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SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
ELENA GIORGI

MR0172678 (30 #2897) 83.53

Penrose, Roger

Gravitational collapse and space-time singularities.

Physical Review Letters **14** (1965), 57–59.

It is shown that even in the absence of spherical symmetry the occurrence and persistence of "trapped surfaces" will, within a finite time, lead to true singularities of the metric field. A trapped surface is a space-like closed 2-surface (topologically equivalent to a spherical surface) which shrinks if it is mapped on the two one-parametric sets of similar 2-surfaces by means of the two pencils of null geodesics passing through the original 2-surface and perpendicular to it, both times proceeding in the future direction. The very notion of the trapped surface appears to be a major conceptual contribution to the clarification of what usually is referred to as Schwarzschild "singularities", the quotes being used because locally the Schwarzschild radius is in no sense singular. Trapped surfaces will occur inside, but not outside, the Schwarzschild radius; this definition is independent of spherical symmetry, of the presence of isometries, and of the choice of coordinate system.

P. G. Bergmann

From MathSciNet, November 2022

MR0293962 (45 #3037) 83.53

Hawking, S. W.

Black holes in general relativity.

Communications in Mathematical Physics **25** (1972), 152–166.

It is assumed that the singularities which occur in gravitational collapse are not visible from outside but are hidden behind an event horizon. This implies that the breakdown of the presently accepted physical theory which occurs at a singularity cannot affect events outside the Schwarzschild radius which corresponds to the Schwarzschild metric outside the star, and one can still predict the future in the exterior region from Cauchy data on a spacelike surface. A black hole on a spacelike surface is defined to be a connected component of the region of the surface bounded by the event horizon; as a direct consequence of this definition it is found that black holes may merge together as time increases, but they can never bifurcate. It is shown that the surface area of a black hole cannot decrease with time and that a rotating stationary black hole must be axi-symmetric. Moreover, it is found that any stationary black hole must possess a topologically spherical boundary, which, when taken in conjunction with earlier results, strongly supports the conjecture that a black hole settles down to a Kerr solution. On the basis of this conjecture the surface area of a black hole is related to its mass, angular momentum, and electric charge. The fact that the surface area cannot decrease with time is used to place upper bounds on the amounts of energy that can be extracted from black holes. These limits suggest that there may be a spin-dependent force between two black holes analogous to that between magnetic dipoles.

H. Rund

From MathSciNet, November 2022

MR0408681 (53 #12444) 83.35

Chandrasekhar, S.

On the equations governing the perturbations of the Schwarzschild black hole.

Proceedings of the Royal Society. London. Series A. Mathematical, Physical and Engineering Sciences **343** (1975), 289–298.

The paper is based on the previous investigations by the author and J. L. Friedman [Astrophys. J. **175** (1972), 379–405; MR0312865]. There, a form of metric (a convenient "gauge") was found that is adequate for treating time-dependent axisymmetric systems. Since any perturbation of the Schwarzschild metric belonging to a particular spherical-harmonic mode is axisymmetric about some axis, the above framework can be used. It leads to a system of three linear, first order, ordinary differential equations for three radial functions. The solution is uniquely determined by appropriate boundary conditions and it determines, in turn, the corresponding perturbation uniquely.

In the second part of the paper, all existing different approaches to the problem [J. M. Bardeen and W. H. Press, J. Mathematical Phys. **14** (1973), 7–19; MR0321475; F. J. Zerilli, Phys. Rev. D (3) **2** (1970), 2141–2160; MR0321491; J. Regge and J. A. Wheeler, Phys. Rev. (2) **108** (1957), 1063–1069; MR0091832] are compared with the aid of the above results. Relations between the different functions, potentials and equations are explicitly written down. In this way not only is a classification of the subject matter achieved, but also the very practical possibility is created of going from one method to another in particular calculations. The hope is also expressed that a similar approach can be useful in the case of the Kerr metric.

Petr Hájíček

From MathSciNet, November 2022

MR0700826 (85c:83002) 83-02; 83C15

Chandrasekhar, Subrahmanyan

The mathematical theory of black holes. (English)

International Series of Monographs on Physics, 69.

The Clarendon Press, *Oxford University Press, New York*, 1983, xxi+646 pp., ISBN 0-19-851291-0

When seeing this new book of the famous author for the first time, curiosity may lead one simply to skip through the pages to get an impression, to find and enjoy small novel pieces of information (perhaps a new theorem), to understand the "why" of the questions and answers, and to have a glance at the main results. But one soon detects that the book is not written that way: it is a result of, needs, and deserves a systematic study. To begin with, one should know about the elements of differential geometry, forms, tetrads, and the Newman-Penrose formalism, all given in detail in the first sixty pages of the book, thus forming a self-consistent introduction into the mathematics of general relativity theory. Then the Schwarzschild metric is derived (in null coordinates) and discussed, followed by a comprehensive and detailed study of its timelike and null geodesics. This study, which includes 17 illustrations showing typical orbits and light rays, is unique in its completeness. Gravitational perturbations of the Schwarzschild black hole follow, caused, e.g., by an incident gravitational wave. It is shown in detail how Einstein's equations, linearized around the Schwarzschild metric, can be decoupled, and how the decoupled partial differential equations are separated. The emphasis is, of course, on the "radial" functions and the one-dimensional scattering problem intimately connected with them. Most of the analysis given here (in sixty-six pages) is a by-product of the study of the Kerr black hole (and is included there as a subcase), but—as the author states—it is the aim of this chapter to present a self-contained, coherent, and unified treatment of the perturbations of the spherically symmetric black hole. On his way to his final goal—the Kerr metric—the author now approaches the Reissner-Nordstrom metric, where, in conformity with the title of the book, only the case of small charges, $|M| > |Q|$, is considered; naked singularities are excluded. Again the metric is derived ab initio, geodesics are studied, and perturbations are dealt with. Naturally, the treatment of the Kerr metric constitutes the very heart of the book; three hundred pages are devoted to its study, showing the difficulties of the task, the bewildering richness of mathematical relations so far obtained, and the multitude of questions still to be answered. Among the topics treated here are: the derivation of the Kerr metric from the assumptions of axisymmetry, stationarity, existence of an event horizon, and asymptotic flatness; the Ernst equation; the uniqueness of the Kerr metric; Killing tensors and geodesic motion; Penrose process and superradiance; electromagnetic waves in Kerr geometry; gravitational perturbations and the reduction of the linearized Einstein equations to decoupled ordinary differential equations; spin-$\frac{1}{2}$ particles in Kerr geometry. Throughout, extensive use is made of the Newman-Penrose formalism. Many results of the author's own research are collected here. The bulk of the text can be understood as being a study of two families of functions defined by two ordinary differential equations of second order, each depending on several parameters (spin of the perturbations in question, separation parameters, etc.). These "Teukolsky functions" generalize the well-known Legendre and Bessel functions which appear when solving wave equations in flat space-time. Some tables give numerical values of these Teukolsky functions for a typical set of parameters. Generally, the book is well organized. Each of the chapters is followed by bibliographical notes. Despite the author's assertion that he has not made any serious search of the literature, but only gave the sources of his information, there are so many references that an author index would be useful. Some inconvenience is caused by the fact that the equation numbers run through each chapter separately, but no chapter numbers are given on the pages: to find an equation given by number and chapter, one has to consult the table of contents. Sometimes one regrets that the author sticks too much to the "mathematics" of black holes and does not include the physical and astrophysical explanations and implications of his calculations, e.g., the relation between electromagnetic scattering modes and the possible vacua of quantum electrodynamics. But certainly this might fill another volume! Altogether, this book offers a magnificent piece of mathematical physics, sometimes overwhelming in its complexity, written by an author who obviously loves and admires his subject. To let him have the last word: "Perhaps, at a later time, the complexity will be unravelled by deeper insights. But meantime, the analysis has led us into a realm of rococo: splendorous, joyful, and immensely ornate."

Hans Stephani
From MathSciNet, November 2022

MR1316662 (95k:83006) 83C05; 35Q75, 58G16, 83C35
Christodoulou, Demetrios; Klainerman, Sergiu
The global nonlinear stability of the Minkowski space. (English)
Princeton Mathematical Series, 41.
Princeton University Press, *Princeton, NJ*, 1993, x+514 pp., ISBN 0-691-08777-6

This book presents the authors' theorem on the stability of Minkowski space, a landmark in the development of mathematical relativity. The book is quite self-contained but it is worth mentioning two useful sources of background information. An article by J.-P. Bourguignon [Astérisque No. 201-203 (1991), Exp. No. 740, 321–358 (1992); MR1157847] provides an introduction to various geometrical aspects of the proof while a paper of the authors [Comm. Pure Appl. Math. **43** (1990), no. 2, 137–199; MR1038141] shows some of the central analytic tools at work in a simpler setting. The main statement of the theorem is, informally, that, given any initial data set for the vacuum Einstein equations which is sufficiently close to the initial data induced on a hyperplane in Minkowski space, there exists a corresponding solution which is global in the sense of being geodesically complete, and whose asymptotic structure resembles that of Minkowski space. In the book there are three statements of versions of the main theorem which are increasingly precise (and technical). The first two are contained in Chapter 1 (the introduction) while the third is contained in Chapter 10, where the highest level steps of the proof are carried out. The book is not easy to read, due to the very technical nature of its contents, but under the circumstances the quality of the exposition is excellent.

It is impossible to give a useful idea of the proof in this review but some elements of its structure will be used to organize the description of the contents of the various chapters which follows. Energy estimates are the motor which drives the machinery of the proof. They are obtained using the Bel-Robinson tensor, a fourth rank tensor quadratic in the curvature. This tensor is such that, given any timelike conformal Killing vector field in a spacetime, it provides a conservation law which represents a weighted L^2 estimate for the curvature. Commuting with vector fields then gives estimates for derivatives of the curvature. (In the context of this proof this step requires a great deal of care.) This part of the proof is carried out in Chapters 7 and 8. Since the proof deals with arbitrary small perturbations of Minkowski space, in general the spacetimes considered possess no Killing vectors. In fact what is used is that these spacetimes do have approximate Killing vectors, which correspond to the exact Killing vectors of Minkowski space. Combining these approximate Killing vectors with the Bel-Robinson tensor leads to approximate conservation laws. The construction of approximate Killing vectors which give useful estimates is very delicate. This construction, and the estimates it gives rise to, forms the content of Chapters 9 and 11–16. As is clear from what has already been said, the primary estimates for the geometry which are obtained are estimates for the curvature. In order to turn this information into bounds for the metric a number of theorems on elliptic equations on two- and three-dimensional Riemannian manifolds are required. These are proved in Chapters 2–6. In Chapter 10 all the estimates mentioned up to now are linked together, revealing the structure of the proof as an enormous bootstrap argument.

The final chapter of the book is devoted to a closer examination of the asymptotic structure of the spacetimes whose existence is asserted by the theorem. It is shown that the laws of gravitational radiation discovered by Bondi and others more than thirty years ago using formal power series expansions are rigorously true in this class of spacetimes. Thus the reader is given some idea of the question which motivated the study of the problem solved in this book.

Alan D. Rendall

From MathSciNet, November 2022

MR1680551 (2000a:83086) 83C75; 35L67, 35L75, 53C50, 83C05

Christodoulou, Demetrios

The instability of naked singularities in the gravitational collapse of a scalar field.

Annals of Mathematics. Second Series **149** (1999), *no.* 1, 183–217.

A naked singularity in a solution of the Einstein equations is a singularity which is visible from infinity. The cosmic censorship hypothesis, asserting the absence of naked singularities in solutions of the Einstein equations under appropriate circumstances, has been a central theme of mathematical relativity theory since it was proposed by R. Penrose thirty years ago [Riv. Nuovo Cimento **1** (1969), Special Issue, 252–276]. This hypothesis is not as it stands a well-defined mathematical conjecture, due to the necessity of determining what "appropriate circumstances" means. Here input is required from physics. As a consequence, work on this subject has been concerned as much with finding the right formulation as with proving the resulting statement. Over the years, counterexamples to overly naive formulations have accumulated. One important point is the choice of matter model. The Einstein equations describe self-gravitating matter and the formulation must include assumptions about the description of the matter. In the paper under review, matter is described by a massless scalar field, which is expected to be a good choice. Another important idea is that a genericity assumption may be necessary: naked singularities may occur, but they should be unstable under perturbations.

The paper under review is the culmination of a series of eight by the author on spherically symmetric solutions of the Einstein equations coupled to a scalar field. Without the assumption of spherical symmetry the problem would be totally out of reach at this time. The scalar field is the simplest choice for the matter model. (In vacuum the problem trivializes, due to Birkhoff's theorem.) The main result is that generic solutions with regular initial data do not develop naked singularities. In fact the set of initial data which do lead to naked singularities is of codimension at least one. In a previous paper [Ann. of Math. (2) **140** (1994), no. 3, 607–653; MR1307898] the author showed the existence of solutions which do develop naked singularities. Now he shows that they are unstable and thus demonstrates for the first time that a mechanism which has been suggested as being crucial for ensuring cosmic censorship really does occur in solutions of the Einstein equations.

In the instability proof an important role is played by solutions with non-smooth initial data. The author previously showed [Comm. Pure Appl. Math. **46** (1993), no. 8, 1131–1220; MR1225895] that existence and uniqueness in the Cauchy problem for the spherically symmetric Einstein-scalar field equations can be obtained for data of bounded variation. He was also able to obtain a sharp continuation criterion for these solutions. These remarkable results required estimates of a quite new type. They allow solutions corresponding to data with a jump discontinuity to be constructed. It is shown in the paper under review, using a criterion from a previous paper by the author [Comm. Pure Appl. Math. **44** (1991), no. 3, 339–373; MR1090436], that when general initial data are perturbed by the introduction of a small jump discontinuity in a particular way the resulting solutions contain a black hole rather than a naked singularity. It is also shown that the same effect can be achieved by introducing a small absolutely continuous perturbation. By this means it can be seen that data giving rise to naked singularities are of codimension at least two in the space of data of bounded variation and of codimension at least one in the space of absolutely continuous data.

The absence of naked singularities means that any singularities formed in gravitational collapse are contained in black holes. Thus cosmic censorship says that under reasonable conditions gravitational collapse leading to the occurrence of a singularity almost always results in the formation of a black hole. That this should be the case is fundamental to the way general relativity is applied today. The work of Christodoulou reviewed here shows how many profound mathematical ideas are required to prove the correctness of this picture even in a restricted special case. It gives an indication that the road to a proof of cosmic censorship is likely to be long and arduous and at the same time it means that one important milestone along the way has been reached.

Alan D. Rendall

From MathSciNet, November 2022

MR2531716 (2010k:83009) 83C05; 53C50, 53C80

Bieri, Lydia; Zipser, Nina

Extensions of the stability theorem of the Minkowski space in general relativity. (English)

AMS/IP Studies in Advanced Mathematics, 45.
American Mathematical Society, *Providence, RI; International Press, Cambridge, MA*, 2009, xxiv+491 pp. pp., ISBN 978-0-8218-4823-4

In the 1990s, D. Christodoulou and S. Klainerman proved the global stability of Minkowski space. The proof is so long that it fills a whole book [*The global nonlinear stability of the Minkowski space*, Princeton Univ. Press, Princeton, NJ, 1993; MR1316662]. In a nutshell, their result says that the initial value problem of the vacuum Einstein equation admits a geodesically complete solution, provided that the initial data satisfy the conditions of global smallness and of strong asymptotic flatness. Global smallness means that, in terms of an appropriately defined norm, the initial data (i.e., the 3-metric g_{ij} and the second fundamental form k_{ij}) are close to the data for Minkowski space ($g_{ij} = \delta_{ij}$ and $k_{ij} = 0$). Strong asymptotic flatness means that $g_{ij} = (1 + 2m/r)\delta_{ij} + o(r^{-3/2})$ and $k_{ij} = o(r^{-5/2})$ for $r \to \infty$. Among other things, the relevance of the Christodoulou-Klainerman result lies in the fact that it demonstrated, for the first time, the existence of geodesically complete solutions to the vacuum Einstein equations other than Minkowski space. To translate the result into a physicist's terminology, one has to observe that solutions to the vacuum Einstein equation describe pure (source-free) gravitational fields, including gravitational waves, and that geodesic completeness implies the absence of singularities in the sense of the Hawking-Penrose singularity theorems. Hence, the Christodoulou-Klainerman result implies that weak gravitational waves that satisfy a certain fall-off condition cannot grow so strongly that they produce singularities.

The book under review consists of two independent parts which present two different extensions of the Christodoulou-Klainerman theorem. In the first part Bieri presents the results of her Ph.D. thesis ["An extension of the stability theorem of the Minkowski space in general relativity"], which was written in 2007 under the supervision of Christodoulou at the ETH in Zürich. She proves that the Christodoulou-Klainerman theorem remains valid if the condition of strong asymptotic flatness is replaced with a weaker fall-off condition. She also works with a somewhat modified condition of global smallness. As with the weaker fall-off conditions some estimates become borderline cases, it is argued that the condition cannot be further relaxed. From a technical point of view, the most important new features are that, with the weaker fall-off condition, the spacetime curvature is no longer bounded in the L^∞ sense and that the angular momentum is no longer defined. (The linear 4-momentum still is.) Christodoulou and Klainerman had used, in their proof, the angular momentum almost Killing vector fields for controlling the angular derivatives of the curvature. In her modified version, Bieri cannot use these tools; however, she is able to show that the desired conditions can be found from the Bianchi identities. There are a great number of technicalities to be worked out, so the whole proof fills almost 300 pages.

In the second part Zipser presents the results of her Ph.D. thesis ["The global nonlinear stability of the trivial solution of the Einstein-Maxwell equations", Harvard Univ., Cambridge, MA], which was written in 2000 under the supervision of S.-T. Yau. She generalises the Christodoulou-Klainerman theorem by replacing the vacuum Einstein equation with the Einstein-Maxwell equations. Correspondingly, she needs smallness and fall-off conditions not only for the gravitational field but also for the electromagnetic field. Introducing appropriately generalised norms and calculating estimates for these norms requires a lot of additional work; the whole proof fills more than 200 pages. In contrast to the first part, the fall-off conditions are chosen in analogy to Christodoulou and Klainerman's. Hence, the same number of almost (conformal) Killing vector fields is available, and they play an important part in the proof.

Both parts are well written. However, as the subject is a highly technical one and the proofs are quite long and involved, the book is a difficult read. The results are important, without any doubt, and the book should be of interest to anyone who is doing research in mathematical relativity.

Volker Perlick

From MathSciNet, November 2022

MR3919495 35Q76; 35B30, 35P25, 83C57

Dafermos, Mihalis; Holzegel, Gustav; Rodnianski, Igor

Boundedness and decay for the Teukolsky equation on Kerr spacetimes I: The case $|a| \ll M$. (English)

Annals of PDE **5** (2019), *no.* 1, Paper No. 2, 118 pp.

The work constitutes a first step towards the proof of the linear stability of Kerr solutions to gravitational perturbations, on the exterior of the black hole region, the ultimate goal being to establish the nonlinear (dynamic) stability of the Kerr family as solutions to the Einstein vacuum equations, without symmetry assumptions, in analogy to the global nonlinear stability of Minkowski space established by D. Christodoulou and S. Klainerman in [*The global nonlinear stability of the Minkowski space*, Princeton Math. Ser., 41, Princeton Univ. Press, Princeton, NJ, 1993; MR1316662].

The present step consists in proving boundedness and polynomial decay results for solutions to the so-called spin ± 2 Teukolsky equation on a Kerr exterior background in the very slowly rotating regime $|a| \ll M$.

As a second step, analogous results to the ones presented here shall be proven for general sub-extremal Kerr parameters $|a| < M$.

Finally, in a third step, these boundedness and decay properties of solutions to the Teukolsky equation will be used to prove the aforementioned linear stability result.

This program generalizes the one followed by the authors in [Acta Math. **222** (2019), no. 1, 1–214; MR3941803] to establish the linear stability of the Schwarzschild solution ($a = 0$) to gravitational perturbations.

Since examining the linear stability of the Einstein vacuum equations around (Schwarzschild or Kerr) solutions requires the imposition of a gauge in which these equations become well-posed, a crucial step in the proof of linear stability consists in the control of the gauge invariant part of the perturbations.

In the general Kerr case, S. A. Teukolsky [Astrophys. J. **185** (1973), 635–648,] identified two gauge invariant quantities, the complex scalars $\alpha^{[\pm 2]}$, which decouple from the full linearization of the Einstein vacuum equations and satisfy a wave equation, viz. the spin ± 2 Teukolsky equation.

In the present paper, the boundedness and decay properties of $\alpha^{[\pm 2]}$ are established by extending the analysis of this equation in [M. Dafermos, G. H. Holzegel and I. Rodnianski, op. cit.], where $a = 0$, to the case $|a| \ll M$.

This is possible by observing that although in the general Kerr case $a \neq 0$ (and in contrast to the Schwarzschild case $a = 0$), certain quantities $P^{[\pm 2]}$ that are associated to the solutions $\alpha^{[\pm 2]}$ do not totally decouple from $\alpha^{[\pm 2]}$; nevertheless, the coupling is in a certain sense weak enough to be controlled, at least when $|a| \ll M$, in particular invoking frequency-localization methods.

Ioannis Giannoulis

From MathSciNet, November 2022

MR3969149 83C75; 35Q75, 35R01, 83C57

Luk, Jonathan; Oh, Sung-Jin

Strong cosmic censorship in spherical symmetry for two-ended asymptotically flat initial data II: the exterior of the black hole region.

Annals of PDE **5** (2019), *no.* 1, Paper No. 6, 194 pp.

This is a follow-up paper for the completion of the proof of the C^2-formulation of the strong cosmic censorship conjecture regarding an Einstein-Maxwell-(real) scalar field system with spherical symmetry for two-ended asymptotically flat initial data. Indeed, the class of spherically symmetric solutions of a system of equations for a real-valued function ϕ satisfying the linear scalar wave equation $\Box_g\phi = 0$ on a four-dimensional manifold M with a Lorentzian metric g and a 2-form F with $dF = 0$ on it are studied in this context. In Part I [Ann. of Math. (2) **190** (2019), no. 1, 1–111; MR3990601], the authors have shown, particularly based on prior decay results by M. Dafermos and I. Rodnianski, that if (M, g, ϕ, F) is a maximal globally hyperbolic future development of an admissible two-ended asymptotically flat smooth spherically symmetric Cauchy initial data set with a smooth and compactly supported scalar field and a non-vanishing charge, then (M, g) is depicted by one of two possible Penrose diagrams and is future@-inextendible with a C^2-Lorentzian metric provided that an L^2-averaged lower bound holds for the scalar field derivative along each horizon. In the present paper, this lower bound result is extended for a generic set of admissible and smooth Cauchy initial data that is open to a (weighted) C^1 topology and dense to a (weighted) C^∞ topology and holds on each of the event horizons, whereby the authors rely on existing results on linear instability of black hole exterior regions, and on lower bounds for solutions to the linear wave equation in such regions.

Thus, starting with the equations of the Einstein-Maxwell-(real) scalar field system in double null coordinates in spherical symmetry and the formulation of the relevant Cauchy problem and the characteristic initial value problem, the admissible Cauchy initial data and some preliminary results on the corresponding maximal globally hyperbolic future development are presented, before a precise statement of the paper's main result and its proof, amounting to three (actually stronger) theorems that imply it, are given. The authors proceed to the derivation of decay estimates in the region exterior to the maximal globally hyperbolic future development arising from an arbitrary admissible Cauchy data set. For these decays, based on the Price law's decay estimates obtained by the Dafermos-Rodnianski theorem, future-normalized (retarded and advanced) null coordinates are used. Next, the detailed proof (by contradiction) of the first of the aforementioned three theorems is given by showing that an L^2-averaged lower bound holds along the event horizon. The detailed proof of the second theorem, a stability theorem, comes next. To this purpose, first a Cauchy stability result is stated for a (spherically symmetric) compact region, for the system of equations under consideration. Then, the stability is examined for a neighborhood of spatial infinity and for a neighborhood of null infinity, and finally, the (mathematically quite demanding) proof of the second theorem is completed by applying a bootstrap argument, and utilizing Hardy-type estimates, Morawetz estimates, and r^p-weighted energy estimates. The paper's final section contains the proof of the third and last theorem. Here, smooth and essentially compactly supported one-parameter perturbations of admissible Cauchy data are constructed by solving the constraint equations (that, due to the spherical symmetry assumed, are reduced to a system of ordinary differential equations) on a Cauchy hypersurface, close to the asymptotically (Minkowski) flat end. It is shown that the leading term near spatial infinity can be well controlled, while the remainder contribution is negligibly small.

Theophanes Grammenos
From MathSciNet, November 2022

MR4150259 83C05; 35Q75, 35Q83, 83C57

Moschidis, Georgios

A proof of the instability of AdS for the Einstein-null dust system with an inner mirror.

Analysis & PDE **13** (2020), *no.* 6, 1671–1754.

In 2006, Dafermos and Holzegel formulated the AdS instability conjecture, which states that there exist arbitrarily small perturbations to AdS initial data which evolve by the Einstein vacuum equations with cosmological constant Λ together with reflecting boundary conditions on conformal infinity $\mathcal{I}$ that lead to the development of trapped surfaces, i.e., black holes. Thus such AdS initial data are nonlinearly unstable. P. Bizoń and A. Rostworowski [Phys. Rev. Lett. **107** (2011), no. 3, 031102], this conjecture in a simple setting of a spherically symmetric Einstein-scalar field. Since then, there have been numerous works, both numerical and heuristic, on this conjecture.

In this paper, the author provides the first rigorous proof of the AdS instability conjecture in the simplest possible setting, namely for the Einstein-massless Vlasov system in spherical symmetry, further reduced to the case when the Vlasov field is supported only on radial geodesics. In fact, this system is equivalent to the spherically symmetric Einstein-null dust system, allowing for both ingoing and outgoing dust. This system has been studied in the $\Lambda = 0$ case by E. Poisson and W. Israel [Phys. Rev. D (3) **41** (1990), no. 6, 1796–1809; MR1048877]. There is a serious problem with the spherically symmetric Einstein-null dust system, which is that it breaks down when the null dust reaches the center $r = 0$. The author points out that for any reasonable initial data, the spherically symmetric Einstein-null dust system is not well-posed and AdS is not a Cauchy stable solution of it. One way to restore the well-posedness of this system is to place an inner mirror at some radius $r = r_0 > 0$ and to study the evolution of the system in the exterior region $r > r_0$ and to allow r_0 to shrink to 0 as the total ADM mass of the initial data shrinks to 0 in order to address this conjecture.

After the introduction and setting up the problem in double null coordinates, the author first establishes Cauchy stability for the Einstein-scalar field system, and the breakdown at $r = 0$ for the Einstein-null dust system. Then the resolution of the problem is to introduce the reflecting boundary conditions. He then refers to the well-posedness and the properties of the maximal development for a system with reflecting boundary conditions, which originates in his companion paper ["The Einstein-null dust system in spherical symmetry with an inner mirror: structure of the maximal development and Cauchy stability", preprint, `arXiv:1704.08685`]. He also introduces an interesting technical idea which involves the arrangement of the null dust into a large number of beams which are successively reflected on the moving boundary and the conformal infinity $\mathcal{I}$, in a configuration that forces the energy of a certain beam to increase after each successive pair of reflections. When r_0 shrinks to 0, the number of reflections before a black hole is formed necessarily goes to infinity.

After each theorem and proposition, the author also provides several technical remarks that elaborate on the conditions required or on possible generalizations.

Kotik K. Lee

From MathSciNet, November 2022