2813532

[19] Robert Young, *Lipschitz connectivity and filling invariants in solvable groups and buildings*, Geom. Topol. **18** (2014), no. 4, 2375–2417, DOI 10.2140/gt.2014.18.2375. MR3268779

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