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NIRENBERG'S CONTRIBUTIONS TO LINEAR PARTIAL DIFFERENTIAL EQUATIONS: PSEUDO-DIFFERENTIAL OPERATORS AND SOLVABILITY

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Abstract. This article is a survey of Louis Nirenberg's contributions to linear partial differential equations, focusing on his groundbreaking work on pseudo-differential operators and solvability.

1. Introduction

One cannot overestimate Louis Nirenberg's impact on twentieth century mathematics, especially on the analysis of both linear and nonlinear partial differential equations. In this article, we shall concentrate on Nirenberg's achievements in linear PDEs, in particular his development (with Kohn) of the calculus of pseudo-differential operators and microlocal analysis. These ΨDOs were developed as tools for the analysis of PDEs, but they have now become indispensible both for analysis and for other areas of mathematics. In connection with this, we shall also treat Nirenberg's work (with Treves) on the solvability of partial and pseudo-differential operators.

As a graduate student, I read Nirenberg's papers [25] and [26] on pseudo-differential operators to learn the subject, and they have been a great inspiration to me. I also heared about Nirenberg's work with Treves [28] about solvability, especially their famous conjecture which came to play an important role in my research. Later I met Nirenberg several times, including at his Abel Prize celebration, but we never had any collaborations.

2. Background

To appreciate Nirenberg's contributions in the development of ΨDOs, one has to know the background which presented the need for these operators. The development of the theory of distributions by Laurent Schwartz [29] at the end of the 1940's revolutionized the analysis of PDEs. Distributions are generalizations of generalized functions, which extend the notion of functions and had been used as weak solutions of PDEs. By defining distributions as functionals on classes of smooth test functions, one could simplify the theory of PDEs and be relatively unrestricted in their use of the Fourier transform. For example, one could now define a fundamental solution to any linear PDE.

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Distributions were not appreciated by all; see for example Bochner's review [3] of Schwartz's book. In spite of the opposition, distributions lead to a quick development of the theory of PDEs and many new results. One example is the proof of existence of fundamental solutions to any constant coefficient PDEs by Ehrenpreis [7] and Malgrange [24] in 1954–55. By using distributions and Fourier transformation, the study of constant coefficient PDEs is often reduced to the study of polynomials and their zeros.

But nonconstant coefficient PDEs presented a more difficult problem. Here singular integral operators became a useful tool, which for example Calderón [4] used to prove the uniqueness of the Cauchy problem in 1958. Singular integral operators on $\mathbf{R}^n$ have the form:

$$Su(x) = a(x)u(x) + \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} h(x,x-y)u(y)\,dy, \tag{2.1}$$

where $a(x) \in C^\infty$, $h(x,y)$ is homogeneous of degree $-n$ in y, $\int_{|y|=1} h(x,y)\,dy = 0$ with appropriate conditions on the regularity of $h(x,y)$. The operator is then given by the principal values of the improper integral in (2.1); a one-dimensional example is the Hilbert transform. These operators are useful tools in analysis, but the calculus is complicated. For example, the composition of two operators involves a mysterious *symbol* $\sigma(S)(x,\xi)$ of the operator; see Seeley's interesting exposition [30].

By work of Calderón and Zygmund, Horwath and Kohn, the symbol turned out to involve the partial Fourier transform of $h(x,y)$ in the y variables:

$$\sigma(S)(x,\xi) = a(x) + \lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<1/\varepsilon} h(x,y)e^{-iy\xi}\,dy, \tag{2.2}$$

which is then homogeneous of degree 0 in the ξ variables; see [30]. This is due to the fact that convolutions correspond to multiplications of the Fourier transforms.

3. Pseudo-differential operators

Kohn and Nirenberg [17] defined pseudo-differential operators in 1965 as

$$p(x,D)u(x) = (2\pi)^{-n}\int e^{i\langle x-y,\xi\rangle}p(x,\xi)u(y)\,dy, \tag{3.1}$$

where the symbol $p(x,\xi)$ is an asymptotic sum of homogeneous terms:

$$p(x,\xi) \sim \sum_{j\ge 0} p_{m-j}(x,\xi). \tag{3.2}$$

Here $p_k(x,\xi)$ is homogeneous of degree k in ξ, and the highest order term p_m is called the principal symbol. Note that $D = \frac{1}{i}\partial$ is the imaginary derivative which is a symmetric operator. This generalized the partial differential operators, where the symbols are polynomials in ξ. For singular integal operators the order $m = 0$, and the generalization to arbitrary order simplifies the calculations. The asymptotic expansion (3.2) is formal, but a result by Whitney [31] gives that there exists a symbol $q(x,\xi)$ such that

$$\left|q(x,\xi) - \sum_{j\ge 0}^{k} p_{m-j}(x,\xi)\right| \le C(1+|\xi|)^{m-k-1} \qquad \forall\, k. \tag{3.3}$$

These operators are called *classical* (or *polyhomogeneous*) ΨDOs.

Kohn and Nirenberg's paper also gave the calculus of ΨDOs, where the composition is done by the Leibnitz formula but now with infinitely many terms. The formulas for adjoints and change of variables were also given, similar to those of PDOs. Also, the principal symbol transforms as a function on the cotangent space $T^*\mathbf{R}^n$ under changes of coordinates.

With ΨDOs one loses the local property, since the support (where the distribution is nonzero) could be increased by the operators. But ΨDOs retain the pseudo-local property, that the singular support (where the distribution is not smooth) is not increased. The pseudo-local property is important, e.g., for the study of singularities and solvability.

The simpler calculus and the easy use of symbols made ΨDOs more convenient to use than the singular integral operators. For example, the order of a fundamental solution to an elliptic equation of order m is a ΨDOs of order $-m$. One could now also localize in cones in the ξ variables using homogeneous symbols, called *microlocalization*.

Kohn and Nirenberg immediately used the techniques of [17] to study elliptic boundary problems in [18]. There were also many properties of ΨDOs to explore, for example (semi)boundedness of operators having (semi)bounded symbols. In 1966 Nirenberg together with Lax [19] proved a Gårding type of lower bound for matrix valued ΨDOs with semidefinite principal symbols.

Kohn and Nirenberg's highly influential paper started a revolution in the analysis of PDEs and initiated the field of *microlocal analysis*, where one can localize in cones in $T^*\mathbf{R}^n$. The *singular support* sing supp$(u) \subseteq \mathbf{R}^n$ was refined by Hörmander [13] to the *wave front set* $\mathrm{WF}(u) \subseteq T^*\mathbf{R}^n$. This set indicates the directions where the localized Fourier transform does not decay of arbitrary order.

Hörmander [13] also proved that for pseudo-differential operators with real principal symbols that are of *principal type*, the wave front sets of the solutions propagate along the bicharacteristics of the principal symbol, which generalizes geometrical optics. Principal type means that the principal symbol vanishes of first order at its zeros, called *characteristics*. The *bicharacteristics* are the flow-out of the Hamilton vector field of the principal symbol on the characteristics; see (4.2).

This refinement simplifies the study of singularities of solutions to ΨDOs since the bicharacteristics foliate the characteristics and never intersect. Results on propagation of singularities lead to results on solvability by duality. In fact, the propagation of singularities to solutions to the homogeneous adjoint equation could prohibit singular solutions from having compact support.

Fourier integral operators were developed by Hörmander [12] to obtain symplectic coordinate transformations of ΨDOs on the cotangent space. With these operators, ΨDOs developed into a powerful tool making it possible to have natural invariant microlocal conditions on the symbols of the operators.

But the development was also towards more general classes of pseudo-differential operators. The classical ΨDOs by Kohn and Nirenberg are sufficient to invert elliptic PDOs, for example, the Laplacian Δ, but not for inverting hypoelliptic PDOs, such as the heat equation $\partial_t + \Delta_x$. In order to treat those cases, Hörmander [11] generalized the calculus in 1966 to the symbol classes $S^m_{\varrho,\delta}$ defined by

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m+|\alpha|\delta-|\beta|\varrho} \qquad \forall \alpha,\beta \in \mathbf{N}^n, \tag{3.4}$$

where $\varrho > 0$, $\delta \ge 0$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for the multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$. When $\varrho > \delta$ we get an asymptotic expansion in powers of $\langle\xi\rangle^{\delta-\varrho}$ of the compositions, and $\varrho \geq \delta$ corresponds to the uncertainty principle in quantum mechanics. It is easy to see that the class of classical symbols by Kohn and Nirenberg is contained in $S^m_{1,0}$. It is not hard to show that the symbol of the inverse of the heat equation is $(i\tau - |\xi|^2)^{-1} \in S^1_{1/2,0}$. But, as we shall see, these symbol classes were not enough for studying some of the harder problems in analysis, like the solvability of PDEs.

4. Solvability

One area where microlocal analysis had a great impact is the solvability of PDOs and ΨDOs of principal type, where Nirenberg together with Treves revolutionized the field. This development started in 1957 when Hans Lewy [23] created a sensation when he found a first order complex vector field that was not solvable anywhere in $\mathbf{R}^3$:

$$L(x, D) = D_{x_1} + iD_{x_2} + 2i(x_1 + ix_2)D_{x_3}. \tag{4.1}$$

This means that the equation $Lu = f$ has no local solution for almost all smooth f. Observe that the Cauchy–Kowalevska theorem gives analytic solutions for any analytic f, which is a dense set. Actually, the vector field L is in suitable coordinates the tangential Cauchy-Riemann operator of the boundary of the strictly pseudoconvex domain

$$\{(z_1, z_2) : \operatorname{Im} z_2 \geq |z_1|^2\} \subset \mathbf{C}^2.$$

Thus, it has an important role in complex analysis in several variables.

The tangential Cauchy–Riemann operators always have a large kernel containing any analytic function of the coordinates $z = (z_1, \dots, z_n) \in \mathbf{C}^n$. But Nirenberg [26] showed in 1972 that there exist arbitrarily small smooth perturbations of (4.1) whose kernels only contain the constant functions. Thus the perturbed vector fields cannot be a tangential Cauchy–Riemann operator of any domain. This was generalized in 1982 by Jacobowitz and Treves [16], showing that there exist arbitrarily small smooth perturbations of any given smooth vector field in $\mathbf{R}^3$ whose kernels only contain the constant functions.

After Lewy's example, Hörmander [9] took up the quest for solvability in 1960 by showing that the lack of solvability is generic for PDEs. In fact, if $P(x, D_x)$ is of order m with principal symbol p_m, then the commutator $[P^*, P]$ of the operator and its adjoint is of order $2m - 1$ and has principal symbol equal to the Poisson bracket

$$2iC_{2m-1} = 2i \sum_j \partial_{\xi_j} \operatorname{Re} p_m \partial_{x_j} \operatorname{Im} p_m - \partial_{x_j} \operatorname{Re} p_m \partial_{\xi_j} \operatorname{Im} p_m = 2iH_{\operatorname{Re} p_m} \operatorname{Im} p_m, \tag{4.2}$$

where $H_{\operatorname{Re} p_m}$ is the Hamilton vector field of $\operatorname{Re} p_m$. Hörmander showed that if this bracket does not vanish on the characteristics, which is a generic condition, then $P(x, D_x)$ is not locally solvable. (In his thesis [8], Hörmander had to assume $C_{2m-1} \equiv 0$ in order to prove solvability.)

The condition that $C_{2m-1} \neq 0$ on $p_m = 0$ means that the operator is of principal type and that the imaginary part of p_m switches sign on the bicharacteristics of the real part, which are the flow-out of the Hamilton vector field $H_{\operatorname{Re} p_m}$.

If P is of principal type and C_{2m-1} is a linear combination of p_m and $\overline{p}_m$, then the operator is *principally normal* and Hörmander [10] showed that these operators are locally solvable. One could also have special conditions on the repeated brackets when studying solvability, but then the situation gets rather complicated.

By using microlocal analysis and preparation theorems, one can reduce a classical ΨDOs of principal type to a first order operator with principal symbol on the normal form

$$p_1(x,\xi) = \tau + if(t,x,\xi), \tag{4.3}$$

where $f \in S^1_{1,0}$ is a real and homogeneous symbol and $(t,x;\tau,\xi) \in T^*(\mathbf{R} \times \mathbf{R}^n)$. In this case, the Poisson bracket (4.2) becomes $2i\,\partial_t f(t,x;\xi)$. *Principally normal* means that $\partial_t f = c \cdot f$ for some homogeous $c(t,x;\xi)$, a differential equation which preserves both signs and zeros of $f(t,x;\xi)$ as t changes.

5. The Nirenberg–Treves conjecture

But Nirenberg and Treves [27] in 1963 changed the perspective and introduced conditions on the sign changes of the imaginary part of the principal symbol of a PDO. The most important is condition (P), which says that the imaginary part of the principal symbol cannot change sign on the bicharacteristics of the real part. For example, for the normal form (4.3) this means that $t \mapsto f(t,x;\xi)$ does not change sign as t varies.

Nirenberg and Treves showed that condition (P) was sufficient for solvability of first order analytic PDOs of principal type, and they conjectured that this condition was both necessary and sufficient for solvability of first order PDOs of principal type.

Armed with the tools of pseudo-differential operators, Nirenberg and Treves [28] in 1970 took up the study of the solvability of PDOs and ΨDOs of principal type. For ΨDOs condition (P) is not the relevant condition. Instead it is condition (Ψ) which prohibits sign changes from $-$ to $+$ of the imaginary part of the principal symbol on the oriented bicharacteristics of the real part. For the normal form (4.3) this condition means that $t \mapsto f(t,x,\xi)$ does not change sign from $-$ to $+$ as t increases. For PDO conditions (P) and (Ψ) are equivalent, since the Poisson bracket in (4.2) has odd order.

Nirenberg and Treves conjectured that condition (Ψ) is necessary and sufficient for solvability of ΨDOs of principal type. They proved sufficiency for PDOs with analytic principal symbols and necessity in the case where the sign changes are of finite order (e.g., for analytic principal symbols).

When the principal symbol is analytic satisfying condition (P), Nirenberg and Treves used microlocal tools and reduced the operator to the normal form,

$$P = D_t + iA(t)B, \tag{5.1}$$

where $t \mapsto A(t) \geq 0$ is uniformly continuous and bounded, $B = B^*$ is constant in t, and the commutators $[B, A(t)]$ and $[B,[B,A(t)]]$ are bounded. The commutator conditions fit well into the calculus if A is operator of order zero and B is an operator of first order.

Under these conditions, Nirenberg and Treves [28] proved the following L^2 estimate:

$$\|u\| \leq CT\|Pu\| \tag{5.2}$$

if $u \in C_0^\infty$ has support where $|t| \leq T \ll 1$. This estimate may then be perturbed with L^2 bounded terms in P for sufficiently small T. But the reduction to the normal form relies on analyticity; for example, it does not work when the imaginary part vanishes of infinite order.

But in 1973, Beals and Fefferman [1] proved the sufficiency of condition (P) for solvability of smooth PDOs of principal type. In the proof, they used more refined symbol classes adapted to the operator to reduce it to the normal form (5.1) microlocally. This made it possible to use the Nirenberg–Treves lemma to obtain L^2 estimates from (5.2). Beals and Fefferman [2] in 1974 developed the calculus for these more general symbols, having symbol classes where $1 + |\xi|$ is replaced by other suitable weights in the symbol estimates (3.4).

Hörmander [14] developed this calculus further in 1979 into the *Weyl calculus*, defining symbols to be uniformly smooth with respect to suitable metrics on $T^*\mathbf{R}^n$. These metrics, called Hörmander metrics, have to satisfy conditions on local equivalence, temperatedness, and the uncertainty principle. By using the Weyl quantization, this calculus has the additional advantages of symplectic invariance and that real symbols correspond to symmetric operators.

The necessity of condition (Ψ) for solvability of pseudo-differential operators of principal type was then established by Hörmander [15] in 1981 by microlocally reducing the operator to the normal form (4.3) and constructing pseudo modes, i.e., localized approximate solutions to the homogeneous adjoint equation.

It was in general assumed that condition (Ψ) would also be sufficient for solvability of ΨDOs of principal type, and Lerner [20] proved this 1988 in dimension 2. There even appeared false proofs of sufficiency in any dimension, claiming estimates like (5.2) where one loses one derivative.

But a counterexample of Lerner [21] in 1994 shows that in general condition (Ψ) does not give the expected solvability with a loss of one derivative, which one has in the case when condition (P) holds. Thus, the estimate (5.2) in general does not hold, and solvability could depend on lower order terms. This example questioned the validity of the Nirenberg–Treves conjecture, whether it is sufficient to have conditions only on the principal symbol.

But in 2004 the sufficiency of condition (Ψ) was proven by the author in [6], giving solvability of principal type operators with a loss of two derivatives, thus finally resolving the Nirenberg–Treves conjecture. The proof uses a multiplier estimate adapted to the operator that can handle lower order terms.

The estimate was then improved in [5] to a loss of $3/2 + \varepsilon$ derivatives, for any $\varepsilon > 0$. Lerner [22] obtained the solvability with a loss of 3/2 derivatives with a similar multiplier estimate. This is currently the best result, but there still remains a gap since the counterexample in [21] only gives a loss of $1 + \varepsilon$ derivatives for any $\varepsilon > 0$.

In closing, the insights of Nirenberg led to a revolution in the analysis of PDEs, to the development of microlocal analysis, and to breakthroughs in the solvability of PDOs and ΨDOs.

About the author

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