MATHEMATICAL PERSPECTIVES

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 60, Number 2, April 2023, Pages 251–269
https://doi.org/10.1090/bull/1795
Article electronically published on January 27, 2023

SELECTED MATHEMATICAL REVIEWS

related to the paper in the previous section by
NILS DENCKER

MR0088770 (19,577a) 53.3X

Newlander, A.; Nirenberg, L.

Complex analytic coordinates in almost complex manifolds.

Annals of Mathematics. Second Series **65** (1957), 391–404.

This paper solves the geometrical problem of proving that a sufficiently differentiable, almost complex manifold is a complex manifold, provided certain "integrability" conditions are satisfied. The problem, in an analytical formulation, is reduced to that of finding n independent (complex-valued) solutions $\zeta^1, \cdots, \zeta^n$ of an overdetermined system of partial differential equations of the form (1) $L_j w \equiv \overline{\partial}_j w - a_j{}^k \partial_k w = 0$ $(j = 1, \cdots, n)$ when the L_j commute. Here, $\partial_j = \partial/\partial z^j$ and $\overline{\partial}_j = \partial/\partial \overline{z}^j$, where the z^i represent complex coordinates and the $\overline{z}^i$ their complex conjugates; the condition $a_j{}^i = 0$ for $z^1 = \cdots = z^n = 0$ is assumed. An arbitrary solution w of (1) would be an analytic function of these ζ^i; hence, in particular, w would satisfy the Cauchy-Riemann equations (2) $\overline{d}_j w = 0$, where $\overline{d}_j = \partial/\partial\overline{\zeta}^j$ (we shall also write $d_j = \partial/\partial\zeta^j$). This shows, in view of the presumed equalities $\overline{d}_j w = \partial_k w \cdot \overline{d}_j z^k + \overline{\partial}_k w \cdot \overline{d}_j \overline{z}^k = \partial_k w(\overline{d}_j z^k + a_m{}^k \overline{d}_j \overline{z}^m)$, obtained from (1), that solving (1) is equivalent to solving the system of equations (3) $\overline{d}_j z^k + a_m{}^k \overline{d}_j \overline{z}^m = 0$. The latter system is simpler than (1) in that but one kind of differentiation enters into each individual equation of the system, and the system is solved by an ingenious iterative procedure based on the fact that an equation of the form $\overline{d}_j z = f$ is inverted by a double integral in the ζ^j plane.

A. Douglis

From MathSciNet, January 2023

MR0131498 (24 #A1348) 26.00

John, F.; Nirenberg, L.

On functions of bounded mean oscillation.

Communications on Pure and Applied Mathematics **14** (1961), 415–426.

The authors define the mean oscillation of an integrable function u in a finite cube Q of n-space, and they establish in terms of it, for a suitable constant c, an estimate of the measure of the subset of Q in which $|u - c| > \sigma$. A variant of this estimate is also given, in which c is replaced by the mean value of u in

Q. The estimate, which is in the order of ideas of arguments used extensively by Hörmander and others, appears to the reviewer capable of many applications, and, as might be expected, its proof depends on the well-known lemma of F. Riesz (in its n-dimensional formulation), i.e., on the lemma which is the basis of the theorem known to analysts for several decades as "Max". In the paper itself, the authors give only an application of the case $n = 1$: it consists in a proof of a recent theorem of Weiss and Zygmund [Nederl. Akad. Wetensch. Proc. Ser. A **62** (1959), 52–58; MR0107122], which was previously derived from a theorem of Littlewood and Paley. The authors' proof really establishes a little more, namely, that if $F(x)$ is periodic and $\varphi(u)$ denotes the supremum for $0 < h < u$ and all x of the expression $|F(x+h) + F(x-h) - 2F(x)|$, then the convergence of $\sum_n n\varphi^2(1/n)$ implies that F is the indefinite integral of a function f such that, for some $b > 0$ and some c, where b, c are constant, the exponential of $b|f - c|$ is integrable.

L. C. Young

From MathSciNet, January 2023

MR0155203 (27 #5142) 34.95; 46.90

Agmon, S.; Nirenberg, L.

Properties of solutions of ordinary differential equations in Banach space.

Communications on Pure and Applied Mathematics **16** (1963), 121–239.

This is a far-reaching study of the properties of the solutions of first-order ordinary differential equations in Banach (or Hilbert) space of the form (1) $Lu = i^{-1}(du/dt) - Au = 0$ (homogeneous case), or (2) $Lu = f$ (inhomogeneous case), or perturbed equations: (3) $|lu| \leq \varphi(t)|u| + b(t)$, A being a closed operator from a Banach space X into another Banach space Y, $X \subset Y$, with continuous injection. The paper contains a wealth of results, mostly new, or at least considerable extensions of known ones. In a way it is a continuation of the fundamental work of P. Lax [same Comm. **9** (1956), 747–766; MR0086991; ibid. **10** (1957), 361–389; MR0093706]. As the authors themselves point out, the methods, with few exceptions, are elementary: Fourier transforms, Phragmén-Lindelöf principle, complex integration. However, it is by no means an easy reading, this mainly due to the often quite involved assumptions. These are usually conditions on the resolvent $R(\lambda) = (\lambda - A)^{-1}$ on the resolvent set ρ; actually, in order to make the theory more easily applicable also to higher-order equations, in some instances, $R(\lambda)$ is replaced by a modified resolvent $R_S(\lambda)$, where S is a closed subspace of Y, which is obtained, roughly, by restricting $(\lambda - A)^{-1}$ to S; similarly, ρ is replaced by a modified resolvent set ρ_S. One has then to impose the auxiliary assumption that $R_S(\lambda)$ is holomorphic in ρ_S. There appears also a certain related assumption called the "ζ hypothesis" which makes truncation possible. Therefore we shall only give below a brief survey of the principal topics discussed without trying to formulate the precise statements.

There are five chapters. Chapter I starts with various results connected with unique continuation of solutions vanishing at a finite point or at infinity (in the sense that $|u(t)| = O(e^{at})$, $t \to \infty$ for all a), all closely related to the theorem of P. Lax [first paper cited]. Then attention is given to obtaining lower bounds for the solution in some special cases. P. J. Cohen and M. Lees [Pacific J. Math. **11** (1961), 1235–1249; MR0133601] showed that if iA is self-adjoint, $X = Y$ being a Hilbert space, then all solutions of (3) with $\varphi \in L_p$ $(1 \leq p \leq 2)$ and $b = 0$, satisfy the inequality $|u(t)| \geq Ke^{-\mu t}$. Now this result is proven anew by a different method and extended to other cases ($p > 2$, etc.). Next the generalization to Banach space is considered. Then iA is assumed to be a multiple of the infinitesimal generator of a strongly continuous group of operators. All these developments depend on "convexity inequalities" for the solution, several of which are derived.

Chapter II deals in the first place with asymptotic expansions in exponential solutions (i.e., solutions of the form $p(t)e^{i\lambda t}$, where $p(t)$ is a polynomial in t with coefficients in X) of solutions of (1) in the interval $(0, \infty)$. In the special case of Hilbert space the condition on $R_S(\lambda)$ is of the form (4) $|R_S(\lambda)| = O(1)$, $|\lambda| \to \infty$, $0 \leq \operatorname{Im} \lambda < a$, $R_S(\lambda)$ being meromorphic in the strip, and the convergence is in the L_2 sense. In the case of Banach space one has to assume that $R(\lambda)$ (now $S = Y$) is holomorphic, not only in the strip, but also in a larger region, e.g., a "logarithmic" region $\operatorname{Im} \lambda \leq C \log |\operatorname{Re} \lambda|$, $|\lambda| \geq C$ or an "angular" region $0 \leq \arg\, (\lambda - c) \leq \theta_1$, $\pi - \theta_2 \leq \arg\, (\lambda + c) \leq \pi$, and exponentially bounded there. One gets instead pointwise convergence. In this connection what P. Lax [second paper cited] calls an "abstract Weinstein principle" is established, i.e., under a proper assumption on $R(\lambda)$ all exponentially bounded (in L_1) solutions of (1) in $(-\infty, \infty)$ are spanned by a finite number of exponential solutions. Finally these asymptotic expansions are used to establish the completeness (in some sense) of the exponential solutions.

Chapter III parts from the observation that the previous chapter yields the exponential decay of the solutions of (1), i.e., an "abstract Phragmén-Lindelöf principle" in the sense of P. Lax [second paper cited]. Now this is extended to equations (3), with an appropriate condition on φ, in the case of Hilbert space. The solutions are estimated in L_p $(1 < p < \infty)$ which makes it necessary to invoke (when $p \neq 2$) the Mihlin multiplier theorem. The condition on $R_S(\lambda)$ is of the type (4).

Chapter IV deals with the regularity of solutions of (2) $(X = Y)$. Necessary and sufficient conditions for the differentiability and analyticity in a finite interval are established. Here $R(\lambda)$ has to be estimated in a two-sided "logarithmic" or two-sided "angular" region (cf. the discussion of Chapter II). Chapter V, finally, gives applications, mainly of results of Chapter II and Chapter III, to partial differential equations of the form

$$(5) \qquad A(x, D_x, D_t)u = \sum_{j=0}^{l} A_j(x, D_x){D_t}^{l-j}u = f, \quad \text{order } A_j \leq 2mj/l,$$

where $x = (x_1, \cdots, x_n)$, $D_x = (i^{-1}\partial/\partial x_1, \cdots, i^{-1}\partial/\partial x_n)$, $D_t = i^{-1}\partial/\partial t$ in a cylindrical domain $\Gamma = G \times \{-\infty < t < \infty\}$ of $(n+1)$-dimensional space, with boundary conditions

$$(6) \qquad B_j(x, D_x)u = 0 \quad (j = 1, \cdots, m)$$

on the boundary $\partial\Gamma = \partial G \times \{-\infty < t < \infty\}$. It is assumed that A is "weighted elliptic of type $(2m, l)$" {which is a special case of "semi-ellipticity" in the sense of F. E. Browder [Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 270–273; MR0090741] so that type $(2m, 2m)$ corresponds to "elliptic" and type $(2m, 1)$ corresponds to "parabolic"} and B_j are assumed to satisfy a "complementary condition" generalizing the known complementary condition of, e.g., S. Agmon, A. Douglis and L. Nirenberg [Comm. Pure Appl. Math. **12** (1959), 623–727; MR0125307] in the elliptic case. Then the following inequality holds

$$(7) \qquad \sum_{j=0}^{2m} |\lambda|^{(2m-j)/d} \|D_x{}^j u\|_{L_p(G)} \leq C\|A(x, D_x, \lambda)u\|_{L_p(G)} \quad (d = 2m/l)$$

for all u satisfying (6) on ∂G and all real λ, $|\lambda| \geq C$. This follows, by an ingenious device due to one of the authors, from the inequalities of, e.g., Agmon, Douglas and Nirenberg [loc. cit.] in the elliptic case. Problem (5)-(6) is now reduced to an "abstract" equation of type (2) via a change of variables $U = (u, D_t u, , \cdots, D_t{}^{l-1} u)$ and, using (7), the resolvent $R(\lambda)$, or the modified resolvent $R_S(\lambda)$, with $S = \{(0, 0, \cdots, f)\}$, is shown to satisfy estimates which permit the application of the "abstract" theory. Asymptotic expansions in exponential solutions, completeness of exponential solutions, exponential decay of the solutions, accordingly, are established.

J. Peetre

From MathSciNet, January 2023

MR0162050 (28 #5252) 35.46

Agmon, S.; Douglis, A.; Nirenberg, L.

Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II.

Communications on Pure and Applied Mathematics **17** (1964), 35–92.

In questo lavoro vengono generalizzati ai problemi al contorno per sistemi ellittici di ordine qualunque i fondamentali risultati ottenuti nel caso di una equazione nella parte I [stessi Comm. **12** (1959), 623–727; MR0125307]; sono infatti dimostrate le maggiorazioni del tipo di Schauder e quelle negli spazi del tipo L_p per una vasta classe di sistemi ellittici lineari e ne sono dedotte alcune interessanti applicazioni quali la regolarizzazione delle soluzioni anche per sistemi di equazioni non lineari.

Nel cap. I viene formulato il problema al contorno. Si consideri il sistema

$$(1) \qquad \sum_{j=1}^{N} l_{ij}(P, \partial) u_j(P) = F_i(P), \quad i = 1, \cdots, N,$$

dove gli $l_{ij}(P, \partial) = l_{ij}(P; \partial/\partial x_1, \cdots, \partial/\partial x_{n+1})$, operatori differenziali lineari, sono polinomi in ∂ a coefficienti a valori complessi dipendenti da P variabile in un dominio D dello spazio euclideo ad $n+1$ dimensioni. L'ordine di tali operatori dipende da due sistemi di pesi interi $s_1, \cdots, s_N$, $t_1, \cdots, t_N$ nel modo seguente: $l_{ij}(P, \xi)$ è un polinomio in ξ di grado $\leq s_i + t_j$, $i, j = 1, \cdots, N$; ovviamente se $s_i + t_j < 0$ allora $l_{ij}(P, \partial) = 0$. Aggiungendo una opportuna costante ad un sistema di interi e togliendola all'altro si può poi assumere $s_i \leq 0$ e poichè non tutti gli $l_{ij}(P, \partial)$ sono $\equiv 0$ allora $t_j \geq 0$. Detta $l_{ij}{}'(P, \xi)$ la parte di grado $s_i + t_j$ del polinomio $l_{ij}(P, \xi)$ la condizione di ellitticità imposta sul sistema (1) è la seguente: $L(P, \xi) = \det \|l_{ij}{}'(P, \xi)\|_{i,j=1,\cdots,N} \neq 0$ per ξ reale $\neq 0$. Nel caso $n = 1$ viene imposta la seguente condizione supplementare su L: $L(P, \xi)$ è un polinimio in ξ di ordine $\sum_{i=1}^{N}(s_i + t_i) = 2m$; per ogni $P \in \dot{D}$ (frontiera di D) se n è la normale a $\dot{D}$ in P e se ξ è un vettore reale $\neq 0$ tangente a $\dot{D}$ in P allora il polinomio $L(P, \xi + \tau n)$, nella variabile complessa τ, ha esattamente m radici $\tau_1{}^+(P, \xi), \cdots, \tau_m{}^+(P, \xi)$ con parte immaginaria positiva. Il caso $m = 0$ si può facilmente risolvere in maniera esplicita [Douglis e Nirenberg, ibid. **8** (1955), 503–538; MR0075417] e quindi si suppone

$m > 0$; inoltre si suppone il sistema uniformemente ellittico. Si considerino poi su $\dot{D}$ le condizioni al contorno espresse nella forma seguente:

$$\sum_{j=1}^{N} B_{hj}(P,\partial)u_j(P) = \varphi_h(P), \quad h = 1, \cdots, m, \tag{2}$$

con $B_{hj}(P,\partial)$ operatori differenziali lineari a coefficienti a valori complessi dipendenti da P. Gli ordini di tali operatori dipendono da due sistemi di pesi interi $t_1, \cdots, t_N$, $r_1, \cdots, r_m$ nel modo seguente: $B_{hj}(P,\xi)$ è un polinomio in ξ di grado $\leq r_h + t_j$, $h = 1, \cdots, m$, $j = 1, \cdots, N$; naturalmente se $r_h + t_j < 0$ allora $B_{hj}(P,\partial) = 0$. Sia $B_{hj}'(P,\xi)$ la parte di $B_{hj}(P,\xi)$ di grado $r_h + t_j$. Indicata con $\|L^{jk}(P,\xi+\tau n)\|_{j,k=1,\cdots,N}$ la matrice aggiunta di $\|l_{ij}'(P,\xi+\tau n)\|_{i,j=1,\cdots,N}$, si impone allora sulle condizioni al contorno (2) la seguente condizione complementare: le righe della matrice

$$\|B_{hj}'(P,\xi+\tau n)\|_{\underset{j}{=}1,\cdots,N\to h=1,\cdots,m} \cdot \|L^{jk}(P,\xi+\tau n)\|_{j,k=1,\cdots,N},$$

i cui elementi sono considerati come polinomi in τ, devono essere linearmente indipendenti modulo il polinomio, in τ, $\prod_{k=1}^{m}(\tau - \tau_k{}^+(P,\xi))$. È naturale chiedersi se, dato un sistema ellittico, esistano delle condizioni al contorno che verificano la condizione complementare; per la risposta completa a tale questione gli autori rinviano ad un lavoro non pubblicato di R. Bott; viene solo da essi dimostrato che il problema di Dirichlet per un sistema fortemente ellittico verifica la condizione complementare. Viene dimostrato, sempre nel cap. I, che ogni sistema ellittico può essere rimodellato, con l'aggiunta di nuove variabili, in modo che sia $s_i + t_j \leq 1$ e che, trasformando di conseguenza le condizioni alla frontiera originarie, la condizione complementare sia ancora verificata.

Il cap. II è dedicato allo studio del sistema (1), (2) con $l_{ij} = l_{ij}'$ ed l_{ij}' a coefficienti costanti e $B_{hj} = B_{hj}'$ e B_{hj}' a coefficienti costanti, D essendo il semispazio $x_{n+1} > 0$. Nel n.4 viene costruita, medianti opportuni nuclei di Poisson, una formula esplicita per la soluzione del problema studiato nel caso $F_i = 0$, $i = 1, \cdots, N$. Tale costruzione è basata sullo studio, fatto nel n.3, del comportamento asintotico delle soluzioni di sistemi di equazioni differenziali ordinarie con condizioni iniziali molto generali. Sempre nel n.4 vengono date alcune maggiorazioni dei nuclei di Poisson che permetteranno nel n.5 di applicare alla soluzione esplicita il teorema di Calderón e Zygmund [Acta Math. **88** (1952), 85–139; MR0052553]. Fondamentale per il seguito è la formula di rappresentazione, ottenuta nel n.6, per le soluzioni del problema non omogeneo. Tale risultato si dimostra usando la formula esplicita ottenuta nel n.4 ed i risultati del n.5 con un ragionamento analogo a quello svolto nella parte I nel caso di una equazione.

Il cap. III è dedicato alla dimostrazione delle maggiorazioni del tipo di Schauder. Tali maggiorazioni vengono dapprima (n.8) ottenute per i problemi considerati nel cap. II, per i quali è stata trovata la formula di rappresentazione, ed infine, con le stesse techniche della parte I, tali maggiorazioni vengono estese al caso generale. Sia l intero ≥ 0 e α reale con $0 < \alpha < 1$; si indica con $C^{l+\alpha}(D)$ lo spazio delle funzioni u continue con le loro derivate fino all'ordine l in $\overline{D} = D \cup \dot{D}$ e inoltre con le derivate di ordine l uniformemente hölderiane di esponente α in D normalizzato da

$$|u|^D_{l+\alpha} = \sup_{|h|\le 1}\left(\sup_{x\in\overline{D}}|\partial^h u(x)|\right) + \sup_{|h|=1}\left(\sup_{\underset{x}{\neq} y\to x, y\in D}\frac{|\partial^l u(x)-\partial^l u(y)|}{|x-y|^\alpha}\right)$$

intendendo che per ogni $(n+1)$-upla $h=(h_1,\cdots,h_{n+1})$ di interi $h_i\ge 0$ è $|h| = \sum_{i=1}^{n+1} h_i$ e $\partial^h u = \partial^{|h|}u/\partial x_1{}^{h_1}\cdots\partial x_{n+1}^{h_{n+1}}$. In modo analogo si definisce lo spazio $C^{l+\alpha}(\dot D)$ con la norma $|u|^{\dot D}_{l+\alpha}$. Si ha allora il Teorema 9.3: sia D un dominio limitato di R^{n+1} di classe $C^{l+\lambda+\alpha}$ con l intero $\ge l_0=\max(0,r_1,\cdots,r_m)$, α reale con $0<\alpha<1$ e $\lambda=\max(t_1,\cdots,t_N,-s_1,\cdots,-s_N,-r_1,\cdots,-r_m)$. Supponiamo che i coefficienti di l_{ij} siano in $C^{l-s_i+\alpha}(\overline{D})$ e quelli di B_{hj} in $C^{l-r_h+\alpha}(\dot D)$. Nelle ipotesi fatte su l_{ij} e B_{hj} nel cap. I, sia $u_1,\cdots,u_N$ una soluzione di (1) in D e di (2) su $\dot D$ con $F_i\in C^{l-s_i+\alpha}(\overline{D})$ e con $\varphi_h\in C^{l-r_h+\alpha}(\dot D)$. Se $u_j\in C^{l_0+t_j+\alpha}(\overline{D})$, allora $u_j\in C^{l+t_j+\alpha}(\overline{D})$ e vale la maggiorazione:

$$|u_j|^D_{l+t_j+\alpha}\le C\left(\sum_{i=1}^N|F_i|^D_{l-s_i+\alpha}+\sum_{h=1}^m|\varphi_h|^{\dot D}_{l-r_h+\alpha}+\sum_{k=1}^N|u_k|_0{}^D\right),\quad j=1,\cdots,N,$$

con C costante che non dipende da $u_1,\cdots,u_N$, $F_1,\cdots,F_N$, $\varphi_1,\cdots,\varphi_m$. Tale risultato è anche valido sotto opportune condizioni nel caso in cui D sia un dominio illimitato.

Il cap. IV è poi dedicato alle maggiorazioni a priori negli spazi H_{j,L_p}. Per j intero >0, $H_{j,L_p}(D)$ è qui inteso come completamento astratto di $C^\infty(\overline{D})$ rispetto alla norma $\|u\|_{j,L_p}=(\sum_{|h|\le j}\int_D|\partial^h u|^p\,dx)^{1/p}$, $p>1$; $H_{j-1/p,L_p}(\dot D)$ è lo spazio delle funzioni φ su $\dot D$ che sono "tracce" su $\dot D$ di funzioni $v\in H_{j,L_p}(D)$ la norma essendovi definita da $\|\varphi\|_{j-1/p,L_p}=\inf\|v\|_{j,L_p}$ fra tutte le $v\in H_{j,L_p}(D)$ aventi φ come traccia su $\dot D$. Tali maggiorazioni sono ottenute sempre a partire dalla formula di rappresentazione stabilita nel n.6, con lo stesso ragionamento della parte I. Il risultato più importante per le maggiorazioni di carattere globale è il seguente Teorema 10.5: sia $l_1=\max(0,r_1+1,\cdots,r_m+1)$ e sia l un intero $\ge l_1$; sia D un dominio limitato di classe $C^{l+\lambda}$, e supponiamo che i coefficienti di l_{ij} siano in $C^{l-s_i}(\overline{D})$ e quelli di B_{hj} in $C^{l-r_h}(\dot D)$. Nelle ipotesi fatte su l_{ij} e B_{hj} nel cap. I, sia $u_1,\cdots,u_N$ una soluzione di (1) con $F_i\in H_{l-s_i,L_p}(D)$ e di (2) con $\varphi_h\in H_{l-r_h-1/p,L_p}(\dot D)$; allora se $u_j\in H_{l_1+t_j,L_p}(D)$ risulta per $j=1,\cdots,N$ $u_j\in H_{l+t_j,L_p}(D)$ e

$$\|u_j\|_{l+t_j,L_p}\le K\left(\sum_{i=1}^N\|F_i\|_{l-s_i,L_p}+\sum_{h=1}^m\|\varphi_h\|_{l-r_h-1/p,L_p}+\sum_{k=1}^N\|u_k\|_{0,L_p}\right)$$

con K costante indipendente da $u_1,\cdots,u_N$, $F_1,\cdots,F_N$, $\varphi_1,\cdots,\varphi_m$. Vengono poi date anche delle maggiorazioni di carattere locale alla frontiera.

Nel cap. V viene dimostrata, con alcuni esempi, la necessità delle ipotesi fatte nel cap. I per avere le maggiorazioni a priori (n. 11) e sono date, nei n. 12, 13, 14, alcune applicazioni dei risultati ottenuti: regolarizazzione di sistemi non lineari, perturbazione di problemi non lineari, maggiorazioni di Schauder per equazioni semi-lineari. Vengono infine (n. 15) costruiti dei nuclei di Poisson "approssimati" per equazioni a coefficienti variabili.

Come è detto nell'introduzione, alcuni dei risultati di questo lavoro sono stati annunciati da vari autori.

G. Geymonat
From MathSciNet, January 2023

MR0163045 (29 #348) 35.01
Nirenberg, L.; Treves, F.
Solvability of a first order linear partial differential equation.
Communications on Pure and Applied Mathematics **16** (1963), 331–351.

A linear partial differential operator L with complex C^∞ coefficients is called solvable on an open set $\Omega \subset R^n$ if for every $f \in {C_0}^\infty(\Omega)$ the equation $Lu = f$ has a solution $u \in D'(\Omega)$. It is solvable at a point ω if it is solvable on some neighborhood of ω.

In 1957 H. Lewy [Ann. of Math. (2) **66** (1957), 155–158; MR0088629] discovered the fact that the operator $L = \partial/\partial x^1 + i\,\partial/\partial x^2 - 2i(x^1 + ix^2)\,\partial/\partial x^3$ is not solvable. In fact, for most C^∞ functions f the equation $Lu = f$ has no solution on any non-void open set.

In 1960 L. Hörmander [Math. Ann. **140** (1960), 124–146; MR0130574; ibid. **140** (1960), 169–173; MR0147765] gave a striking necessary condition for an operator L to be solvable at a point. Let L be of order m and let C_{2m-1} be the terms of order exactly $2m-1$ in the commutator $L\overline{L} - \overline{L}L$. (The coefficients in $\overline{L}$ are the conjugates of those in L.) If L is solvable at ω, then (for the characteristic polynomials) $C_{2m-1}(x;\xi) = 0$ if $L_0(x;\xi) = 0$ for x in a neighborhood of ω and $\xi \in R^n$. When L has order 1 Hörmander's condition becomes simply that C_1 is a linear combination of the leading parts L_0 and $\overline{L}_0$. If L_0 and $\overline{L}_0$ are linearly independent at ω, then this condition is also sufficient [L. Nirenberg, Seminars on Analytic Functions, Vol. I, pp. 172–189, Inst. for Advanced Study, Princeton, N.J., 1957].

In this paper the authors study the first-order case very carefully. They formulate a certain condition—(P) below—and show that it is sufficient for solvability, and another one, (P′), between (P) and the condition of Hörmander, and show that it is necessary for solvability. They conjecture that (P) is both necessary and sufficient.

Let C^k be the commutator $L_0C^{k-1} - C^{k-1}L_0\,(C^1 = C_1)$ and let $k(x)$ be the first value of k (if there is one) such that, at x, C^k is not a linear combination of L_0 and $\overline{L}_0$. The condition (P′) is as follows. (P′): In some neighborhood of ω, $k(x)$ if finite is even. Hörmander's condition is just that in some neighborhood of ω, $k(x)$ is never 1. In order to state condition (P) choose new (local) coordinates so that L_0 takes the form

$$L_0 = g\left(\frac{\partial}{\partial x^n} + i\sum_1^{n-1} b^k \frac{\partial}{\partial x^k}\right),$$

where g is a complex C^∞ function and the b^k are real C^∞ functions. The authors show that this is always possible and that the condition (P) does not depend on the way in which it is done. (It is assumed, of course, that L is strictly first-order at ω.)

(P): In some neighborhood of ω, the direction of the vector b is independent of x^n. In the new coordinates, L_0 and $\overline{L}_0$ are linearly independent at a point x if and only if $b(x) \neq 0$. In this case, (P), (P′), and the condition of Hörmander are all equivalent. At a point x where L_0 and $\overline{L}_0$ are linearly dependent, $k(x)$ is the least k such that $(\partial/\partial x^n)^k b \neq 0$, so that (P′) has a simple interpretation in terms of b.

In addition to the main theorem that (P) is sufficient for solvability and (P′) is necessary, there are interesting corollaries that give general circumstances under which (P) and (P′) are equivalent. One such is when b has no zero of infinite order as a function of x^n. In particular, (P) and (P′) are equivalent if the leading coefficients of L are analytic.

As an example, the theorems show that the operator $\partial/\partial t + it^k\,\partial/\partial x$ is solvable at points on the x-axis if and only if k is even (for in this case clearly $k(x) = k$).

K. T. Smith

From MathSciNet, January 2023

MR0176362 (31 #636) 47.70; 35.23

Kohn, J. J.; Nirenberg, L.

An algebra of pseudo-differential operators.

Communications on Pure and Applied Mathematics **18** (1965), 269–305.

This article is the first to give a fairly complete discussion of a class of operators which has recently been developed in connection with partial differential equations. An earlier announcement was made by Unterberger and Bokobza [#635a-b above], and further references are given in the review of their articles.

The operators in question act on the space S of tempered distributions on R^n, and are estimated in terms of the norms $(\|f\|_s)^2 = \int |f\hat{}(\xi)|^2(1+|\xi|^2)^s d\xi$, with $f\hat{}$ the Fourier transform of f.

A linear transformation A on S is of order r if $\|Af\|_s \leq \text{const}\ {}_s\|f\|_{s+r}$ for all s. The authors choose a fixed C^∞ function ζ, such that $\zeta(\xi)$ vanishes for small ξ and $\zeta(\xi)^{-1}$ vanishes for large ξ, and they let $\zeta_s(\xi) = |\xi|^s\zeta(\xi)$. Then the operator $\zeta_s(D)$ defined by $(\zeta_s(D)f)\hat{}(\xi) = \zeta_s(\xi)f\hat{}(\xi)$ is of order s. A function $a(x,\xi)$ is a symbol if $a(x,t\xi) = a(x,\xi)$ for $t > 0$, and if there is a C^∞ function $a(\infty,\xi)$, likewise homogeneous of degree zero, such that for each p, α, and β, $(1+|x|^p)D_x{}^\alpha D_\xi{}^\beta(a(x,\xi) - a(\infty,\xi))$ is bounded for x in R^n and $|\xi| = 1$. An operator A is canonical of order s if it has the form

$$(1)\qquad (Af)\hat{}(\xi) = \zeta_s(\xi)(2\pi)^{-n/2}\int e^{-ix\cdot\xi}a(x,\xi)f(x)\,dx.$$

Then the symbol of A is $\sigma(A)(x,\xi) = |\xi|^s a(x,\xi)$. An operator A is a pseudo-differential operator if there is a sequence of real numbers $s_j \to -\infty$, and a sequence of operators A_j, canonical of order s_j, such that $A - \sum_0^N A_j$ has order $< s_N$. The symbol of A is then the formal series $\sum_0^\infty \sigma(A_j)$. The set L of pseudo-differential operators forms a *-algebra containing the differential operators with constant coefficients and with coefficients in S. Moreover, operations can be defined for the symbols so that σ is a homomorphism of L onto the set of all such formal series, whose kernel consists of the ideal $L_{-\infty}$ of operators of order $-\infty$. The inf for $T \in L_{-\infty}$ of $\sup\|Af + Tf\|_s/\|f\|_{s+s_0}$ is given by $\sup|\sigma(A_0)(x,\xi)|$ for $x \in R^n$, $|\xi| = 1$, when $\sigma(A) = \sum_0^\infty \sigma(A_j)$ and A_0 is canonical of order s_0; in particular, the A_j and $\sigma(A_j)$ are determined by A. Further details are given, such as a Gårding inequality, and a proof of compactness of various operators, in particular, of $\Phi A\Psi$, when A is in L and Φ and Ψ denote multiplications by C^∞ functions φ and ψ such that $\varphi\psi = 0$. In the final section, the authors consider a larger class of operators, having a representation of the form (1) with $s = 0$, but without any assumption of homogeneity for the function a appearing there. The result is a class of operators A for which $\Phi A \Psi$ is compact when Φ and Ψ are multipliers with disjoint support.

The significance of these results is that they take account of all "lower order terms", neglected in previous treatments. Moreover, the assumptions on a in (1) allow the authors to deduce all their results from (1), without recourse to singular integrals; this applies also to a discussion of coordinate changes, not treated in the present article.

R. T. Seeley

From MathSciNet, January 2023

MR0181815 (31 #6041) 35.00; 47.65

Kohn, J. J.; Nirenberg, L.

Non-coercive boundary value problems.

Communications on Pure and Applied Mathematics **18** (1965), 443–492.

The authors consider equations of the form $(*)$ $Q(u,v) - \lambda(u,v) = (\alpha, v)$ for all $v \in B$, where Q is the integral over a C^∞ bounded domain $\mathcal{M}$ in R^{n+1} of a bilinear form in the vector functions u and $\overline{v}$ and their first derivatives with coefficients in C^∞ $(\mathcal{M} \cup b\mathcal{M})$, (,) denotes the inner product in L_2, $\alpha \in L_2$, and B is a linear space of vectors such that (a) $B \supset {C_c}^\infty(\mathcal{M})$, (b) each x_0 on $b\mathcal{M}$ is in a neighborhood U on $\mathcal{M} \cup b\mathcal{M}$ on which there exists a y-coordinate system ($\in C^\infty$) such that $y^{n+1} = 0$ on $U \cap b\mathcal{M}$, and for each neighborhood V such that $V \cup bV \subset U$, there exists a real-valued ζ of class C^∞ on U, vanishing near $bU \cap \mathcal{M}$ with $\zeta = 1$ on V, such that $\zeta u \in B$ if u does, and (c) in U, the u^j can be replaced by suitable linear combinations so that the boundary conditions induced by B are invariant under translations in the y-coordinates. If $Q = Q_0 + iQ'$, Q_0 being the hermitian part of Q, it is assumed that (i) $Q_0(u,u) \geq \|u\|^2$, and (ii) the derivatives of u and $\overline{v}$ enter linearly in Q'. Additional conditions on Q are: (ii)$'|Q'(u,v)|^2 \leq CQ_0(u,u)Q_0(v,v)$, and (iii) the boundary is nowhere characteristic with respect to Q_0. K denotes the Hilbert space obtained by completing B with respect to the norm ${Q_0}^{1/2}$.

The authors prove the following results of a global character. (1) Suppose that the unit ball in K is precompact in L_2 and that (iii) holds. Then for each $\alpha \in C^\infty(\mathcal{M} \cup b\mathcal{M})$, there exists a unique solution $u \in K \cap {H_2}^1(\mathcal{M})$ of $(*)$ with $\lambda = 0$ and $u \in C^\infty$ $(\mathcal{M} \cup b\mathcal{M})$. If (ii)$'$also holds, u is unique in K. (2) Under the hypotheses in (1), the set of λ for which the homogeneous equation $(*)$ has non-zero solutions is denumerable and has no limit point in the plane, and for each λ, the manifold of "eigen-vectors" is finite dimensional. Additional results are proved, including some of a local character and some in which Q involves derivatives of higher order. These results include recent results of the first author on the $\overline{\partial}$-Neumann problem [Ann. of Math. (2) **78** (1963), 112–148; MR0153030; ibid. (2) **79** (1964), 450–472]. Most of the results are stated in the introduction and in Sections 1 and 2.

C. B. Morrey Jr.

From MathSciNet, January 2023

MR0264470 (41 #9064a) 47.70; 35.01

Nirenberg, Louis; Trèves, François

On local solvability of linear partial differential equations. I. Necessary conditions.

Communications on Pure and Applied Mathematics **23** (1970), 1–38.

MR0264471 (41 #9064b) 47.70; 35.01

Nirenberg, Louis; Trèves, François

On local solvability of linear partial differential equations. II. Sufficient conditions.

Communications on Pure and Applied Mathematics **23** (1970), 459–509.

One of the major problems in the theory of linear partial differential equations has been the determination of those equations that are locally solvable. In this two-part paper the authors solve the problem for partial differential operators of principal type with analytic principal part. They also provide new insight into the general problem and its generalization to pseudo-differential operators. The authors have already given an excellent summary of their results [C. R. Acad. Sci. Paris Sér. A-B **269** (1969), A774-A777; MR0264469 above; ibid. **269** (1969), A853-A856; MR0262661].

In Part I the authors consider pseudo-differential operators of the form $P = P_m + P_{m-1}$ on a C^∞ manifold X, where P_{m-1} has order at most $m-1$ and P_m has a symbol p_m that is C^∞ and fiber-homogeneous of degree m (at least outside a neighborhood of the zero section of the cotangent bundle $T^*(X)$). They also assume that P is of principal type, i.e., if ξ is a (real) non-zero cotangent vector at x in X such that $p_m(x,\xi)=0$, then $d_\xi p_m(x,\xi)\neq 0$.

A pseudo-differential operator P is said to be locally solvable at a point $x_0\in X$ if there exist relatively compact neighborhoods $U\subset V$ of x_0 such that, for every $f\in {C_c}^\infty(U)$, there exists a distribution u with support in V such that $Pu=f$ on U.

The basic conditions for local solvability involve the behavior of p_m along certain bicharacteristics. In general, if A is any real C^∞ function defined on some open set Σ in the cotangent bundle over X, then an (oriented) curve in Σ is said to be a null bicharacteristic for A if it is a solution of the system $\dot x = \operatorname{grad}_\xi A$, $\dot\xi = -\operatorname{grad}_x A$ on which $A=0$ but $\operatorname{grad}_\xi A\neq 0$.

The main theorem of Part I states that if P is locally solvable at x_0 then there is a neighborhood of x_0 such that on the cotangent bundle of this neighborhood (minus the zero section) p_m satisfies the condition (Φ): On the null bicharacteristics of $\operatorname{Re} p_m$ the function $\operatorname{Im} p_m$ can change sign at a zero of finite order only if the change is from positive to negative. This improves the results of L. Hörmander [Math. Ann. **140** (1960), 169–173; MR0147765; Ann. of Math. (2) **83** (1966), 129–209; MR0233064], and extends the authors' results for first order equations [Comm. Pure Appl. Math. **16** (1963), 331–351; MR0163045] and the second author's recent results for equations in two variables [Amer. J. Math. **92** (1970), 174–204; MR0259336]. A more general necessary condition has recently been obtained by Ju. V. Egorov [Dokl. Akad. Nauk SSSR **186** (1969), 1006–1007; MR0249808], who replaced the principal type assumption by the condition that $dp_m(x,\xi)\neq 0$ when $p_m(x,\xi)=0$.

In the case in which X is analytic and p_m is analytic (outside the zero section), the condition (Φ) is equivalent to (Ψ): On the null bicharacteristics of $\operatorname{Re} p_m$, if $\operatorname{Im} p_m$ is negative at some point then it is non-positive from then on. If p_m is also a partial differential operator, then these conditions are equivalent to $(\mathcal{P})$: $\operatorname{Im} p_m$ does not change sign on the null bicharacteristics of $\operatorname{Re} p_m$.

The proof of the main theorem is based on a significant refinement of the original method of Hörmander [loc. cit.]. An important feature in this proof is that (Φ) and other related conditions are essentially invariant under multiplication of p_m by non-vanishing C^∞ functions. More precisely, it is shown that if p and q are C^∞ functions on an open set in $T^*(X)$ such that $q \neq 0$, and both $d_\xi \operatorname{Re} p$ and $d_\xi \operatorname{Re}(pq)$ are non-zero when $p = 0$, then p satisfies the condition (Φ) if and only if pq does. (In the appendix to Part II, the same invariance result is established for the conditions $(\mathcal{P})$ and (Ψ) instead of (Φ).)

Now, observe that if $d_\xi \operatorname{Re} p_m$ and p_m both vanish at some point, then the conditions (Φ), (Ψ) and $(\mathcal{P})$ as stated do not imply conditions on p_m at that point even though they do for non-real multiples of p_m. However, for the sake of simplicity, the authors implicitly interpret (Φ), (Ψ) or $(\mathcal{P})$ at such points to be the corresponding conditions for some (and hence all) non-real multiples of p_m. With this understanding (Φ) is a necessary condition for local solvability and is invariant under multiplication of p_m by non-zero functions.

In Part II, the authors restrict their attention to partial differential operators of principal type with analytic principal part (on an analytic manifold). The main theorem here is that the condition $(\mathcal{P})$ holds on the cotangent bundle over some neighborhood of x_0 if and only if there is a neighborhood U of x_0 such that, for every $f \in \mathcal{L}^2(U)$, $Pu = f$ has a solution $u \in \mathcal{L}^2(U)$ such that all its (distribution) derivatives of order at most $m-1$ are in $\mathcal{L}^2(U)$. Combining this with the results of Part I, we see that both conditions are equivalent to local solvability at x_0.

The main part of the proof of this theorem involves proving the local existence of solutions in $\mathcal{L}^2$ for first order pseudo-differential equations of the form $D_t u - a(t,x,D_x)u - ib(t,x,D_x)u = f$, where $f \in \mathcal{L}^2$. Here, for each real t near zero, $a(t,x,D_x)$ and $b(t,x,D_x)$ are real pseudo-differential operators near the origin in R^n, and $D_t - a - ib$ satisfies the condition $(\mathcal{P})$ near the origin in R^{n+1}. If $A_0(t)$ and $B_0(t)$ denote the self-adjoint parts of these operators (t fixed), and $U(t)$ is the solution of $D_t U(t) = A_0(t)U(t)$, $U(0) = I$, then the proof reduces to the local solution of the equation $D_t v - iU^{-1}B_0 U v = g$ in $\mathcal{L}^2$. The authors then prove that $U^{-1}B_0 U$ is a pseudo-differential operator that can be factored (modulo lower order terms) in such a way that solvability is assured. They remark that analyticity is needed only for this factorization and that, in the previous treatments of the first order and two variable cases (cited above), the condition $(\mathcal{P})$ was actually shown to be sufficient for local solvability without the assumption of analyticity.

The proof of the fact that $U^{-1}B_0 U$ is a pseudo-differential operator centers on the study of $U(t)$ as a Fourier integral operator (modulo certain smoothing operators). Such operators were recently introduced and studied by Hörmander [Acta Math. **121** (1968), 193–218; "Fourier integral operators", lecture notes, Nordic Summer School of Mathematics, Univ. Göteborg, Gothenburg, 1969]. The operators of interest here are of the form

$$K(t)u = \int_{\mathbf{R}^n} k(t,x,\xi) e^{ih(t,x,\xi)} \hat{u}(\xi)\, d\xi,$$

where $h_t = a(t, x, h_x)$, $h|_{t=0} = x \cdot \xi$ and k has a suitable asymptotic expansion in terms of functions that are homogeneous in ξ. Such operators are studied in some detail. One of the principal results is that, for any compactly supported pseudo-differential operator Q in R^n, $U(t)^{-1}QU(t)$ is a pseudo-differential operator on R^n depending smoothly on t whose principal symbol at (t, x, ξ) is the principal symbol of Q at time t on the (null) bicharacteristic of $\tau - a$ that starts at (x, ξ).

It should be remarked that an important consequence of the results of Part II is that local solvability is equivalent to local solvability "with the loss of one derivative". Indeed, the main theorem is also valid when $\mathcal{L}^2$ is replaced by any Sobolev space of functions with $\mathcal{L}^2$ derivatives.

Some extensions of the results to systems of partial differential equations are given but the results are far from complete, due to the requirement that the scalar operators be of principal type.

Finally, the authors conjecture that (Ψ) is sufficient for the local solvability of pseudo-differential equations.

R. D. Moyer

From MathSciNet, January 2023

MR0488102 (58 #7672) 58-02; 47HXX

Nirenberg, L.

Topics in nonlinear functional analysis. (English)

With a chapter by E. Zehnder. Notes by R. A. Artino, Lecture Notes, 1973–1974.
Courant Institute of Mathematical Sciences, *New York University, New York*, 1974, viii+259 pp., \$6.75

For students and researchers in nonlinear analysis this volume of lecture notes is the most useful introduction to the subject currently available. It is short, concise and to the point, and the proofs are unusually elegant, always with a geometric flavor and the best available.

Much of the flavor of the notes centers on the use of degree theory. The introduction of the notion of degree and the development of its basic properties in the span of some fifty pages is a triumph in itself. This is used in Chapters III and IV in the discussion of bifurcation theory (the highlight being a complete proof of Rabinowitz' global bifurcation theorem) and the solution of nonlinear partial differential equations (the highlight being the global theorem of Landesman and Lazer).

The discussion of bifurcation at simple eigenvalues using the Morse lemma (suggested by H. Duistermaat) is an elegant treatment of well-known results on bifurcation at simple eigenvalues. It has inspired others to use normal forms as well. (Compare works by J. K. Hale [see MR0488104 below] and the reviewer [Bull. Amer. Math. Soc. **84** (1978), no. 6, 1125–1148].)

Extensions of the topological approach to include framed cobordism, cohomotopy groups, etc. (as developed by Elworthy, Tromba, Gęba, Granas, Berger, et al.) are discussed in detail in Chapter IV. (Two of the sections were written by J. Izé.)

Chapter V discusses methods of monotone operators and work of the author and H. Brezis on the minimax principle. (There is a section on the single-valuedness of monotone operators, written by N. Bitzenhofer.)

Finally, Chapter VI (written by E. Zehnder) deals with "hard" implicit function theorems and applications.

The notes are already out of date in certain areas. The reader may wish to consult, for example, works by J. Izé [Mem. Amer. Math. Soc. **7** (1976), no. 174; MR0425696], M. S. Berger [see MR0488101 above] and S. Klainerman ["Global existence for nonlinear wave equations", Ph.D. Thesis, Courant Inst. Math. Sci., New York Univ., New York, 1978], and to scour recent issues of Math Reviews for papers of Rabinowitz, Nirenberg and Brezis. Despite these recent advances, these notes remain the best introduction to nonlinear analysis available.

J. E. Marsden

From MathSciNet, January 2023

MR0609039 (83e:58015) 58E05; 35B99, 47H15, 58E07

Nirenberg, L.

Variational and topological methods in nonlinear problems.

American Mathematical Society. Bulletin. New Series **4** (1981), *no.* 3, 267–302.

The author's aim is to give an informal introduction to some new results on the existence of solutions for nonlinear problems of the type $Fu = y$. Here F is a continuous (and usually smooth) operator which maps a Banach space X into another one Y. In the first section the author deals with the main topological ideas used to prove the existence of solutions: the theory of topological degree, homotopy theory and the method of topological continuation. For the most part the author presents the Elworthy-Tromba theory of Fredholm operators of index $\nu \geq 0$ and its recent generalisations. The last part of this section is devoted to the Poincaré operator and its analogues, related to the translation operator along trajectories of autonomous differential systems; in connection with it the author presents some results on Hopf bifurcation. The author discusses the results of Ambrosetti and Mancini, Sylvester, Berger, Fuller, Amann and Zehnder, Alexander and Yorke and others. In the second part the author presents a sequence of results on the existence of critical points for smooth functionals satisfying the Palais-Smale condition and which are based on the "mountain pass lemma" of P. Rabinowitz and its generalizations. The abstract constructions are illustrated using a simple boundary value problem for partial differential equations. Further the author gives an account of Rabinowitz' results on the existence of solutions for Hamiltonian systems and the nonlinear string equation. The last part contains a short survey of the methods of studying local problems and in particular the Nash-Moser implicit function theorem. To illustrate the latter theorem the author presents Klaĭnerman's results on the existence of smooth solutions for all time of certain nonlinear hyperbolic initial value problems.

P. P. Zabreĭko

From MathSciNet, January 2023

MR0634248 (84a:35083) 35J60; 53C05, 58G20

Gidas, B.; Ni, Wei Ming; Nirenberg, L.

Symmetry of positive solutions of nonlinear elliptic equations in $\mathbf{R}^n$.

Mathematical analysis and applications, Part A, pp. 369–402, *Adv. in Math. Suppl. Stud., 7a, Academic Press, New York-London*, 1981.

The authors follow up their celebrated paper [Comm. Math. Phys. **68** (1979), no. 3, 209–243; MR0544879] by studying positive solutions of nonlinear elliptic equations in the whole of $\mathbf{R}^n$ and give conditions sufficient to ensure that the solutions are spherically symmetric. Three theorems will illustrate the scope of their work. Theorem 1: Let $u \in C^2(\mathbf{R}^n)$ be a positive solution of $-\Delta u = g(u)$ in $\mathbf{R}^n$ $(n \geq 3)$ with $u(x) = O(|x|^{-m})$ at infinity $(m > 0)$, and suppose that (i) on the interval $[0, u_0]$, where $u_0 = \max\,\{u(x)\colon x \in \mathbf{R}^n\}$, g can be written as $g_1 + g_2$, where $g_1 \in C^1$ and g_2 is continuous and nondecreasing: (ii) near 0, $g(s) = O(s^\alpha)$ for some $\alpha > \max\{(n+1)/m, (2/m)+1\}$. Then $u(x)$ is spherically symmetric about some point in $\mathbf{R}^n$, $u_r < 0$ for $r > 0$ (r is the radial coordinate about that point), and $|x|^{n-2}u(x) \to k > 0$ as $|x| \to \infty$. Theorem 2: Let $u \in C^2(\mathbf{R}^n)$ be a positive solution of $-\Delta u + m^2 u = g(u)$ in $\mathbf{R}^n$ $(n \geq 2, m > 0)$ with $u(x) \to 0$ as $|x| \to \infty$ and g continuous, $g(s) = O(s^\alpha)$ (for some $\alpha > 1$) near 0. Suppose that on $[0, u_0]$, $g = g_1 + g_2$ with g_2 nondecreasing and $g_1 \in C^1$ such that $|g_1(s) - g_1(t)| \leq C|s-t|/|\log \min(s,t)|^p$ $(s, t \in [0, u_0])$ for some $C > 0, p > 1$. Then $u(x)$ is spherically symmetric about some point in $\mathbf{R}^n$, $u_r < 0$ for $r > 0$, and $r^{(n-1)/2}e^r u(r) \to 0$ as $r \to \infty$. Theorem 3: Let $u \in C^2(\mathbf{R}^n)$ be a positive solution of $-\Delta u = g(u)$ in $\mathbf{R}^n \setminus \{0\}$ such that $u(x) = O(|x|^{-m})$ as $|x| \to \infty$ $(m > 0)$ and $u(x) \to \infty$ as $x \to 0$. Suppose that (i) g is continuous and nondecreasing on $[0, \infty)$, and for some $\alpha > (n+1)/m$, $g(s) = O(s^\alpha)$ near 0; (ii) $\liminf_{s\to\infty} g(s)s^{-p} > 0$ for some $p > n/(n-2)$. Then $u(x)$ is spherically symmetric about 0 and $u_r < 0$ for $r > 0$. The techniques used are adaptations of the ingenious ones used in the authors' earlier paper [op. cit.].

{For the collection containing this paper see MR0634233}

D. E. Edmunds

From MathSciNet, January 2023

MR0673830 (84m:35097) 35Q10; 76D05

Caffarelli, L.; Kohn, R.; Nirenberg, L.

Partial regularity of suitable weak solutions of the Navier-Stokes equations.

Communications on Pure and Applied Mathematics **35** (1982), *no.* 6, 771–831.

The paper makes significant improvements on earlier works of Scheffer on the partial regularity of solutions for Navier-Stokes equations (1) $u_t^i + u \cdot \nabla u^i - \Delta u^i + \nabla_i p = f^i$ $(i = 1, 2, 3)$, $\nabla \cdot u = 0$. Let P^1 be the one-dimensional Hausdorff measure defined by parabolic cylinders: $Q_r(x,t) = \{|y - r| < r, t - r^2 < \tau < t\}$. One of the main results states that the singular set S of a "suitable" weak solution satisfies $P^1(S) = 0$. It is known that when $u(x,t)$ and $p(x,t)$ solve (1), then $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$ solve (1) with $f_\lambda(x,t) = \lambda^3 f(\lambda x, \lambda^2 t)$ (which suggests the use of parabolic cylinders). This, combined with the local regularity estimates, such as the following, yields the partial regularity results: there exist $\varepsilon_1, \varepsilon_2 = \varepsilon_2(q)$ and C_1 such that, when

$$\iint\limits_{Q_1} \left(|u|^3 + |u|\,|p|\right) + \int_{-1}^{0} \left(\int_{|x|<1} |p|\,dx \right)^{5/4} dt \leq \varepsilon_1$$

and $\iint_{Q_1} |f|^q \leq \varepsilon_2$ for some $q > 5/2$, then u is regular on $Q_{1/2}$.

Tai Ping Liu

From MathSciNet, January 2023

MR0709644 (84h:35059) 35J65; 35B99

Brézis, Haïm; Nirenberg, Louis

Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents.

Communications on Pure and Applied Mathematics **36** (1983), *no.* 4, 437–477.

The authors obtain various existence and nonexistence results for the nonlinear elliptic problem $(*)$ $\Delta u + f(x,u) + u^{(n+2)/(n-2)} = 0$, $u > 0$ in Ω and $u = 0$ on $\partial\Omega$, where Ω is a bounded smooth domain in $\mathbf{R}^n$, $n \geq 3$, and $f(x,u)$ is a lower-order perturbation of the term $u^{(n+2)/(n-2)}$. One interesting feature of $(*)$ is that the exponent $(n+2)/(n-2)$ is the so-called Sobolev cut-off exponent, i.e., the embedding $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is not compact if $p = (n+2)/(n-2)$. Thus, $(*)$ resembles some variational problems in geometry and physics, e.g. Yamabe's problem, where lack of compactness also occurs.

In the case $f \equiv 0$, the well-known Pokhozhaev identity implies that $(*)$ has no solution if Ω is star-shaped. In this paper, it is shown that if $f \not\equiv 0$, then this situation may be reversed. As an example of various results proved here, we set $f(x,u) \equiv \lambda u$, where $\lambda > 0$ is a parameter. The authors prove in this case that: (i) when $n \geq 4$, $(*)$ has a solution for every $\lambda \in (0, \lambda_1)$ where λ_1 is the first eigenvalue of $-\Delta$; moreover, $(*)$ has no solution if $\lambda \notin (0, \lambda_1)$ and Ω is star-shaped; (ii) when $n = 3$, and Ω is a ball, $(*)$ has a solution if and only if $\lambda_1/4 < \lambda < \lambda_1$.

Wei Ming Ni

From MathSciNet, January 2023

MR0739925; 87f:35096 35J65; 58G30

Caffarelli, L.; Nirenberg, L.; Spruck, J.

The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation.

Communications on Pure and Applied Mathematics **37** (1984), *no.* 3, 369–402.

The main result of this article is the following: Given a bounded strictly convex domain Ω in $\mathbf{R}^n$, and given two functions $\psi \in C^\infty(\overline{\Omega})$, $\psi > 0$ in $\overline{\Omega}$, and $\varphi \in C^\infty(\overline{\Omega})$, there exists a unique strictly convex solution $u \in C^\infty(\overline{\Omega})$ of the Dirichlet problem for the Monge-Ampère equation, $(*)$ $\det(u_{ij}) = \psi$ in Ω, $u = \varphi$ on $\partial\Omega$.

Existence in $C^\infty(\Omega)$ had previously been obtained geometrically by proving, first, the existence of a generalized solution in the sense of A. D. Aleksandrov and then its interior regularity. More recently, purely analytic proofs were also given by P.-L. Lions [C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), no. 12, 589–592; MR0647688] using the penalty method, and by S. T. Yau and S. Y. Cheng [*Beijing symposium on differential geometry and differential equations, Vol. 1, 2, 3* (Beijing, 1980), 339–370, Science Press, Beijing, 1982; MR0714338] via complete Kähler bounded geometry in $(\Omega \oplus i\mathbf{R}^n)$.

Here, the classical continuity method is used, with the maximum principle of Protter-Weinberger; the authors build up all the barriers required for the "a priori estimates part" of the method and they obtain a solution smooth up to the boundary. In 1974, such a solution was announced by Nirenberg [*Proceedings of the International Congress of Mathematicians, Vol. 2* (Vancouver, B.C., 1974), 275–279, Canad. Math. Congr., Montreal, Que., 1975; MR0442291], but a gap appeared to the authors in the estimation of the third derivatives of u on $\partial\Omega$ (E.

Calabi, Nirenberg). This difficulty is eventually overcome in Sections 5 and 6 of this article, where a logarithmic modulus of continuity is carried out for the second derivatives of u on $\partial\Omega$. The key lemma is Lemma 5.1 which goes as follows: Let $L = a^{ij}(x)\partial_{ij} + a^i(x)\partial_i + a(x)$, with $a \le 0$, $M^{-1}|\xi|^2 \le a^{ij}\xi_i\xi_j \le M|\xi|^2$, $|a^i|$, $|a| \le M$, in a half-ball $B_R^+ = \{x \in \mathbf{R}^n : |x| < R,\ x_n > 0\}$; let $v \in C^2(B_R^+) \cap C^1(\overline{B}_R^+)$ satisfy the following conditions: $Lv \le C$, $|v| \le C$ in B_R^+, $|\nabla_{x'} v(x', 0)| \le C$, $v_n(x', 0) \le C$, $v(x', 0)$ is convex. Then $|\nabla_{x'} v(x', 0) - \nabla_{x'} v(\bar{x}', 0)| \le \overline{C}/(1 + \left|\mathrm{Log}|x' - \bar{x}'|\right|)$ for $|x'|$, $|\bar{x}'| \le \frac{1}{2}R$, where $\overline{C}$ depends only on n, M, R and C.

This new result yields further regularity by careful application of elliptic theory; alternatively, it enables one to complete bare-handed the a priori estimation of the third derivatives of u on $\overline{\Omega}$, as noticed by the authors and as done by the reviewer [C. R. Acad. Sci. Paris Sér. I Math. **296** (1983), no. 5, 253–256; MR0693786].

In the final section of this article, the Dirichlet problem is treated for more general real elliptic Monge-Ampère equations of the form $\det(u_{ij}) = \psi(x, u, \nabla u)$ and $\det(a_{ij} + u_{ij}) = \psi(x)$.

This is a self-contained paper, the first of a series: The case of complex Monge-Ampère equations is treated in part II [L. A. Caffarelli et al., Comm. Pure Appl. Math. **38** (1985), no. 2, 209–252; MR0780073]. Part III deals with functions of the eigenvalues of the Hessian [L. A. Caffarelli, L. Nirenberg and J. Spruck, Acta Math. **155** (1985), no. 3-4, 261–301; MR0806416].

In the 2-dimensional case one should mention the concise attempt of T. Aubin [J. Funct. Anal. **41** (1981), no. 3, 354–377, part I; MR0619958], that of the reviewer [C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), no. 21, 693–696; MR0666620], and the complete treatment given in the monograph of D. Gilbarg and N. S. Trudinger [*Elliptic partial differential equations of second order*, Springer, Berlin, 1983; MR0737190]; all use the continuity method. Concerning the Monge-Ampère equation in two dimensions, in the case when it is no longer elliptic, one must mention the recent local results of C.-S. Lin, dealing with variable signature ["The local isometric embedding in $\mathbf{R}^3$ of two-dimensional Riemannian manifolds with Gaussian curvature changing sign cleanly", Ph.D. Thesis, Courant Inst., New York Univ., New York, 1983] and degeneracy [J. Differential Geom. **21** (1985), no. 2, 213–230], derived via the Nash-Moser technique.

Last, but not least, a globally smooth elliptic solution of $(*)$ (as well as of more general equations, e.g., $\det(u_{ij}) = A^{ij}u_{ij} + \psi$, $\psi > 0$, $(A^{ij}) \ge 0$) was also produced by N. V. Krylov [Math. USSR-Sb. **48** (1984), no. 2, 307–326; MR0691980; ibid. **49** (1984), no. 1, 207–228; MR0703325] using diffusion processes and new techniques of estimation.

Among the possible differential-geometric applications of these results, one can mention the solvability of the Dirichlet problem for equations of prescribed Gauss curvature, carried out by Trudinger and J. I. E. Urbas [Bull. Austral. Math. Soc. **28** (1983), no. 2, 217–231; MR0729009].

Philippe Delanoë

From MathSciNet, January 2023

MR0780073 (87f:35097) 35J65; 58G30

Caffarelli, L.; Kohn, J. J.; Nirenberg, L.; Spruck, J.

The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampère, and uniformly elliptic, equations.

Communications on Pure and Applied Mathematics **38** (1985), *no.* 2, 209–252.

This is the second paper in a series [Part I, L. A. Caffarelli, L. Nirenberg and J. Spruck, Comm. Pure Appl. Math. **37** (1984), no. 3, 369–402; MR0739925; Part III, Acta Math. **155** (1985), no. 3-4, 261–301; MR0806416] devoted to the Dirichlet problem for fully nonlinear elliptic second-order equations: (1) $F(x, u, Du, D^2u) = 0$ in Ω, $u = \varphi$ on $\partial\Omega$, $\Omega \subset \mathbf{R}^n$ a bounded smooth domain, φ and F given smooth real-valued functions, F elliptic and concave in D^2u. The solution of (1) relies on a global $C^{2+\alpha}$ a priori estimate on u, for some $\alpha \in (0, 1)$. Providing such an estimate, existence of a smooth solution can be established: if $F_u \leq 0$, by means of the continuity method, and the solution is unique; in the general case, by means of a degree theory argument (p. 243). Moreover, the $C^{2+\alpha}$ a priori estimate carried out here enables one to solve (1) by approximation when $F = f(u, Du, D^2u) - \psi(x)$ with f only Lipschitz in its arguments, but still concave in D^2u: this applies for instance to the Pucci equation [C. Pucci, Ann. Math. Pura Appl. (4) **72** (1966), 141–170; MR0208150].

The $C^{2+\alpha}$ a priori estimate on u is divided into two steps: (2) a C^2 estimate on u, and, assuming the latter, (3) a C^α estimate on D^2u. Step (2) is carried out, respectively, for the complex Monge-Ampère equation and for the general equation (1), in the first and the third part of this paper. In the second part, step (3) is carried out for the general equation (1).

In the first part the equation under consideration is $\det(u_{z^i\bar{z}^j}) = \psi(z, \bar{z}, u, Du) > 0$ in Ω, $u = \varphi$ on $\partial\Omega$. Here Ω is a strongly pseudoconvex domain in $\mathbf{C}^n$ and u is to be strictly plurisubharmonic. The case where $\psi = \psi(z, \bar{z})$ vanishes on $\partial\Omega$ with φ constant is also treated, under assumptions on $(\psi)^{1/2n}$ somewhat analogous to those in a paper by N. S. Trudinger and J. I. E. Urbas [Bull. Austral. Math. Soc. **28** (1983), no. 2, 217–231, Section 4; MR0729009]. Regarding the C^0 estimate on D^2u, the authors first establish that, for some controlled constant C, $\max_{\overline{\Omega}}|D^2u| \leq \max_{\partial\Omega}|D^2u| + C$. They proceed then to estimating $\max|D^2u|$ on $\partial\Omega$ by means of classical barrier arguments, carefully choosing local coordinates near $\partial\Omega$; this is the technical heart of the matter. A purely interior C^2 estimate à la Pogorelov has been derived by F. Schulz [J. Reine Angew. Math. **348** (1984), 88–93; MR0733924].

In the second part, the main contribution is the completion of Evans' interior estimate [L. C. Evans, Comm. Pure Appl. Math. **35** (1982), no. 3, 333–363; MR0649348; Trudinger, Trans. Amer. Math. Soc. **278** (1983), no. 2, 751–769; MR0701522] by a C^α bound on D^2u near $\partial\Omega$: actually, assuming a C^2 bound (step (2)), the ellipticity and concavity of F, a logarithmic modulus of continuity of D^2u is derived near $\partial\Omega$ which yields the desired global bound by standard elliptic regularity theory. Another possible approach is that of N. V. Krylov [Izv. Akad. Nauk SSSR Ser. Mat. **47** (1983), no. 1, 75–108; MR0688919], also presented by J. L. Kazdan [*Prescribing the curvature of a Riemannian manifold*, see pp. 42–45, Conf. Board Math. Sci., Washington, D.C., 1985; MR0787227]. Whereas the authors here use equation (1) differentiated twice, φ and $\partial\Omega$ in C^4 and the concavity hypothesis, Krylov used (1) differentiated only once, φ and $\partial\Omega$ only C^3 and assumed no concavity on F but its uniform ellipticity.

In the third part, it is necessary of course to assume certain structure conditions on the nonlinearity F, among them its uniform ellipticity, as done by Trudinger [op. cit.] and Krylov [op. cit.].

Philippe Delanoë
From MathSciNet, January 2023

MR0806416 (87f:35098) 35J65; 53C40, 58G30

Caffarelli, L.; Nirenberg, L.; Spruck, J.

The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian.

Acta Mathematica **155** (1985), *no.* 3-4, 261–301.

This paper is a sequel to parts I and II [L. A. Caffarelli, L. Nirenberg and J. Spruck, Comm. Pure Appl. Math. **37** (1984), no. 3, 369–402; MR0739925; L. A. Caffarelli et al., Comm. Pure Appl. Math. **38** (1985), no. 2, 209–252; MR0780073]. It deals with the Dirichlet problem for a special class of second-order concave elliptic differential operators: those which factor as symmetric functions of the eigenvalues of the Hessian. No crude assumption of uniform ellipticity should be made anymore. Instead, it is necessary to understand the geometry that relates an operator to the domain over which it acts, in order for the Dirichlet problem to be well posed. This deep relationship, necessary in case of constant boundary data, sufficient in all cases, yields the crucial C^2 a priori estimates that guarantee uniform ellipticity. General devices, established both in part II [op. cit.] and in a paper by N. V. Krylov [Izv. Akad. Nauk SSSR Ser. Mat. **47** (1983), no. 1, 75–108; MR0688919], provide then the global $C^{2,\alpha}$ estimate sufficient to conclude by means of the continuity method.

The enlightening model of an operator comes from L. Gårding's study of hyperbolic polynomials [J. Math. Mech. **8** (1959), 957–965; MR0113978]; the drastic importance of hyperbolicity, for stable a priori ellipticity, was already hinted at in a paper by N. M. Ivochkina [Mat. Sb. (N.S.) **122**(**164**) (1983), no. 2, 265–275; MR0717679]. Moreover, according to Gårding [op. cit.], hyperbolicity implies concavity.

An important application to special Lagrangian geometry [R. Harvey and H. B. Lawson, Jr., Acta Math. **148** (1982), 47–157, Section III.2.A; MR0666108] is presented, with an extra study of the solution set of Im $\prod_{j=1}^{n}(1+i\lambda_j)=0$.

Philippe Delanoë
From MathSciNet, January 2023

MR1258192 (95h:35053) 35J15; 35B50, 35J20, 35P15

Berestycki, H.; Nirenberg, L.; Varadhan, S. R. S.

The principal eigenvalue and maximum principle for second-order elliptic operators in general domains.

Communications on Pure and Applied Mathematics **47** (1994), *no.* 1, 47–92.

Let Ω be a bounded domain in $\mathbf{R}^n$, and let L be a uniformly elliptic operator of second order in Ω with real coefficients: $L = M + c(x)$, where $M = a_{ij}\partial_{ij} + b_i(x)\partial_i$. It is well known that if the boundary $\partial\Omega$ and the coefficients are sufficiently regular, then there exist a principal eigenvalue λ_1 and an associated principal eigenfunction ϕ_1 such that $(L + \lambda_1)\phi_1 = 0$ in Ω, $\phi_1 = 0$ on $\partial\Omega$; λ_1 is a simple eigenvalue, and it lies to the left of other eigenvalues of the Dirichlet problem; $\phi_1 > 0$ in Ω, and only for the eigenvalue λ_1 does there exist a positive eigenfunction. It is also well known that the condition $\lambda_1 > 0$ is equivalent to the validity of the maximum principle: $Lw \geq 0$ in Ω and $\limsup_{x\to\partial\Omega} w(x) \leq 0$ imply $w \leq 0$ in Ω. The authors investigate the nonsmooth case. They assume that Ω is arbitrary, the a_{ij} are continuous, the b_i and c are bounded, and $w \in W^{2,n}_{\mathrm{loc}}$. The principal eigenvalue is defined by the formula $\lambda_1 = \sup\{\lambda \in \mathbf{R}\colon$ there exists $\phi > 0$ in Ω satisfying $(L + \lambda)\phi \leq 0\}$. The authors prove the existence of the principal eigenfunction and obtain some general estimates for λ_1. They give a definition of a refined maximum principle and prove that it is true if and only if $\lambda_1 > 0$. The definition has the following form: $Lw \geq 0$ in Ω, $w \leq$ const, and $\limsup w(x_j) \leq 0$ if $u_0(x_j) \to 0$ imply $w \leq 0$ in Ω. Here u_0 is a solution to the equation $Mu = -1$ vanishing on the boundary, in some sense.

M. S. Agranovich

From MathSciNet, January 2023