BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 60, Number 2, April 2023, Pages 277–282
https://doi.org/10.1090/bull/1788
Article electronically published on November 3, 2022

Modern analysis of automorphic forms by example. Vol. 1, by Paul Garrett, Cambridge Studies in Advanced Mathematics, Vol. 173, Cambridge University Press, Cambridge, 2018, xxii+384 pp., ISBN 978-1-107-15400-1

Modern analysis of automorphic forms by example. Vol. 2, by Paul Garrett, Cambridge Studies in Advanced Mathematics, Vol. 174, Cambridge University Press, Cambridge, 2018, xxii+343 pp., ISBN 978-1-108-47384-2

1. Motivation

Automorphic forms appear in many branches of mathematics and have especially important applications in number theory. To help motivate the subject for beginners and outsiders to the area, this review will begin with some of these applications. Garrett's volumes focus mainly on the analytic issues surrounding automorphic forms, and so in keeping with this theme, these applications will also be analytic in nature. There is also great interest of automorphic forms in other directions, such as in algebraic number theory or in spectral geometry, and to which the reader may consult [1, 5, 6, 13] for some launching points.

A prototype for the analysis of automorphic forms is classical Fourier analysis in Euclidean space. Fourier analysis has long had applications in number theory. Many proofs of the evaluation of the Riemann zeta function at positive even integers, first performed by Euler, use some form of Fourier analysis. Riemann himself used the Poisson summation formula to derive the functional equation of the zeta function. Another line of thinking uses Fourier analysis to deduce accurate approximations to the number of integer lattice points in expanding regions, such as in the Gauss circle problem, which asks for an estimate on the number of these points inside a large circle in $\mathbb{R}^2$ centered at the origin. In more modern terminology, we may interpret these uses of Fourier analysis as applications of the spectral decomposition of the Laplacian on $L^2(\mathbb{Z}\backslash\mathbb{R})$ or $L^2(\mathbb{R})$.

Some advanced applications use analysis on more exotic spaces, such as $\Gamma\backslash\mathcal{H}$, where Γ is a discrete subgroup of $SL_2(\mathbb{R})$ (such as $SL_2(\mathbb{Z})$), and $\mathcal{H}$ is the upper half-plane acted on by Möbius transformations. A sample application in this direction is to count asymptotically the number of integer matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $ad - bc = 1$ and $a^2 + b^2 + c^2 + d^2 \leq X$, as $X \to \infty$. This is an instance of a hyperbolic lattice point counting problem. A related variant of this problem concerns the shifted divisor problem. This problem tasks us with developing an asymptotic formula for $\sum_{n \leq X} d(n)d(n + h)$, where $d(n)$ is the divisor function, and h is a positive integer. The shifted divisor problem is in turn vital for understanding the fourth moment of the Riemann zeta function, that is, $\int_0^T |\zeta(1/2 + it)|^4 dt$. The state of the art on these problems relies on the spectral theory of the Laplacian on $SL_2(\mathbb{Z})\backslash\mathcal{H}$. It has long been hoped, but so far not yet achieved, that generalizations to $SL_d(\mathbb{Z})$ should further our understanding of the $2d$-th moment of zeta, for $d \geq 3$. One barrier (among many) to progress on this tantalizing problem is that automorphic forms on $SL_d(\mathbb{Z})$ with $d \geq 3$ are much more complicated than for $d = 2$.

Euclidean harmonic analysis is also a powerful tool for understanding equidistribution questions in number theory. The original version of Weyl's criterion reduces an equidistribution question of a sequence x_j modulo 1 to a nontrivial bound on the exponential sum $\sum_{j=1}^{J} \exp(2\pi i n x_j)$, for each nonzero integer n. A simple yet fundamental example considers $x_j = \alpha j$ with α irrational, in which case the exponential sum is recognizable as a geometric series and is then easy to bound. Weyl's equidistribution criterion may be generalized to $\Gamma\backslash\mathcal{H}$, but the basic exponentials must be replaced with a complete system of eigenfunctions of the Laplacian on the space $\Gamma\backslash\mathcal{H}$.

Our final motivational example, the Selberg trace formula, has probably been the most influential. Selberg's trace formula gives a connection between spectral data of $\Gamma\backslash\mathcal{H}$ on the one side, and geometric data on the other. The formula is simpler when the space $\Gamma\backslash\mathcal{H}$ is compact, in which case the spectrum consists of a countably infinite discrete sequence of eigenvalues. The trace formula then has an application to count asymptotically the number of eigenvalues up to a given bound X, which is called Weyl's law. When $\Gamma\backslash\mathcal{H}$ is not compact but has finite hyperbolic area, then there is both a countable collection of eigenvalues as well as a continuous family furnished by Eisenstein series. In this situation, the trace formula is not able to directly separate the continuous spectrum from the discrete spectrum, but instead gives an asymptotic formula for some measure of both spectra in a combined form. Without further input, it is thus not even clear if the discrete spectrum is infinite or finite (see [13] for more on this issue). However, for arithmetical groups Γ, such as $SL_2(\mathbb{Z})$, Selberg directly showed that the continuous spectrum by itself does not contribute enough to match the leading term on the geometric side of the trace formula. Therefore, the discrete spectrum is infinite in these arithmetical cases, and the trace formula recovers Weyl's law.

The Selberg trace formula has a nice analogy to the explicit formula for the Riemann zeta function, which connects the zeros of the zeta function on the one side with primes on the other. We have much more information about the spectral side of the Selberg trace formula, and for instance for $SL_2(\mathbb{Z})$, the analogue of the Riemann hypothesis is known. It has long been hoped that one can prove the Riemann hypothesis by finding a spectral interpretation of the zeros of the zeta function, and Selberg's trace formula has helped fuel this hope. See [3] for more discussion.

2. Definitions, by example

A potential reader may have a different expectation of what constitutes an "example" than what appears Garrett's book. One might think that an example could mean a particular automorphic form, such as the Ramanujan delta function or holomorphic Eisenstein series of small weight, but in fact Garrett's examples consist of the full infinite families of automorphic forms on some relatively simple spaces. These example families themselves have been the complete scope of some other textbooks.

2.1. **The "small examples".** Garrett considers four spaces of the form $\Gamma\backslash X = \Gamma\backslash G/K$ where G is a group, K is a compact subgroup, and Γ is a discrete subgroup. The two cases most easy to describe have $G = SL_2(\mathbb{R})$ and $G = SL_2(\mathbb{C})$. The next two cases use matrices with elements in the Hamiltonian quaternions $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$. The third case is a symplectic group $G = Sp^*_{1,1} = \{g \in GL_2(\mathbb{H}) : g^* S g = S\}$, where $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and g^* is the conjugate transpose of g.

The fourth case takes $G = SL_2(\mathbb{H})$. For $j = 1, 2$, let $Sp_j^* = \{g \in GL_j(\mathbb{H}) : g^*g = 1\}$. The compact group K is defined as

$$K = \begin{cases} SO_2(\mathbb{R}), & \text{for } G = SL_2(\mathbb{R}), \\ SU_2, & \text{for } G = SL_2(\mathbb{C}), \\ Sp_1^* \times Sp_1^*, & \text{for } G = Sp_{1,1}^*, \\ Sp_2^*, & \text{for } G = SL_2(\mathbb{H}). \end{cases}$$

Next we define Γ. For $SL_2(\mathbb{R})$ we have $\Gamma = SL_2(\mathbb{Z})$, and for $SL_2(\mathbb{C})$ we have $\Gamma = SL_2(\mathbb{Z}[i])$, where $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ is the set of Gaussian integers, a Euclidean ring. Define the ring of Hurwitz integers $\mathfrak{o} \subset \mathbb{H}$ by

$$\mathfrak{o} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}k + \mathbb{Z} \cdot \frac{1 + i + j + k}{2}.$$

The Hurwitz integers are also Euclidean of a sort (though noncommutative). For $G = Sp_{1,1}^*$ and $G = Sp_2^*$, Γ consists of the elements of G with entries in $\mathfrak{o}$.

In these four examples, the space $\Gamma\backslash G/K$ is not compact, but it has finite G-invariant (Haar-type) measure, and there is a single cusp. The Laplacian Δ is a second-order differential operator that commutes with the action of the group G.

An *automorphic function* is a function on $\Gamma\backslash X$, which can be viewed as a function on G that is left-invariant by Γ and right-invariant by K. There is particular interest in the automorphic functions that are also eigenfunctions of the Laplacian.

There is an important and pervasive dichotomy regarding the behavior of an automorphic function f with respect to the cusp. This is measured most naturally in terms of the *constant term* of f, defined as an integral $\int_{(N\cap\Gamma)\backslash N} f(n.x)dn$, where $N \subset G$ is the subset of elements of the form $\{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$. A *cusp form* is an automorphic function with vanishing constant term. The complementary notion to a cusp form is that of an *Eisenstein series*, defined as

$$E(z, s) = E_s(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \eta(\gamma z)^s,$$

where Γ_∞ is the stabilizer of ∞, and η is a certain well-defined function on the quotient. In the $SL_2(\mathbb{R})$ case, $\eta(x + iy) = y$, and the definition for the other small examples is similar. The Eisenstein series is an eigenfunction of Δ, with eigenvalue $s(s-1)$, but it is not a cusp form.

2.2. **The adelic example.** Let k be a number field, and let $\mathbb{A}_k$ be the ring of adeles over k. Tate [14] developed harmonic analysis on the adeles, and derived the functional equation of the Dedekind zeta function. Since then, the adelic framework of automorphic forms and their L-functions has flourished. One of the advantages of the adelic approach is that it may treat all number fields in a relatively uniform fashion. Garrett's adelic example is the development of the theory of automorphic forms on $GL_2(k)\backslash GL_2(\mathbb{A}_k)$. His exposition proceeds in a parallel fashion with the four small examples.

2.3. **The "larger examples".** Garrett's third family of examples consists of $\Gamma = SL_n(\mathbb{Z})$, $G = SL_n(\mathbb{R})$, and $K = SO_n(\mathbb{R})$, with $n \geq 3$. However, Garrett develops much of the theory in greater generality, working adelically.

One of the main new issues arising in this third family of examples is the much larger array of Eisenstein series. More general Eisenstein series may be induced from the various parabolic subgroups contained in G, via cuspidal automorphic forms from lower-dimensional groups.

3. Main theorems

To get a feel for the focus and scope of Garrett's books, we next highlight a few of the most important results that are proved therein. For simplicity of this review, consider the setting of the four small examples. Define $L^2_0(\Gamma\backslash X)$, with $X = G/K$, to be the space of square-integrable cusp forms. The following theorem says that this space is discretely decomposed into Δ-eigenfunctions.

Theorem 3.1. *The space* $L^2_0(\Gamma\backslash X)$ *is a closed subspace of* $L^2(\Gamma\backslash X)$ *and has an orthonormal basis of* Δ*-eigenfunctions. Each eigenspace is finite dimensional, and the number of eigenvalues below a given bound is finite.*

The full spectral decomposition is harder and needs additional information about the Eisenstein series:

Theorem 3.2. *The Eisenstein series* E_s *has meromorphic continuation to* $s \in \mathbb{C}$. *There are only finitely many poles in the region* $\mathrm{Re}(s) \geq 1/2$*, which all lie on the segment* $(1/2, 1]$.

For the special case of $SL_2(\mathbb{Z})$, the meromorphic continuation may be derived by an explicit calculation of the Fourier expansion, and it relies on the meromorphic continuation of the Riemann zeta function. However, this argument does not generalize. Moreover, it is desirable to reverse this reasoning, and use properties of the Eisenstein series to deduce facts about L-functions. This is the strategy of the important Langlands–Shahidi method; see [6, Section 8] for a gentle introduction. Garrett gives a thorough proof of the crucial Theorem 3.2.

Finally, we state the full spectral decomposition.

Theorem 3.3. *Suppose* $f \in L^2(\Gamma\backslash X)$. *Let* ϕ_j *run over a complete orthonormal basis of* Δ*-eigenfunctions that are cusp forms. Then* f *has the expansion*

$$f = \sum_j \langle f, \phi_j \rangle + \frac{1}{4\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \langle f, E_s \rangle E_s ds + \sum_{s_0} \langle f, \mathrm{Res}_{s_0} E_s \rangle \mathrm{Res}_{s_0} E_s.$$

The above versions of the results were stated for the four smallest examples, but Garrett also produces the analogues for the adelic example and the family of larger examples.

4. Garrett's books

4.1. **Organization and prerequisites.** Garrett's system of presentation begins by devoting the first three chapters to developing definitions and stating the main results mentioned in Section 3 for the three families of examples. The proofs are postponed to later in the book. The two volumes combined weigh in at over 700 pages, in part due to the extensive background that is provided in later chapters. Even with this amount of background, it is inevitable that some readers may be missing some implicit prerequisites. For instance, Garrett assumes familiarity with the adeles over a general number field. The reader is also assumed to have some familiarity with topological groups and, in particular, with the existence of Haar measure on a locally compact group. Some experience with functional analysis would be very helpful, though Volume 2 contains a streamlined presentation of the necessary background. The main prerequisite for the book is a level of maturity that one might expect from more advanced graduate students.

The upshot of stating the main results early in the exposition is that it draws attention to the main goals, and helps the reader avoid getting lost in the myriad details. A downside is that it can be difficult to determine when a proof of one of these results is complete. To highlight this problem, let's consider a thought experiment where a curious but inexperienced reader encounters a claim in Chapter 1.10, stating that an Eisenstein series is well approximated by its constant term. Garrett refers to Chapter 8.1 for a proof of this fact. Browsing Chapter 8.1, the reader finds a proof, but with references to some material from Chapter 14, on vector-valued integrals. Perhaps Chapter 14 is an appendix that needs mastery prior to reading Chapter 8? Now we imagine the curious reader begins reading Chapter 14, which immediately begins with some previously undefined vocabulary, such as "quasi-complete", "locally convex topological vector space", "Hahn–Banach theorem", etc. With some digging around in the index, one finds that the term locally convex is defined in Chapter 13.11, the Hahn–Banach theorem is presented in Chapter 14.A (an appendix itself), and quasi-complete is defined on p. 214, in Chapter 13.8. Somewhat distressingly, the term quasi-complete appears on pages 188 and 211 as well, prior to its definition and without any indicator that a definition is yet to appear. I would imagine that this hypothetical reader is left at a loss as to what is the larger-scale logical order of the chapters. This problem could have been at least partially alleviated with the inclusion of a logical dependency graph and with more pointers to the location of definitions. On a local level, upon encountering some new vocabulary, this reader would be uncertain about what is to be defined later in the book and what is truly taken as a prerequisite.

4.2. **Style.** Garrett's writing style is distinctive. He tends to use long sentences with multiple clauses, and liberally uses italics for emphasis. How much one enjoys or dislikes it is plainly a matter of taste. I personally found some of these stylistic issues distracting, but not a serious barrier to reading the volumes.

4.3. **Comparison and contrast to other books.** The main attraction of Garrett's book(s) is its wealth of content that is difficult (if not impossible) to find in other sources. A generation of researchers in analytic number theory have used Iwaniec's book [10] for the spectral theory of $\Gamma\backslash\mathcal{H}$, for Γ a discrete subgroup of $SL_2(\mathbb{R})$. The book [4] treats $\Gamma = SL_2(\mathbb{Z}[i])$. I am not aware of similar references for Garrett's other two "small" examples. A nice feature of Garrett's work is that it typically presents arguments that can treat all four small examples at once.

For Garrett's adelic example, Bump's book [2] has been the standard reference since it appeared in 1996. More recently, Goldfeld and Hundley's volumes [8, 9] have been a welcome addition to the literature. Both of these references are more focused on L-functions and local representation theory than Garrett.

Garrett's most difficult example of $SL_n(\mathbb{Z})$, $n \geq 3$, has its closest comparison with Goldfeld's book [7]. Goldfeld's work is a more gentle introduction, and focuses more on the construction of L-functions. I think many students will prefer to read Goldfeld's book, and turn next to Garrett for a more serious and thorough treatment of the spectral decomposition, which is not treated in full detail by Goldfeld. Indeed, Goldfeld states the spectral decomposition carefully only for $SL_3(\mathbb{Z})$, and the proof is light on details compared to Garrett. On the other hand, compared to [11] and [12], Garrett's exposition has fewer prerequisites and provides more details.

It should also be pointed out that Garrett does not treat some important topics, such as L-functions, automorphic representations, and the Arthur–Selberg trace formula, each of which may be found in some of the other sources that are more limited in scope in other respects. Considering the length of Garrett's volumes, these omissions are not objectionable.

4.4. **Conclusion.** Garrett's volumes fill an important void in the literature of automorphic forms. Their main weakness may be the organizational issues highlighted earlier in this review. Nevertheless, a dedicated and mature student can overcome these issues and will surely find these volumes invaluable.

References

[1] D. Bump, J. W. Cogdell, E. de Shalit, D. Gaitsgory, E. Kowalski, and S. S. Kudla, *An introduction to the Langlands program*, Birkhäuser Boston, Inc., Boston, MA, 2003. Lectures presented at the Hebrew University of Jerusalem, Jerusalem, March 12–16, 2001; Edited by Joseph Bernstein and Stephen Gelbart. MR1990371

[2] Daniel Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997, DOI 10.1017/CBO9780511609572. MR1431508

[3] J. Brian Conrey, *The Riemann hypothesis*, Notices Amer. Math. Soc. **50** (2003), no. 3, 341–353. MR1954010

[4] J. Elstrodt, F. Grunewald, and J. Mennicke, *Groups acting on hyperbolic space*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. Harmonic analysis and number theory, DOI 10.1007/978-3-662-03626-6. MR1483315

[5] Stephen Gelbart, *An elementary introduction to the Langlands program*, Bull. Amer. Math. Soc. (N.S.) **10** (1984), no. 2, 177–219, DOI 10.1090/S0273-0979-1984-15237-6. MR733692

[6] Stephen S. Gelbart and Stephen D. Miller, *Riemann's zeta function and beyond*, Bull. Amer. Math. Soc. (N.S.) **41** (2004), no. 1, 59–112, DOI 10.1090/S0273-0979-03-00995-9. MR2015450

[7] Dorian Goldfeld, *Automorphic forms and L-functions for the group* $\mathrm{GL}(n, \mathbf{R})$, Cambridge Studies in Advanced Mathematics, vol. 99, Cambridge University Press, Cambridge, 2006. With an appendix by Kevin A. Broughan, DOI 10.1017/CBO9780511542923. MR2254662

[8] Dorian Goldfeld and Joseph Hundley, *Automorphic representations and L-functions for the general linear group. Volume II*, Cambridge Studies in Advanced Mathematics, vol. 130, Cambridge University Press, Cambridge, 2011. With exercises and a preface by Xander Faber, DOI 10.1017/CBO9780511973628. MR2808915

[9] Goldfeld, Dorian and Joseph Hundley, *Automorphic representations and L-functions for the general linear group. Volume II*, Cambridge Studies in Advanced Mathematics, vol. 130, Cambridge University Press, Cambridge, 2011, With exercises and a preface by Xander Faber. MR2808915

[10] Henryk Iwaniec, *Spectral methods of automorphic forms*, 2nd ed., Graduate Studies in Mathematics, vol. 53, American Mathematical Society, Providence, RI; Revista Matemática Iberoamericana, Madrid, 2002, DOI 10.1090/gsm/053. MR1942691

[11] Robert P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, Vol. 544, Springer-Verlag, Berlin-New York, 1976. MR0579181

[12] C. Mœglin and J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*: *Une paraphrase de l'Écriture [A paraphrase of Scripture]*, Cambridge Tracts in Mathematics, vol. 113, Cambridge University Press, Cambridge, 1995, DOI 10.1017/CBO9780511470905. MR1361168

[13] Peter Sarnak, *Spectra of hyperbolic surfaces*, Bull. Amer. Math. Soc. (N.S.) **40** (2003), no. 4, 441–478, DOI 10.1090/S0273-0979-03-00991-1. MR1997348

[14] J. T. Tate, *Fourier analysis in number fields, and Hecke's zeta-functions*, Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), Thompson, Washington, D.C., 1967, pp. 305–347. MR0217026

Matthew P. Young
Department of Mathematics
Texas A&M University