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F. S. MACAULAY: FROM PLANE CURVES TO GORENSTEIN RINGS

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ABSTRACT. Francis Sowerby Macaulay began his career working on Brill and Noether's theory of algebraic plane curves and their interpretation of the Riemann–Roch and Cayley–Bacharach theorems; in fact it is Macaulay who first stated and proved the modern form of the Cayley–Bacharach theorem. Later in his career Macaulay developed ideas and results that have become important in modern commutative algebra, such as the notions of unmixedness, perfection (the Cohen–Macaulay property), and super-perfection (the Gorenstein property), work that was appreciated by Emmy Noether and the people around her. He also discovered results that are now fundamental in the theory of linkage, but this work was forgotten and independently rediscovered much later. The name of a computer algebra program (now Macaulay2) recognizes that much of his work is based on examples created by refined computation.

Though he never spoke of the connection, the threads of Macaulay's work lead directly from the problems on plane curves to his later theorems. In this paper we will explain what Macaulay did, and how his results are connected.

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1. Introduction

By 1850 a theory of algebraic curves in the complex projective plane was being developed. Riemann's work in 1857 introduced a radical new point of view: the central object of his theory was a (topological) finite branched cover of the sphere, carrying many possible complex structures, each of which could be represented as an algebraic plane curve (with singular points). A central result, connecting the topology of the covering space with the analytic properties of the meromorphic functions defined on it, is now called the Riemann–Roch theorem. Among other applications, it is the key to understanding maps from an algebraic curve to projective space, and its generalizations still have a central place in algebraic geometry.

However, Riemann's theory depended on what he named the "Dirichlet principle".[1] This asserts that the solution to a certain differential equation is a minimizer of an appropriate energy function. Riemann, however, asserted the existence of a minimizer, though this actually fails in some other cases, and this was regarded with suspicion. In the 1870s, 1880s, and 1890s, Alexander Brill and Max Noether, the leading German algebraic geometers of the day, set out to re-interpret the Riemann–Roch theorem and the closely related Cayley–Bacharach theorem in the context of an algebraic curve C in the complex projective plane without reference to the branched coverings. Their project necessarily involved the study of singular plane curves, and they encountered many difficulties associated with what they thought of as the "point groups" that arise when curves intersect at singular points, objects that would now be treated as finite subschemes of the plane.

From his first paper [Mac95] until 1904, Macaulay sought to extend this work, apparently without direct contact with the group around Brill and Noether. He did subtle and partly definitive work on the problems that arose in this theory, which we will describe in some detail below. One of the high points of his work was a result in [Mac00, §4] that he called the "Generalized Riemann–Roch theorem", and which is now well-known as the Cayley–Bacharach theorem (misattributed to Bacharach in [EGH96]); we give Macaulay's statement, in Theorem 1, below. Macaulay's work in this period was recognized with an invitation to speak at the Heidelberg International Congress of Mathematicians in 1904 where he laid out conjectures that would have greatly extended some of Noether's work, but which were incorrect because of the possibilities of what would soon be understood as embedded primes in a primary decomposition.

The appearance of Lasker's paper [Las05] introducing primary decomposition (and much else) changed the landscape of the theory of polynomial rings, and the possibilities for extending Macaulay's ideas. The second period of Macaulay's research began with a ground-breaking paper published in 1913 in which Macaulay took full account of these possibilities. It is there that the concept of a perfect ideal (now Cohen–Macaulay ideal) was introduced, and many results that now belong to the theory of linkage were proven.[2]

[1] See [Mon75] and [Bot86].

[2] In 1972 the first author of this note was in Oberwolfach, when he happened to meet Alexander Ostrowski, who was already active in Göttingen at the time of the First World War. Ostrowski, who worked on number theory, said that he'd originally been interested in commutative algebra, and indeed Macaulay cites some of his work (see §8, below). Such was the originality of Macaulay's work that, hearing about it during the war with little chance of having direct contact with Macaulay, Ostrowski said he felt that he had better switch fields.

By the time of Macaulay's last paper [Mac34] he had introduced and explored several of the central concepts of modern commutative algebra and was in contact with the Göttingen school; he was the first to write about Emmy Noether's work in English.

At least three of Macaulay's contributions are well known today: the definition of perfect ideals (roughly, those defining Cohen–Macaulay rings); the proof that ideals of maximal minors of a matrix are perfect if they have generic codimension; and the characterization of Hilbert functions (all in the context of homogeneous polynomial ideals). Other results, such as those on Gorenstein ideals and linkage, were forgotten and subsequently discovered independently, and Macaulay's insight there is rarely credited. Macaulay's only well-known work today was published as the Cambridge Tract in Mathematics No. 19 [Mac16], to which we will refer as the *Tract*. Unlike the work of Hilbert before him or Noether after him, the style and language now seem old-fashioned, and the book is quite difficult for a modern algebraist to read.

Part of the difficulty in reading Macaulay's work comes from his extensive use of a method for representing an ideal[3] by a sorted vector space basis that could be thought of as an early version of a Gröbner basis. His name is now known to many through the computer program Macaulay2 [GS93], which makes such computations vastly easier. Though Macaulay did not himself point to the connections, the results and ideas of his later work seem from a modern point of view to be direct responses to the problems with which he and others had struggled many years before. The purpose of this note is to explain what we see as the long arc of Macaulay's work, and to show how his later work can be seen as responding to—and solving—the problems he encountered as a beginner. In [EG] (in preparation) we trace the historical development in more detail.

Macaulay's career. Born in 1852, Macaulay was educated at St John's College, Cambridge, graduating 8th wrangler in January 1883 (this means he graduated 8th in order of merit). In 1891 he got a B.Sc. from the University of London, and a D.Sc. in 1898 on the strength of his first published paper.

Macaulay became a highly successful and professionally active high-school teacher at the prestigious public school—to use the British term—St. Paul's in London. Several of his students were outstanding, among them J. E. Littlewood, who compared Macaulay's teaching favorably with what he later received in Cambridge. In the early period of his research, up to 1904, Macaulay worked on matters related to the Riemann–Roch and Cayley–Bacharach theorems in the theory of possibly singular plane algebraic curves, building on the work of the group around Alexander Brill and Max Noether. When Macaulay was unexpectedly passed over for the post of Head of Mathematics at his school, he retired in 1911 at age 49, and devoted himself to research.

Macaulay had begun his research career under the guidance, as he tells us near the start of his first paper, of Charlotte Angas Scott, who had famously done well

[3]A note on terminology: In the period under discussion what we would call an "ideal" was generally called a "module"; outside of quotations, we have replaced the term "module", by "ideal".

in the Cambridge Tripos Examinations in 1880, and they kept in correspondence about her work and his for some time.[4]

Macaulay's recognition in England was slight. After his retirement from St. Paul's he eventually moved to Cambridge, but probably did not become a member of the circle around Henry Frederick Baker, the leading English algebraic geometer of the time. Macaulay's work went in a different direction from Baker's, and Baker wrote a somewhat dismissive obituary [Bak38] when Macaulay died.

Macaulay was better appreciated by Emmy Noether and her school. Van der Waerden reports [vdW75, p. 33] that when he arrived in Göttingen to work with Emmy Noether, Macaulay's Cambridge *Tract* was one of three sources he had to study in addition to Noether's own work, and he credits sections of Macaulay's book as the sources of his treatment of parts of ring theory and elimination theory.

2. Early work: The Riemann–Roch theorem and the Clebsch–Brill–Noether program

The work of Bernard Riemann in the 1850s and 1860s changed the direction of the theory of algebraic curves, replacing curves in the complex projective plane with branched coverings of the complex projective line—the Riemann sphere—and questions about tangents and secants by questions about spaces of meromorphic functions with bounded pole orders. Riemann's starting point was the existence of two independent functions, s, z on the covering space, where z is the coordinate on the Riemann sphere and s is a multivalued function on the sphere, connected by an algebraic equation $f(s, z) = 0$. From a modern point of view, the Riemann surface is then the normalization $\widetilde{C}$ of the closure $C \subset \mathbb{P}^2$ of the affine plane curve defined by f.

However, Riemann's work, depending on the then-unproven "Dirichlet principle", was far from completely accepted. First Alfred Clebsch and then, after Clebsch's death in 1872, Alexander Brill and Max Noether set out to reprove the Riemann–Roch theorem algebraically, within the theory of plane curves. Macaulay's later work was closely related to what Max Noether called the Fundamental Theorem (stated in [Noe73]) and reproved and named in [Noe87, p. 410], and Macaulay contributed importantly to the crucial question of residuation, as well, work that would reappear in a much wider context in Macaulay's later papers.

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[4]Scott was ranked between the 7th and the 8th students in order of merit in the Cambridge Tripos of 1880, but only unofficially because Cambridge at that time did not allow women to read for a degree. This sufficiently embarrassed the University authorities that they allowed women to sit the exams officially in future (but not to take a degree, not until 1945!). Scott then obtained a D.Sc. from the University of London in 1885, but unable to get a job in Britain she left for America, and became one of the first professors at the new, women's College of Bryn Mawr, Pennsylvania in September 1885 (the college opened officially on 23 September that year). She went on to have a distinguished career. She was the dissertation advisor to seven students, putting Bryn Mawr third, behind Chicago and Cornell, at a time when women were winning three times the percentage of PhDs in America that they were to win in the 1950s. Scott alone directed three of the nine PhDs successfully completed by American women in the 19th century. She was influential from the start in the American Mathematical Society and became its vice-president in 1905–06. For information on Scott, see Macaulay's obituary of her (Macaulay 1932) [25], (Kenschaft 1989) [Ken89], and the references cited therein.

2.1. **Noether's Fundamental Theorem.** This result, sometimes referred to as the "$AF + BG$" theorem, was the central result necessary to prove that the linear systems cut out on a curve C by adjoint curves are complete. The Fundamental Theorem says that if a projective plane curve $E : H(X, Y, Z) = 0$ contains all the intersections of the curves $C : F(X, Y, Z) = 0$ and $D : G(X, Y, Z) = 0$, which have no common components, then H is in the homogeneous ideal generated by F and G; in other words, it gives a local test, for each point of the set-theoretic intersection of C and D, for membership in the homogeneous ideal.

The first step in such a result is to define what it means for E to contain the intersection of C and D at a point p. Since the question is local around p, we may assume that the point p is $(0, 0, 1)$ and replace F, G, H by their dehomogenized versions $f(x, y)$, $g(x, y)$, $h(x, y)$, where $x = X/Z$, $y = Y/Z$. Noether said that E contains the intersection of C, D at p if h is in the ideal generated by f, g *in the power series ring* $\mathbb{C}[[x, y]]$. This coincides with the modern scheme-theoretic definition.

The proof Noether gave first asserted that it was enough to treat the affine case, and then used the theory of resultants—relatively unfamiliar at the time—to prove the theorem. The result attracted quick attention; for example [Vos87] and [Ber89] offered what they claimed were improved or clarified proofs, and Noether himself returned to give what he considered more complete proofs several times, though we found his expositions difficult to read. He also maintained that his orginal paper contained an adequate proof of the case where the intersections are not too complicated.

Primary decomposition was first established by Lasker in 1905, more than 30 years after Noether enunciated the Fundamental Theorem; but in Lasker's terms, Noether's assertion was that H is in the homogeneous ideal (F, G) if it belongs to every isolated component of (F, G). The truth of the assertion thus depends on the statement that (F, G) has no primary component associated to the "irrelevant" ideal (X, Y, Z). The Fundamental Theorem *cannot* be reduced to the affine case without somehow taking care of this extra possibility and the corresponding result that would decide whether $H \in (F_1, F_2, F_3)$ is true in the affine and false in the projective case, something that went unremarked or unnoticed.

In modern terms, the absence of an (X, Y, Z)-primary component follows from the fact that coordinates can be chosen so that none of the intersection points lies on the line $Z = 0$, and that the three forms (F, G, Z) are a regular sequence. This is essentially what Brill and Noether finally use in [BN94, p. 353] to deduce the projective case from the affine case. Here is the way they explain it.

Suppose that $F(X, Y, Z)$ and $G(X, Y, Z)$ are homogeneous polynomials with no common factors, and no common zeros on the line $Z = 0$, and that $H(X, Y, Z)$ is a homogeneous polynomial containing the clusters of points common to the curves $F = 0$ and $G = 0$. Let $f(x, y), g(x, y), h(x, y)$ be the polynomials that arise from F, G, H by setting $Z = 1$, and suppose we have proven that $h = af + bg$, where a, b are polynomials. We must show that there are homogeneous polynomials A, B such that $H = AF + BG$.

Note that $h(x, y)$ may be written as $Z^{\deg H} h(X/Z, Y/Z)$, and similarly for F, G. Moreover, a, b are dehomogenizations of some homogeneous polynomials A', B'.

Thus we may write

$$\begin{aligned} H(X,Y,Z) &= Z^{\deg H} h(X/Z, Y/Z) \\ &= Z^{\deg H}\big(a(X/Z,Y/Z)f(X/Z,Y/Z) + b(X/Z,Y/Z)g(X/Z,Y/Z)\big) \\ &= Z^{\deg H}\big(Z^{-\deg A' - \deg F}A'F + Z^{-\deg B' - \deg G}B'G\big), \end{aligned}$$

and thus, clearing the negative powers of Z,

$$Z^m H = A''F + B''G$$

with A'', B'' homogeneous, for some $m \geq 0$, which we may take to be minimal.

If $m > 0$, this leads to a contradiction: Since F, G have no common zero on the line $Z = 0$, the polynomials $F(X,Y,0)$ and $G(X,Y,0)$ have no common factors. Thus $A''(X,Y,0) = C(X,Y)G(X,Y,0)$, so $A''(X,Y,Z) = C(X,Y)G(X,Y,Z) + ZA'''$ for some form A'''. It follows that $Z^m H = ZA'''F + B'''G$ for some homogeneous B'''. But now Z must also divide B''', and dividing both sides by Z, we have reduced m. Since m was assumed minimal, this is a contradiction; so $m = 0$, completing the argument.

2.2. **Residuation.** Brill and Noether represented divisors on the normalization $\widetilde{C}$ of C as differences of intersections of C with other plane curves D having specified behavior at the singular points of C ("satisfying the conditions of adjunction"). These intersections could occur at singular points of C and D, have arbitrary behavior there, and in particular could have high multiplicity. Macaulay called such intersections "clusters" of points. A cluster at (say) the origin corresponding to an intersection of curves $f_1(x,y) = 0, \dots, f_s(x,y) = 0$ was considered as corresponding to the ideal generated by $f_1 = 0, \dots, f_s = 0$ in the power series ring—today we would identify this as a finite subscheme of $\mathbb{P}^2$. The statement of Brill and Noether's proposed version of the Riemann–Roch theorem required that one could form the difference, or *residual*, of one cluster in another $\gamma'' := \gamma \setminus \gamma'$ when γ' was contained in γ, with the property that $\gamma \setminus \gamma'' = \gamma'$.

There is no difficulty about this when C is smooth, and it is possible to do this in the singular case, as Macaulay showed, when $\gamma = C \cap D$ is the intersection of two curves, C, D, as was necessary for the Riemann–Roch theorem; but if $\gamma = C \cap D \cap D'$ is the common intersection of C with two (or more) curves D, D', it may not be possible. Here is a concrete example.

Example 1. Consider the three irreducible curves that intersect at the origin in the affine plane which are defined by the three polynomials $x^2 + y^3, xy + (x+y)^3,\ y^2 + x^3$. In the power series ring $\mathbb{C}[[x,y]]$ these three generate the ideal $I = (x^2, xy, y^2)$, corresponding to a cluster of points γ of multiplicity 3. The intersection of the line $x = 0$ with γ is a cluster β represented by the ideal (x, y^2), which has multiplicity 2, so the residual cluster $\gamma \setminus \beta$ should have multiplicity 1, and thus must be the unique cluster α of multiplicity 1 contained in γ, represented by the ideal (x, y). So far so good: but if β' is the cluster in which $y = 0$ intersects γ, then $\gamma - \beta' = \gamma$ as well, and since $\beta \neq \beta'$, we cannot hope for

$$\gamma \setminus (\gamma \setminus \beta) = \beta.$$

Macaulay studied a cluster of points γ by studying the corresponding ideal $I \subset \mathbb{C}[[x,y]]$ of the power series ring. If $f \in \mathbb{C}[[x,y]]$, then the condition $f \in I$ can be written as a finite set of linear equations on the coefficients of f, which he called the modular equations of the cluster. Macaulay observed that some of these equations could be derived from others, and called γ a t-set point if the minimum number of equations necessary to derive all the others is t. From a modern point of view, t is the dimension of the socle $(I : (x, y))/I$ of $\mathbb{C}[[x, y]]/I$. In [Mac99, p. 407] he proved that residuation in a t-set point is possible if and only if $t = 1$; again, from a modern point of view, this condition means that $\mathbb{C}[[x, y]]/I$ is Gorenstein. He also showed that the intersection of two curves without common components always consists of clusters that are 1-set points, and that any 1-set point has a Noether ideal with just two generators; in modern language, these are complete intersections, and complete intersections are the only Gorenstein ideals of codimension 2.

Finally, he showed that if γ is a 1-set point and $\alpha \subset \gamma$ is a t-set point, then the residual $\gamma - \alpha$ is an s-set point, where $s = t - 1, t$ or $t + 1$, anticipating part of a theorem of Gaeta [Gae52]). He was to develop these ideas far more broadly in his later work.

2.3. **Riemann–Roch and Cayley–Bacharach.** In a paper of 1843 [Cay09, p. 211], Arthur Cayley famously quoted Chasles' 1837 proof that Pascal's theorem on the sides of a hexagon inscribed in a conic could be subsumed in the result that if Γ is the set of nine points of intersection of two cubic curves in the plane, then any cubic through eight points of Γ automatically passes through the ninth (see [Cha89]). Cayley went on to state a generalization to the set of points of intersection of two curves of any degree. His generalization depended on the assumption that the points of intersection were "sufficiently general"—an assumption that can fail. Isaak Bacharach, a student of Brill, used Noether's Fundamental Theorem to prove a corrected version, in a restricted case, in [Bac86].

In [Mac00, p. 424] Macaulay (again using Noether's Fundamental Theorem) proved a much more general version, which is now usually referred to as the Cayley–Bacharach theorem (and misattributed to Bacharach in [EGH96]). Rather than using the name Cayley–Bacharach, Macaulay referred to it as the "Generalized Riemann–Roch Theorem" a name (nearly) justified by the close connection of the two results. Macaulay began with two definitions. Here N represents a cluster of points and n a positive integer:

> The n (called an n-ic) defect of N is the degree-of-freedom of a curve of degree n-ic containing N; that is, one less than the dimension of $I(N)_n$, the degree n part of the homogeneous ideal of N. The n-ic excess of N is the excess of the number N over the number of independent conditions supplied by N for an n-ic, that is, N minus the codimension of $I(N)_n$ in the space of all forms of degree n.

In these terms, Macaulay's result is the following:

Theorem 1. *If the point-base forming the whole intersection of two curves C_ℓ, C_m, which have no common factor and no intersection at infinity, is divided into any two residual point-bases N, N' and if d_n, d'_n, D_n are the n-ic defects, and e_n, e'_n, E_n the n-ic excesses, of $N, N', N+N'$, respectively, then $d'_{n'} = e_n + D_{n'}$ and $e'_{n'} = d_n - D_n$, where $n + n' = \ell + m - 3$.*

For example, in the case treated by Bacharach, the points of Δ lie on a curve of degree $\gamma - 3$ if and only if they fail to impose independent conditions on curves of this degree, that is, if and only if their $(\gamma - 3)$-ic defect is at least 1; and in this case Macaulay's theorem says precisely that the number of conditions on forms of degree r imposed by the points of $C_m \cap C_n \setminus \Delta$ is strictly less than the number of conditions imposed on forms of degree r by the points of $C_m \cap C_n$, so that some curves of degree r that contain $C_m \cap C_n \setminus \Delta$ will in fact not contain $C_m \cap C_n$. By this time Macaulay had already done his work on residuals in 1-set points, and proven that every component of a complete intersection of two curves was a 1-set point, so in principle he could have allowed the intersection of C_m and C_n to be nontransverse. But he did not speak of this, and it seems likely that he thought just of the transverse case.

Like the Riemann–Roch theorem, this version of the Cayley–Bacharach theorem is about residuals of clusters in complete intersections of curves in the plane, and for points on a smooth plane curve it is easily seen to be equivalent to the Riemann–Roch theorem, once one understands the canonical divisor. Macaulay saw it as a more general result than the Riemann–Roch theorem, and he ignored the questions about the canonical divisors that would have made the two truly equivalent. For a detailed modern treatment of the Cayley–Bacharach theorem, with (partially incorrect) history and generalizations, see [EGH96]. Here is a sketch of the equivalence of the Riemann–Roch and Cayley–Bacharach theorems in the case of smooth curves:

2.3.1. *Riemann–Roch, with the Fundamental Theorem, implies the Cayley–Bacharach theorem.* To avoid the problems of the conditions of adjunction, we will consider only the case of Cayley–Bacharach where C is nonsingular (the case where C and D have no singular points in common follows easily from this, but the general case is more delicate). Let $\Gamma = C \cap D$ as a divisor on C, and suppose that $\Gamma = \Gamma' + \Gamma''$, that is, Γ'' is residual to Γ' in Γ. Write n, d for the degrees of the curves C, D, respectively, and let k be an integer with $0 \leq k \leq n + d - 3$. We write H for the divisor on C given by the intersection with a line, so that Γ is linearly equivalent to dH, and $\gamma = nd, \gamma', \gamma''$ for the degrees of $\Gamma, \Gamma', \Gamma''$, respectively. Set $g = (n(n-3)/2) + 1$, the genus of C. Write $K := (d-3)H$ for the canonical divisor of C. Using the completeness of the hypersurface series, which follows from the Fundamental Theorem, we can reformulate the Cayley–Bacharach theorem in terms accessible to the Riemann–Roch theorem:

- The number of conditions imposed by Γ minus the number imposed by Γ' on forms of degree k is
$$N := L(kH - \Gamma') - L((k-d)H) = L(kH - \Gamma') - L(kH - \Gamma).$$
- The failure of Γ'' to impose independent conditions on forms of degree $d + e - 3 - k$ is
$$\begin{aligned} M &:= \gamma'' - (L((n+d-3-k)H) - L((n+d-3-k)H - \Gamma'')) \\ &= \gamma'' - L(K + (d-k)H) + L(K + (d-k)H - \Gamma'')). \end{aligned}$$

Using the Riemann–Roch theorem and the fact that the degree of the divisor kH is kd, we see that

$$\begin{aligned} N &= kd - \gamma' - g + 1 + L(K - kH + \Gamma') - \big(kd - \gamma - g + 1 + L(K - kH + \Gamma)\big) \\ &= \gamma'' + L(K - kH + \Gamma') - L(K - kH + \Gamma) \\ &= \gamma'' + L(K + (d-k)H - \Gamma'') - L(K - (d-k)H) \\ &= M, \end{aligned}$$

as required.

2.3.2. *The Cayley–Bacharach theorem implies the Riemann–Roch theorem.* Again, let C be a smooth plane curve of degree n. Let $\Gamma = K = (n-3)H$, where K is the canonical divisor and H is the class of the intersection of C with a line. Let Γ'' be a divisor on C of degree γ'', and set $\Gamma' = K - \Gamma''$. We allow the possibility that $\gamma' = n(n-3)-\gamma''$, the degree of Γ', is negative, and interpret the Cayley–Bacharach theorem as asserting the equality of the quantities M, N above in any case. Taking $d = k = n-3$, we have

$$N = L(kH - \Gamma') - L(0H) = L(K - \Gamma') - 1 = L(\Gamma'') - 1.$$

On the other hand

$$M = \gamma'' - L(K) + L(K - \Gamma'')$$

and $L(K) = g$, whence $L(\Gamma'') = \gamma'' - g + 1 + L(K - \Gamma'')$, as required.

3. 1903 and 1923: Two papers on resultants

The image under a general linear projection $\pi : \mathbb{P}^n \to \mathbb{P}^m$ of a k-dimensional variety X will be a k-dimensional variety again if $k \leq m$. Algebraically, the projection is represented by an inclusion

$$\mathbb{C}[x_0, \ldots, x_m] \subset S := \mathbb{C}[x_0, \ldots, x_m, y_{m+1}, \ldots, y_n],$$

and the defining ideal of $\pi(X)$ is the intersection of the defining ideal of X with the subring $\mathbb{C}[x_0, \ldots, x_m]$. (In the affine case the image may not be closed; the intersection ideal defines its closure.) The computation of the ideal of $\pi(X)$ goes under the name "elimination theory", since it involves eliminating variables from the equations of X. A special case that plays a major role in early investigations of polynomial algebra occurs when $m = k+1$ or $m = k$; then $\pi(X)$ is defined by a single equation, which is 0 in the latter case. In the case where X is defined by an ideal generated by $c := \operatorname{codim} X = n - k$ equations $f_1, \ldots, f_c$, this single equation is called the "resultant" of $f_1, \ldots, f_c$ with respect to the $m+1$ variables $x_0, \ldots, x_m$. Etienne Bézout [Béz79] in 1779 published a determinantal formula for the resultant in the case $c = 2$ of two polynomials; and in a paper of 1848 [Cay09, pp. 370–374] Arthur Cayley announced, without proof, a formula for the resultant in the general case. Cayley's formula, however, expressed the resultant as a ratio of one complicated product of determinants by another such product. Cayley's idea was expounded by George Salmon in [Sal85, pp. 80–83]. Although the denominator in Cayley's expression must be a factor of the numerator in this product, making the division explicit is an open problem to this day. (See, for example, [ES03] for a modern view of the situation.)

The resultant was particularly important in early 20th century commutative algebra because an ideal generated by c homogeneous polynomials of positive degree cannot have codimension greater than c, and has codimension exactly c if and only if the resultant with respect to $m+1 = k+1$ general variables is 0 (in the affine case this is the resultant with respect to k general variables). Thus the resultant provides a criterion under which, in modern terms, c homogeneous forms are a regular sequence, and this is the use to which it is put in Lasker's, and later Macaulay's, generalization of Noether's Fundamental Theorem.

In [Mac03] Macaulay, always interested in explicit computation and recognizing the difficulty in using Cayley's formula, proposed a potentially simpler computation, writing the resultant as the quotient of a single determinant by one of its minors of a certain size. Mathematical interest in this sort of expression continues: a search in MathSciNet for Macaulay and resultant together yields many relevant references.

The basis of Macaulay's method is the theorem on p. 9 of [Mac03]. Write S_ℓ for the vector space of forms of degree ℓ in the polynomial ring S. Macaulay considered the number $d = \sum_1^c \deg f_i - c + 1$ and the matrix D with $\binom{n+d}{d}$ rows that is the degree d component of the map whose ith component is multiplication by f_i,

$$\bigoplus_{i=1}^{c} S_{d-\deg f_i} \to S.$$

Macaulay proved:

Theorem 2. *If the coefficients in each of the polynomials $f_1, \ldots, f_c$ are taken to be indeterminates, and the matrix D is expressed in terms of these indeterminates, then the resultant of $f_1, \ldots, f_c$ is the greatest common divisor of the minors of size $\binom{n+d}{d}$ of D.*

The formula is not difficult to understand set-theoretically: the resultant, evaluated at the coefficients of the f_i, is nonzero if and only if the ideal generated by the f_i has finite colength, which says that the degree d part of the map above is surjective for $d \gg 0$. Cayley's original insight was that this would be the case if and only if the map was surjective for $d = \sum_1^c \deg f_i - c + 1$, and this is the case if and only if the maximal minors of D generate the unit ideal. Moreover, the resultant must vanish on a set of n-tuples of forms of codimension 1 (the condition is that f_n vanishes on one of the points cut out by $f_1, \ldots, f_{n-1}$, which is a linear condition on the coefficients of f_n), so the annihilator of the cokernel of D has codimension 1. It follows that this annihilator has at least the same radical as the greatest common divisor of the maximal minors of D.

Starting from this theoretical insight, the paper [Mac03] is quite computational. Modern treatments of the computation can be found in [Jou95] and [ES03]. In the case when all the forms f_i are homogeneous of the same degree, $f_1, \ldots, f_c$ can be thought of as defining a rational map of $\mathbb{P}^{n-1}$ to $\mathbb{P}^{c-1}$, and the variety defined by the f_i appears as the base locus of the map. In [Mac27] Macaulay gave a "more symmetric" expression for the resultant in this case. In [Mac23] he returned to this problem, giving "a simpler and more symmetrical form for the quotient" when $f_1, \ldots, f_c$ are all homogeneous of the same degree.

4. The Congress of 1904 and the work of König and Lasker

The year 1904 was a watershed in Macaulay's career. He was invited to give an address at the International Congress of Mathematicians (ICM) in Heidelberg—no small honor for a school teacher. There he met Brill and Noether, probably for the first time. The paper [Macaulay, 1905][5] [13] in the proceedings volume described his work on plane curves, and sketched a far-reaching generalization to higher dimensions. But it is clear that he was unaware of a fundamental difficulty, caused by the possible presence of an embedded primary component. In a footnote

[5]For the citations of Macaulay's own work, see the separate listing of all of his papers, before the main bibliogrphy at the end of this paper.

on p. 285 he wrote:

> Since writing this paper Professor Noether and Professor Brill have kindly drawn my attention to the recently published "Einleitung in die allgemeine Theorie der algebraischen Größen" by Julius König (B. G. Teubner, Leipzig, 1903). This work, remarkable for its precision and comprehensiveness and the large additions it makes to the subject, contains a much desired proof of the extension of Noether's theorem to the case of k polynomials $M_1, \ldots, M_k$ in k variables, when [the variety defined by $M_1, \ldots, M_k$] is of zero dimensions, i.e., when the equations $M_1 = M_1 = \cdots = M_k = 0$ have only a finite number of solutions. I assume throughout my paper a still further, and what I regard as a fundamental, extension, viz., to the case of k polynomials in n variables, where $k > n$. The theorem is that if, for each and every point of intersection of $M_1 = M_2 = \cdots = M_k = 0$ taken as origin, a given polynomial M can be expressed in the form $M_1P_1 + \cdots + M_kP_k$, where $P_1, P_2, \ldots, P_k$ are undetermined integral power series, then $M \equiv 0(\text{mod.}M_1, \ldots, M_k)$. The theorem can be finally extended so as to be free of all restrictions with respect to (M), k, or n.

The claimed result is false because the ideal $(M_1, \ldots, M_k)$ may have an irrelevant component.

Lasker's paper [Las05] finally provided the right tools and language to properly understand the situation. Perhaps from this, Macaulay seems to have realized his error, and he published no further research until [Mac13] by which time, as we shall see in §5, he was master of the situation.

4.1. **König.** Gyula (Julius) König is a major figure in the history of mathematics in Hungary, and he is perhaps best known internationally for his work on Cantor's set theory. In his *Introduction to the general Theory of algebraic Quantities* (Hungarian edition 1902, German edition 1903) he attempted to make Kronecker's difficult theory of modular equations in [Kro82] more accessible. To do so, he set out a broad account of algebra, defining fields of characteristic zero (which he called "holoid domains") and rings with no zero divisors ("orthoid domains"), and greatly elaborating the theory of resultants and resolvents. Resultants allow one to eliminate variables from a system of polynomial equations; resolvents allow one to test for membership of a polynomial ideal. In his account, the statement that a form f belongs to what we would call an ideal M was written as the congruence modulo $f \equiv 0$ mod. M, as in Gauss's introduction of modular arithmetic.

Using these tools, König generalized Noether's ideal membership condition and some of its properties to the case of k polynomials in k variables that define a set of points in affine space.

4.2. **Lasker.** Emanuel Lasker was the world chess champion from 1894 to 1921, and he wrote about chess, bridge, and other games. He was also a great mathematician and originally hoped for an academic career: in between chess competitions he wrote in the 1890s about rational normal curves in n-dimensional space and the convergence of infinite series of functions, for which he was awarded a doctorate under Max Noether in Erlangen. Noether later wrote, in Lasker's words, "a very flattering recommendation"[6] in the context of Lasker's application for an academic job in Pittsburgh that never materialized [Ros18]. Albert Einstein, in the Foreword to [Han91] called Lasker "one of the most interesting people I came to know in my later life", and wrote that

> I had the impression that to him chess was a means of livelihood rather than the real object of life. What he really yearned for was some scientific understanding and that beauty peculiar to the process of logical creation..."

Lasker's most important mathematical work was the great paper [Las05], which established the theory of primary decomposition for ideals in polynomial rings, and also two central properties of regular sequences: that all their syzygies have the form given by the Koszul complex and (using this) that their associated primes have the same dimension[7] [Las05, Satz XI and Satz XXVII, respectively]. He based his work on four great results of Hilbert (which he labelled *Theorems* 1 to 4, in contrast with his own Sätze), among them the Hilbert basis theorem (Theorem 1), which was fundamental to his proof of the existence of primary decompositions for polynomial ideals, as it was in the more general work of Emmy Noether. Lasker also justified Max Noether's ideal membership test with a version of what is now called the Krull intersection theorem (but see Section 6.3). His work deeply influenced Macaulay's later work. We pause to give an account of three theorems from [Las05] that were particularly important for Macaulay.

4.2.1. *Syzygies of a regular sequence.* After preliminaries on resultants, Lasker began with the syzygies of a regular sequence:

Theorem 3 (Satz I, p 24)**.** *If $u_1, \dots, u_h$ are forms in m variables, with $h \leq m$, such that the resultant of $u_1, \dots, u_h$ and $m - h$ linear forms with indeterminate coefficients does not vanish identically, and if there is an identical relation $p_1 u_1 + \cdots + p_h u_h = 0$ where the p_i are forms, then there are forms $q_{i,j} = -q_{j,i}$ such that*

$$p_i = q_{i,1} u_1 + \cdots + q_{i,h} u_h. \tag{1}$$

The resultant condition means that the ideal generated by the u_i and $m - h$ general linear forms has no zeros in projective space; that is, it contains a power of the maximal homogeneous ideal. The condition is equivalent to the modern statement that the ideal $(u_1, \dots, u_h)$ has codimension (at least) h. Thus Satz I, in modern language, says that if h forms generate an ideal of codimension h, then every syzygy of the forms is the product of a skew symmetric matrix and the row of forms themselves; or still more succinctly, the first homology of the Koszul complex of the forms vanishes.[8]

4.2.2. *Primary decomposition.* The most famous result of [Las05] is the existence of primary decomposition; even the definition of a primary ideal was new with Lasker.

Theorem 4 (Satz VII, p. 51)**.** *Every ideal of forms in a polynomial ring has a primary decomposition.*

[6] quoted in [Han91, p. 208]

[7] Recall that the associated primes of an ideal in a Noetherian commutative ring are the radicals of the primary ideals in any minimal primary decomposition.

[8] Despite the name, the Koszul complex was known to Cayley, and appears in a special case in Hilbert's work as well.

Proof. Given a homogeneous ideal $M \subset \mathbb{C}[x_1, \ldots, x_n]$, Lasker's proof started with the geometry of the set of points C in $\mathbb{C}^n$ at which all the elements of I vanish. As in the modern usage, he defined the residual of an ideal M with respect to an ideal N to be the ideal of forms f such that $fM \subseteq N$; we denote it $N : M$. He wrote $C_1 \cup \cdots \cup C_j$ for the union of the maximal dimensional components of C and defined M_{C_i} to be the set of forms F such that the residual ideal $M : F$ contains forms not vanishing on C_i—the primary component of M associated to C_i. If $M = \bigcap M_{C_i}$, he was done; otherwise he defined $M'_{C_i} := M : M_{C_i}$ and chose a form $\Phi \in \sum_i M'_{C_i}$ that does not vanish on any of the C_i. He then proved that

$$M = \bigcap_i M_{C_i} \cap (M, \Phi)$$

and asserted the existence of a primary decomposition of (M, Φ) by analysing (M, Φ) in the same way and by tacitly appealing to the ascending chain condition in the form that Hilbert had proved for polynomial rings—the result Lasker had cited as Theorem I. □

4.2.3. *Equidimensionality.* Perhaps the result of Lasker's that influenced Macaulay the most was this:

Theorem 5 (Satz XI, p. 58). *If $u_1, \ldots, u_h$ are forms in m variables, with $h \leq m$, such that the resultant of $u_1, \ldots, u_h$ and $m - h$ linear forms with indeterminate coefficients does not vanish identically, then $(u_1, \ldots, u_h)$ has a primary decomposition in which all the primary ideals have the same dimension $m - h$.*

In the language Macaulay introduced in his *Tract*: such an ideal is *unmixed.* Lasker's proof relied on his Satz I, on the syzygies of a regular sequence.

Lasker proved Satz XI as follows. By the primary decomposition theorem, the ideal M can be written as

$$M = M_{C_1} \cap M_{C_2} \cap \cdots \cap M_{C_j} \cap N,$$

where $C_1, C_2, \ldots, C_j$ are varieties of the highest dimension which are satisfied by all the forms in M, and N is a variety of dimension lower than that of M. Let $M = (u_1, u_2, \ldots, u_h)$ and let Φ be any form in N whose corresponding variety contains none of the $C_1, C_2, \ldots, C_j$. Let $F \in M_{C_1} \cap M_{C_2} \cap \cdots \cap M_{C_j}$. Then

$$F\Phi \in M_{C_1} \cap M_{C_2} \cap \cdots \cap M_{C_j} \cap N = M.$$

But now the resultant of $\Phi, u_1, u_2, \cdots, u_h$ and $m - h - 1$ linear forms does not vanish identically, and so, by Satz I, $F \in M$. Lasker also proved analogous results over the integers, and for power series rings.

5. Macaulay's great paper of 1913: Perfect ideals, Gorenstein ideals, and linkage

After his report in the ICM of 1904 Macaulay published nothing relevant to our story for nine years; but his next work on algebra is by far his most remarkable. It contains the work for which he is now most famous, the definition and study of *perfect* ideals, as well as work that was largely rediscovered, much later, on what are now called *Gorenstein* ideals and linkage. Perhaps harking back to his early work on Noether's Fundamental Theorem, or to his error at the time of his ICM talk in 1904, Macaulay began by recapitulating some of Lasker's 1905 paper, with a focus on unmixedness.[9] To handle this conveniently, he introduced, in [Mac13, §39], the definitions that will be familiar to a modern reader (except that current usage, and ours below, replaces "imbedded" with "embedded" and "module" with "ideal"):

> **Resolution of any given H-module.** The primary modules of which any given module M is composed are of two kinds, which can be distinguished by the terms isolated and imbedded. An isolated primary module or spread is one which contains points outside all the other primary modules or spreads of M, although it may interpenetrate these other spreads. An imbedded primary module or spread is one whose points are all contained in one or more of the other spreads of M of lower rank. The isolated primary modules into which M resolves are unique, but the imbedded ones are not.

Always concerned with actual computation, Macaulay then explained how to find all the primary components using resolvents. The distinction between isolated and embedded components was central in Macaulay's treatment. In [Mac13, §57] he reinterpreted Lasker's Theorem XI: but where Lasker said that the primary decomposition of the ideal generated by a regular sequence (Macaulay called this "an ideal of the *principal class*") has only the primary components "of highest dimension" (höchster Mannigfaltigkeit"), Macaulay said that the ideal is unmixed. He extended the result to inhomogeneous ideals of the principal class and also proved that powers of such ideals are unmixed.

5.1. **Perfect ideals.** Macaulay's first major step beyond the work of Lasker, and certainly his most influential contribution, is the definition of *perfect* ideals in [Mac13, §66]. His definition there uses what he called the "Hilbert numbers" (we would say the Hilbert function) $H_{S/I}(d)$. Defined only in the case when I is homogeneous, $H_{S/I}(d)$ is the dimension of the dth graded component of S/I. In modern terms, Macaulay observed that if I is a homogeneous ideal in S, then for any linear form $y \in S$ the degree d component of the quotient $S/(I, y)$ is the quotient of the degree d component of S/I by the degree d component of $(S/I)y \subset S/I$. Thus

$$H_{S/I}(d) \leq H_{S/(I,y)}(d) + H_{S/(I,y)}(d-1)$$

with equality for all d if and only if y is a nonzero divisor on S/I. Since the set of zero divisors on S/I is the union of the associated primes of I, the equality holds for all d if and only if y is not contained in any associated prime of I. In [Mac13, §66]—always assuming implicitly[10] that the coordinates $x_1, \dots, x_n$ are chosen generally with respect to I—observed that if the codimension of I is c, then the ideal $J := (I, x_{c+1}, \dots, x_n)$ will contain a power of the maximal ideal $(x_1, \dots, x_n)$, and generalizing the displayed relation above,

$$H_{S/I}(d) \leq H_{S/J}(d) + \binom{n-c}{1} H_{S/J}(d-1) + \binom{n-c+1}{2} H_{S/J}(d-2) \cdots .$$

[9]Though there is a fleeting reference to a "mixed" ideal in [Mac13, p. 43], Macaulay does not use the term "unmixed" until the exposition in [Mac16].

[10]The assumption that the coordinates are chosen generally is so automatic for Macaulay that it is almost never mentioned in his text. General position arguments can be quite deep and difficult to justify. For example, Brill and Noether's "proof" of the famous "Brill–Noether theorem" on the existence of linear series with prescribed degree and dimension depends on a general position assertion that was not verified until 100 years later. However, Macaulay's use of general position in this context is harmless and easy to make rigorous.

He then defined a homogeneous ideal to be *perfect* if the maximum value of $H_{S/I}(d)$ allowed by the expression above is attained for all d, and defined an arbitrary ideal to be perfect if its "equivalent H-ideal"—that is, its homogenization—is perfect.[11] It is interesting to note that this definition remains the most efficient known method for determining whether a homogeneous ideal is perfect by computation in a program such as Macaulay2 [GS93].

In the case of homogeneous ideals, Macaulay's definition is equivalent to the modern notion, since the equality holds if and only if $x_{c+1}, \dots, x_n$ is a regular sequence modulo I. However, Macaulay said that a not-necessarily-homogeneous ideal I is perfect if the "equivalent" homogeneous ideal is perfect—or in modern language, if the ideal representing the closure of the affine variety in projective space is perfect. This is considerably stronger than the modern notion, which would be defined today through localization, a technique that seems to have been first introduced in [Kru38]. Indeed, Macaulay pointed out that even ideals of the principal class (that is, ideals generated by regular sequences) need not be perfect in his sense if they are not homogeneous, though they would be considered perfect in the modern sense.

Example 2. Consider the affine quartic space curve C, given parametrically by

$$\mathbb{A}^1 \ni t \mapsto (t, t^3, t^4) \in \mathbb{A}^3$$

with ideal $I = (x_2 - x_1^3,\ x_1x_2 - x_3)$. Since the curve has codimension 2, this ideal is a complete intersection, and the localized ideal I_P is perfect for every prime ideal P; but the equivalent homogeneous ideal to I is

$$J = (x_0x_3 - x_1x_2,\ x_2^3 - x_1x_3^2,\ x_0x_2^2 - x_1^2x_3,\ x_1^3 - x_0^2x_2),$$

which is the ideal of the closure $\overline{C}$ of C in $\mathbb{P}^3$, the image of

$$\mathbb{P}^1 \ni (s,t) \mapsto (s^4,\ s^3t,\ st^3,\ t^4) \in \mathbb{P}^3.$$

In this case the linear forms x_0, x_3 are general enough, and the factor ring

$$S/(J, x_0, x_3) = k[x_1, x_2]/(x_1x_2, x_1^3, x_2^3)$$

has basis $1, x_1, x_2, x_1^2, x_2^2$, so its Hilbert function has values $1, 2, 2, 0, 0, \dots$. But $H_{S/J}(3) = 13 < 0 + 2 \times 2 + 3 \times 2 + 4 \times 1 = 14$, whereas, if J were perfect, then $1, 2, 2, 0$ would be the second difference function of $H_{S/J}$, and we would have

$$H_{S/J}(3) = (4 \cdot 1) + (3 \cdot 2) + (2 \cdot 2) + 0 = 14$$

instead. Thus, ideal I would not be perfect according to Macaulay's definition.

The homogeneous ideal J in this example played an interesting role in another context. One of the few letters from Macaulay that have been preserved is the letter to David Hilbert proposing the manuscript that became [Mac13] for consideration in the *Mathematische Annalen*.[12] In his letter Macaulay expressed his frustration in trying to prove that homogeneous prime ideals are perfect, and asked whether Hilbert could shed light on this question. He wrote "It would go some way towards proving that a prime ideal is perfect if it were known that (M, x_n) is prime when M is a prime H-ideal of rank $< n - 2$.[13] It would still have to be proved that a

[11]Perhaps he found this awkward: in the *Tract* Macaulay offered a different but equivalent definition in which homogenization figures only indirectly.

[12]Cited with permission from the Niedersachische Staats und Universtät Bibliothek Göttingen, Cod. Ms. D. Hilbert, 136. The full correspondence will appear in [EG].

[13]Macaulay used the term "rank" where we would say "height" or "codimension".

prime H-ideal of rank $n-2$ was perfect." By the time of publication, Macaulay knew why this had been difficult to prove: he gave the example of the projective curve $\overline{C}$ above to show that *not* all homogeneous prime ideals are perfect!

Lasker in his paper of 1905 does not mention the paper [BN94], but his proof that all the primary components of a regular sequence in $\mathbb{C}[x_1, \dots, x_n]$ have the same dimension starts from the same place: he begins by computing generators for the first syzygies of a regular sequence.

Macaulay, in his turn, defined a homogeneous ideal of dimension d in $\mathbb{C}[x_1, \dots, x_n]$ to be perfect if, in generic coordinates, x_{i+1} is a nonzero divisor modulo $I + (x_1, \dots x_i)$ for $i = 0, \dots, c-1$; again, effectively a statement on the syzygies.

Macaulay immediately connected this numerical condition to primary decomposition. To do this, he followed what would also be the simplest modern proof, noting that the numerical condition is equivalent to the condition that x_{c+1} is a nonzero divisor modulo I and further, x_{c+i} is a nonzero divisor modulo $(I, x_{c+1}, \dots, x_{c+i-1})$ for each $i = 2, \dots, n-c$. Given the general choice of coordinates, this condition holds if and only if the ideals

$$I, (I, x_{c+1}), \dots, (I, x_{c+1}, \dots, x_{n-1})$$

have no (embedded) component primary to $(x_1, \dots, x_n)$, what Macaulay called a "relevant simple spread". This idea, called "prime avoidance" in modern texts, is apparently self-evident for Macaulay: presumably he would have argued that the possible associated primes other than $(x_1, \dots, x_n)$ intersect the vector space of linear forms in proper subspaces, so their union cannot contain the general linear form.

Already in [Mac13, §57], Macaulay had stated that homogeneous ideals of the principal class are perfect (noting that inhomogeneous ideals of the principal class need not be perfect in his sense). After he gave the definition in [Mac13, §66], his first goal was to show that there are many other further examples. It is obvious from his definition that any zero-dimensional ideal

$$I = (f_1, \dots, f_t) \subset k[x_1, \dots, x_n]$$

is perfect; and Macaulay pointed out that the ideal $J \subset k[x_1, \dots, x_n, y_1, \dots y_m]$ obtained from I by substituting $x_n + y_1 + \cdots + y_m$ for x_n is a perfect ideal of codimension n in $n+m$ variables. Macaulay next connected perfection with unmixedness although he did not introduce the term, showing in [Mac13, §68–69] that (always assuming the variables x_i are chosen generally) not only is a perfect ideal $I \subset k[x_1, \dots, x_n]$ of codimension c unmixed, but also the ideals

$$(I, x_1), (I, x_1, x_2) \cdots (I, x_1, \dots, x_{n-c})$$

are perfect, and thus unmixed. In [Mac13, §69] he proved that in fact a homogeneous ideal I is perfect if and only if the penultimate ideal in this series, $(I, x_1, \dots, x_{n-c-1})$, is unmixed. (This would not be true if the coordinates were not generally chosen. For example, consider the ideal $I = (x^2, xy) = (x) \cap (x^2, y) \subset \mathbb{C}[x, y, z]$, which defines a line with an embedded point in the projective plane. Since it is mixed, this ideal is not perfect, and modulo z or any general linear form it remains mixed. But $(I, x) = (x)$, which is unmixed.)

5.1.1. *The importance of perfection.* Perfect ideals may be thought of as the ideals that satisfy the ultimate generalization of Noether's Fundamental Theorem: if I is a perfect homogeneous ideal defining a projective variety X of dimension d and $F_1, \ldots, F_d$ are forms that vanish simultaneously on X only in a finite set Γ of points, then membership in the ideal $I + (F_1, \ldots, F_d)$ can be tested by Noether's criterion applied at the points of Γ—the original result is the case in three variables with $d = 1$.

Also, the perfection of I can be thought of as taking place entirely in the ring R/I. Whereas the notion of an ideal of the principal class—that is, one generated by a regular sequence—depends on building up the ideal from 0 in the ring R, the definition of a perfect ideal depends only on the behavior of elements of R as elements of R/I; it is actually a property of R/I that is independent of R in the sense that if $R/I \cong R'/I'$, then I and I' are either both perfect or both not perfect.

In the modern period the property of perfection has far-reaching importance in algebraic geometry because of its relation to Serre's fundamental idea of flatness. For example, I is perfect of (affine) dimension d if and only if, whenever $R/(I + (F_1, \ldots, F_d))$ is zero dimensional, R/I is a finitely generated free (equivalently, flat) module over the ring $\mathbb{C}[F_1, \ldots, F_d]$.

Perfection has also played an important role in combinatorics starting with Richard Stanley's proof of the Upper Bound conjecture for spheres [Sta75] via the formula for the Hilbert function above.

We would say that a homogeneous ideal I in a polynomial ring R is perfect in Macaulay's sense if and only if any partial system of parameters in each localization of R' is unmixed; this is the definition of a "Cohen–Macaulay" ring. The name of Irvin Sol Cohen enters because he gave an analysis of the structure of complete local rings and used it to prove the unmixedness theorem for ideals of the principal class (which he mistakenly attributed to Macaulay rather than Lasker) in regular local rings. For this he used a case-by-case analysis using the theory of complete local rings developed in his 1942 Johns Hopkins thesis under Oscar Zariski [Coh42] and [Coh46]. Irving Kaplansky felt that one should speak simply of Macaulay rings, and that is the terminology used in [Kap74].

5.2. **Linkage: a general theory of residuation.** Macaulay described the example of the projective curve $\overline{C}$ above by "residuation", in the special case that we would call "linkage", or "liaison": he described J as "the prime ideal of order 4 whose variety is the curve in space of three dimensions in which a quadric and cubic surface drawn through two non-coplanar lines intersect again" (the "non-coplanar lines" can be taken to two lines from the same ruling of a nonsingular quadric, so this describes J as the ideal of a divisor of type $(3, 1)$ on the quadric.)

The idea of describing curves in 3-space in this way had already been extensively used by Georges-Henri Halphen. In [Hal82] he considered a range of cases in which the intersection of two surfaces in $\mathbb{P}^3$ is the union of two reduced curves, and deduced properties of one of the curves from properties of the other one. This is similar to the use of residuation for the transverse intersection of two curves in the plane. Macaulay had been concerned in his early work exactly in the case when the intersection of two curves is not transverse, and in 1913 he was able to do the same thing in a far more general setting—one that has been studied extensively in modern commutative algebra.

The fundamental idea is to replace set-theoretic subtraction with residuation, defined by an ideal quotient: if $M \subset M'$ are ideals of a ring R, then the quotient $M : M'$ is defined to be $\{f \in R \mid fM' \subset M\}$. In the special case when M' is unmixed and M is one of its primary components, it is easy to see that $M : M'$ is the intersection of the other primary components: geometrically, the closure of $V(M') \setminus V(M)$ is $V(M : M')$ (here $V(M)$ denotes the "spread" of M in Macaulay's sense, that is, the locus in $\mathbb{C}^n$ where the polynomials in M all vanish.)

The ideal quotient is generally the best approximation to a subtraction operation, but, as in the case of intersections of curves in the plane, it often does not have the good properties of subtraction. The most important questions are whether $M : (M : M') = M'$ and whether the invariants of $(M : M')$ can be deduced in a simple way from those of M and M'. Macaulay proved (always in the context of polynomial ideals) that these properties are satisfied when M is an ideal of the principal class and M' is unmixed of the same codimension as M; in this case, $M : M'$ is said today to be "linked" to M'. The properties of this construction were proven independently in the setting of Gorenstein local rings in [PS74].

The last sections of [Mac13] are all concerned with these questions. He first proved a general elementary result:

Theorem 6 ([Mac13, §53])**.** *If $M : M' = M''$ and $M : M'' = M'''$, then $M : M''' = M''$, i.e., M'', M''' are doubly residual with respect to M.*

He generalized his result that a cluster defined by the intersection of two plane curves is a 1-set point, showing that it is a principal system:

Theorem 7 ([Mac13, §62])**.** *A primary ideal of the principal N-class has a single principal modular equation, i.e., all its modular equations consist of a single equation and its derivates.*

Sketch of Macaulay's proof. To say that a primary ideal I is "of the principal N-class" means that I contains a power $\mathfrak{m}^\gamma$ of the homogeneous maximal ideal $\mathfrak{m} := (x_1, \ldots, x_n)$ and that the ideal I is the primary component of an ideal generated by $n = \dim R$ general elements $F_1, \ldots, F_n \in I$ having the same $\mathfrak{m}$-primary component as I; or, in modern terms, that the localization of I at $\mathfrak{m}$ is a complete intersection. To prove the result, Macaulay observed that one may add any terms of degree $\geq \gamma$ to the F_i without changing the situation; thus $I' := (F_1, \ldots, F_n)$ may be assumed to have no common zeros at infinity, that is, their highest degree terms may be assumed to form a regular sequence as well, say of degrees $\ell_1, \ldots, \ell_n > \gamma$. This has the effect that every monomial of degree $> \sum \ell_i - n$ is contained in I, while the forms G of degree equal to $\sum \ell_i - n$ form a hyperplane in the space of all forms of that degree, defined by a single linear equation $L = 0$ on the coefficients of G.

Macaulay next proved by induction on the number of variables that if $\mathfrak{m}F \in I'$ and $\deg F < \sum_i \ell_i - n$, then $F \in I'$; this is equivalent to saying that the *socle* $(I : \mathfrak{m})/I\ R/I$ is in degree $\geq \sum(\ell_i - 1)$, the degree of the socle of $R/(x_1^{\ell_1}, \ldots, x_n^{\ell_n})$. Together, these statements imply the theorem, because they show that the linear conditions on the coefficients of a form F of degree e to lie in I—the modular equations of degree e—are all derived from the single equation and this that implies that the coefficients of $G = Fm$ satisfy the condition $L = 0$ for every monomial m of degree $\sum \ell_i - n - (d - e)$. ☐

At the end of [Mac13, §62] Macaulay mentioned the fact that, in two variables, the converse holds: every simple N-ideal with a principal modular equation (that is, any $\mathfrak{m}$-primary Gorenstein ideal) is generated by two elements, so it is a complete intersection. The generalization of this to all Gorenstein ideals of codimension 2 in regular local rings is attributed to Serre in [Bas63] (Bass seems to have been unaware of Macaulay's work in this area). But Macaulay stated that in three or more variables this is no longer true. In [Mac16] he gave the example in three variables of the ideal $(x^2, y^2, z^2 - xy, xz, yz)$ to demonstrate this fact.[14]

In [Mac13, §63] Macaulay turned to the application that originally motivated his work on ideals with a principal modular equation in [Mac99].

Theorem 8 ([Mac13, §63])**.** *If M_μ is a simple K-N-ideal (that is, an ideal of finite colength μ) whose modular equations consist of a single principal equation and its derivates, and M_{μ_1} is any ideal containing M_μ, and $M_\mu : M_{\mu_1} = M_{\mu_2}$, then*

$$M_\mu : M_{\mu_2} = M_{\mu_1} \text{ and } \mu_1 + \mu_2 = \mu.$$

In the final pages of [Mac13] Macaulay turned to what we would consider the theory of homogeneous ideals that define Gorenstein factor rings of the polynomial ring. Though he didn't give them a name in this paper, he called them *superperfect* rings in [Mac34]; we will use the modern term “Gorenstein”. Macaulay's definition is that they are perfect ideals M of codimension r such that “$(M, x_{r+1}, \ldots, x_n)$ has a single principal modular equation”.

For example, in [Mac13, §64], he highlighted the symmetry of the Hilbert function of a finite-dimensional graded Gorenstein ring, writing that, for any primary homogeneous ideal with a single principal modular equation,

> We have the rather remarkable result that the numbers of independent derivates of successive degrees of any homogeneous N-equation, or the Hilbert numbers of any simple H N-module with a single principal modular equation ... are the same as when reversed in order....

In [Mac13, §71], Macaulay proved a result that is at the foundation of the theory of linkage: if $M \subset M' \subset R$ are perfect homogeneous ideals of the same codimension in the polynomial ring R, and M is Gorenstein, then $M'' := M : M'$ is again perfect, and $M' = M : M''$.

Theorem 9 ([Mac13, §71])**.** *If M is a perfect H-ideal of codimension r such that $(M, x_{r+1}, \ldots, x_n)$ has a single principal modular equation, and if M' is a perfect H-ideal of codimension r containing M, then $M : M'$ is a perfect ideal M''.*

In the final section, [Mac13, §72] Macaulay derived a central formula in the theory of linkage: under the hypotheses of [Mac13, §71], he computed the Hilbert function of M'' in terms of the Hilbert functions of M and M'.

6. The *Tract*: Absorbing primary decomposition and introducing inverse systems

Macaulay's best known work today is undoubtedly *The Algebraic Theory of Modular Systems*, published in the series Cambridge Tracts in Mathematics and Mathematical Physics [Mac16], which we will refer to simply as the *Tract*. It was republished by Cambridge University Press, with a masterful introduction by

[14]In the early 1970s David Buchsbaum and the first author, quite unaware of Macaulay's work in this direction, studied and discovered the structure of Gorenstein ideals in codimension 3, thus extending the structure result of Macaulay and Bass [BE77].

Paul Roberts, in 1994. Much of the *Tract* is taken up with an exposition of the material that was already presented in [Mac13], but there are several new results and emphases, and we will concentrate on these. Macaulay always focused on concrete computations, and these were usually based on resultants, the theory of which occupies the first sections. Then Macaulay gave an exposition of Lasker's theory of primary decomposition. Whereas in [Mac13] he spoke of ideals with no embedded primary components, he now made this condition central by introducing the term *unmixed*. In [Mac16, §41] he wrote

> An *unmixed module* is usually understood to be one whose *isolated* irreducible spreads are all of the same dimensions; but it is clear from the above [Lasker's theory] that this cannot be regarded as a satisfactory view. It should be defined as follows:
> **Definition.** An *unmixed module* is one whose relevant spreads [associated primes] both isolated [minimal] and imbedded, are all of the same dimensions;[15] and a *mixed module* is one having at least two relevant spreads of different dimensions.

In these sections Macaulay gave many examples showing how resultants and resolvents can be used to partially describe the primary decomposition of an ideal, but do not give the full picture. This must have been a concern; Emmy Noether famously asked whether the decomposition could be effected at all algorithmically ("in endlich vielen Schritten"), a question that her student Grete Hermann resolved positively in characteristic 0. (There are now many such algorithms, in all characteristics and even over the integers; see for example [IPS15] and the references there.)

6.1. **The number of generators of a prime ideal.** It is an easy consequence of Hilbert's Nullstellensatz that any prime ideal in $\mathbb{C}[x, y]$ is generated by just two elements; and Kronecker in [Kro82, p. 85] had given an argument showing that any prime ideal in $\mathbb{C}[x_1, \ldots, x_n]$ is generated *up to radical* by $n+1$ elements. Macaulay filled in this picture by defining, for every $\ell \geq 3$, an ideal $P_\ell \in \mathbb{C}[x, y, z]$ that requires at least ℓ generators; in fact, Macaulay's argument in § 34 shows that even the localization of P_ℓ at a suitable maximal ideal requires at least ℓ generators. The idea is that the variety C_ℓ of P_ℓ should be a curve in 3-space, singular at the origin and having tangent cone there consisting of $\binom{\ell}{2}$ general lines. Since the ideal of $\binom{\ell}{2}$ general points in the plane requires ℓ generators, this implies that P_ℓ requires at least ℓ generators, even locally at the origin. Unfortunately, the claim that P_ℓ is prime depends on properties of "general" choices that are not demonstrated, so for a long time these examples were not regarded as definitive. However, Shreeram Abhyankar successfully reworked the examples—he asserted in [Abh73] that to do so he had to "rediscover" the proof. The fact that the tangent cone of C_ℓ is reducible implies that $P_\ell\mathbb{C}[[x, y, z]]$ is no longer prime—that is, C_ℓ is *analytically reducible*—leaving the open the question of whether such examples were possible in the power series ring. Finally, in [Moh74], Tzuong-Tsieng Moh put the question to rest with a new family of examples of analytically irreducible affine curves.

6.2. **Powers of regular sequences and the perfection of determinantal ideals.** Lasker had proved that ideals of the principal class are unmixed, and

[15] It follows that there are no imbedded primes at all!

Macaulay observed that this implies that they are perfect: if $I = (a_1, \dots, a_r)$ is of codimension r and $J = I + (b_1, \dots, b_s)$ is of codimension $r+s$ (assuming that the a_i and b_j are homogeneous), then J is also of the principal class, and thus unmixed. Macaulay generalized this example of a perfect ideal in two important and related ways:

(1) He proved in [Mac16, §50] that if I is an ideal of the principal class, then all powers I^γ are unmixed; and in the case when I is radical, he identified I^γ as the set of functions that vanish to order at least γ at every point of the variety determined by I. (He observed in a footnote in [Mac16, §52] that, for any prime ideal P, the ideal of functions vanishing to order at least γ on the points where all functions in P vanish is primary to P—a special case of Zariski's "Main Lemma on Holomorphic Functions" [Zar49].)

(2) He proved in [Mac16, §53] that the ideal of $r \times r$ minors of an $r \times s$ matrix (with $r \leq s$) is unmixed, (and even perfect [Mac16, §93]), if the ideal has codimension $s - r + 1$, at the same time computing the first syzygies of this ideal (the first step in the Eagon–Northcott complex), by induction on the size of the matrix (in case $r = 1$ this is an ideal of the principal class).

Macaulay's [Mac16, §52] begins with an interesting assertion about when the powers of a prime P are unmixed. He considered the case of a prime ideal M of codimension r and a linked prime $M' = (f_1, \dots, f_r) : M$ also of codimension r, allowing the f_i and M' to vary. He asserted that

> if M' does not cut M in a fixed spread, then the powers M^γ are unmixed. In the contrary case some power M^γ is mixed, and will have the fixed spread in which M' cuts M as an embedded component.

From a modern point of view, the primes containing all possible ideals $M' + M$ are the primes Q such that the localization M_Q is not a complete intersection, that is, primes in the noncomplete intersection locus; and this, with assertion (1) above, shows that any embedded prime of a power of M must indeed contain all $M + M'$. It follows from a modern theorem of [CN76] that the converse is true for primes Q of codimension $r+1$ containing $M + M'$; but it is false even for minimal primes of the noncomplete intersection locus that have codimension $r+2$. For example, if M is the ideal of the twisted cubic curve in $\mathbb{P}^3$, then the ideal of all the four variables is the unique prime such that M_Q is not a complete intersection; but M^γ is unmixed for all γ [Con98].

We can only speculate on the source of Macaulay's error. He made many computations and was perhaps too quick to assume that the examples he was able to compute represented general behavior. Perhaps he relied too much on the examples of affine curves; for these examples, he would have seen precisely the behavior he described, since the only prime ideals that could be associated to the powers would have been the ideals of points (covered by the theorem of Cowsik and Nori). Since Macaulay worked in isolation in England, he could not test his assertions with others. At the 1904 ICM, one might guess that Brill or Max Noether were important in telling him of Lasker's work and helping him understand his error. But back in England he had no such support, and there is no evidence that he was in contact with the mathematicians in his field on the continent at this time; such contact would have been difficult, in any case, because of the First World War.

Macaulay proposed two examples of powers of primes that are mixed, one based on a geometric idea and the other (it seems) on algebraic computation. Macaulay asserted that if a curve in 3-space has a spatial triple point (that is, a triple point where every line meets the curve at least twice), then the powers of the ideal of the curve are mixed, and thus (he said) the square of the ideal of the curve contains the equation of a surface that is double at every point of the curve. He did not provide a proof, but rather a clever example: Let C be the curve defined by the ideal I of 2×2 minors of the matrix

$$M := \begin{pmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1^2 \end{pmatrix},$$

and let $f(x_1, x_2, x_3) = (x_2x_3 - x_1^3)^2 - (x_2^2 - x_1x_3)(x_3^2 - x_1^2x_2)$. Note that this polynomial lies in the square of the ideal of minors (indeed, the polynomials

$$(x_2x_3 - x_1^3),\ (x_2^2 - x_1x_3),\ (x_3^2 - x_1^2x_2)$$

are the three 2×2 minors of M). Thus f vanishes to order 2 everywhere along the curve. Note that f is divisible by x_1, which does not vanish on the curve except at the origin; thus $g = f/x_1$ vanishes to order 2 at every point of the curve. But g is not in the square of the ideal of minors, since g has only a triple point at the origin, whereas every element of I^2 has order at least 4 at the origin.

The curve C is a rational curve with parametrization

$$\mathbb{A}^1 \to \mathbb{A}^3 : t \mapsto (t^3, t^4, t^5),$$

and is thus reduced and irreducible; but if we allow reducible curves, a much simpler example is possible. Consider the union of the three coordinate axes, whose ideal is

$$I = (x_1, x_2) \cap (x_2, x_3) \cap (x_1, x_3) = (x_1x_2, x_1x_3, x_2x_3).$$

It is obvious that the cubic polynomial $x_1x_2x_3$ vanishes to order 2 on all three axes, but cannot be in I^2 since the generators of I^2 have degree 4. Presumably, Macaulay would not have considered such an example a genuine curve.

The second example [Mac16, §52, Example ii] is the ideal of 3×3 minors of a linear 3×4 matrix in four variables; Macaulay asserted, roughly indicating a computation, that the unmixed part of the cube of this ideal contains a form of degree 8. It is now known ([Tru79], and in a wider context in [Hun81]) that every power of the ideal of 3×3 minors of a generic matrix (in twelve variables) (and indeed of the ideal of maximal minors of a generic matrix of any size) is unmixed. Thus Macaulay's example shows that (unlike in the case of a regular sequence), the specialization of a perfect ideal with unmixed powers need not have unmixed powers.

6.3. **Power series and polynomials.** In [Mac16, §55] Macaulay commented on a generalization of what he calls (Max) Noether's "fundamental theorem in algebraic functions" (different from, though related to, what we have referred to above as Noether's Fundamental Theorem). In [Mac16, §56] Macaulay gave a more general version as follows: if a form F can be written as $F = \sum_{i=1}^{k} P_iF_i$, where the F_i are forms and the P_i are power series, then there is a polynomial ϕ such that ϕF is in the ideal generated by the F_i in the ring of polynomials. We would now describe this phenomenon by saying that the power series ring $\mathbb{C}[[x_1, \dots, x_n]]$ is faithfully flat over the localization $\mathbb{C}[x_1, \dots x_n]_{(x_1,\dots x_n)}$. According to Macaulay, this result was proven by Noether for the case $k = n = 2$, by König for $k = n$ arbitrary, and Lasker [Las05, Satz XXVII, p. 95] for arbitrary k. Macaulay called this the Lasker–Noether theorem but finds that Lasker's proof "seems to be faulty" in one point having to do with general position, which he claimed to correct. (He later felt that his correction, too, was flawed, and referred to a proof by Wolfgang Krull; see Section 8.)

6.4. **Inverse systems and residuation.** Already in his early work, Macaulay studied the homogeneous linear equations that defined the ideals in the power series ring of a "cluster" of points—the "Noether equations" of the cluster— and the linear equations defining set of forms of a given degree d in a given homogeneous ideal—the "modular equations" of the ideal. Chapter IV of the *Tract* formalizes this study through the theory of *inverse systems.* Macaulay emphasized that this was a new idea, and he drew from it several consequences which have been later rediscovered and extended. The idea of inverse systems has two components:

(1) A formal power series
$$E \in P := \mathbb{C}[[x_1^{-1}, \dots, x_n^{-1}]]$$
in the *inverse variables* x_i^{-1} acts as a linear functional on the polynomial ring
$$R := \mathbb{C}[x_1, \dots, x_n]$$
by the rule that if $F \in R$, then $E(F)$ is the constant coefficient of the product EF. For example
$$x_i^{-2}(x_i^2) = 1 \quad \text{while } x_i^{-2}(x_i) = x_i^{-2}(x_i^3) = 0.$$
The *inverse system associated to an ideal* $I \subset \mathbb{C}[x_1, \dots, x_n]$ is the set of elements of P whose associated functionals vanish identically on I. Macaulay also calls this set the set of "modular equations" of I. Macaulay pointed out that I is determined by its modular equations.

(2) Elements of the polynomial ring R act as operators on P by ordinary multiplication, understanding that positive powers in the product are set to 0. For example,
$$x_i(x_i^{-2}) = x_i^{-1} \quad \text{while } x_i^3(x_i^{-2}) = 0.$$
Thus if $F \in R, E \in P$, then $F(E)$ is the sum of the terms of nonpositive degree in the product FE.

The key remark that makes these constructions useful is that if $p \in N \subset P$ is an element of the inverse system of an ideal I and $r \in R$, then rp, which Macaulay called "the r-derivate of p", is again in N: that is, *N is an R-submodule of P.* Though vector spaces with operators were already current in work on group representations, this is surely a very early, if not the first, example of an interesting $k[x_1, \dots, x_n]$-module other than an ideal. In his early work Macaulay spoke more vaguely of deriving one Noether equation of a cluster of points from another. The action of R on $P = \mathrm{Hom}_{\mathbb{C}}(R, \mathbb{C})$ codified and extended this idea.

Example 3. Consider the ideal $(g, h)S = (y^2 - x, x^2)S$ in $S = \mathbb{C}[x, y]$. The equations that imply that F in x, y should belong to the ideal $(g, h)S$ can be written:
$$1(F) = 0$$
$$(y^{-1})(F) = 0$$
$$(y^{-2} + x^{-1})(F) = 0$$
$$(y^{-3} + x^{-1}y^{-1})(F) = 0,$$

where we need no further equations because $(x,y)^4 \subset (g,h)$. These four equations can all be derived from the last one, represented by the inverse polynomial $E := y^{-3} + x^{-1}y^{-1}$ through the action of R on P:

$$y^3(E) = 1$$
$$x(E) = y^{-1}$$
$$y(E) = y^{-2} + x^{-1}.$$

Because all the equations are derived from just one, Macaulay called this a "one-set point", and more generally he defined a t-set point to be one where all the Noether equations are derived from t but not fewer.

Macaulay presented several examples and remarks about the inverse functions. If I is homogeneous, then its inverse functions are generated (as a vector space) by homogeneous functions. If I is zero dimensional, centered at the origin, the inverse functions may be taken to be polynomials, but in general actual power series are required. As a simple example, consider the ideal $(x-1) \subset \mathbb{C}[x]$. The inverse function $p := 1+x^{-1}$ vanishes on $x-1$ since $x^{-1}(x-1) = 1-0$ and $1(x-1) = 0-1$. But $p(x(x-1)) = p(x^2-x) = -1 \neq 0$. As Macaulay explained, any inverse function of degree $-t$ that vanishes on elements of I up to degree t can be continued to a power series that vanishes on all of I. In this case $p' = 1 + x^{-1} + x^{-2}$ vanishes on $x^2 - x$, and more generally $p'' = \sum_{i\geq 0} x^{-i}$ vanishes on all $x^m(x-1)$ and thus on the whole ideal I. An ideal I of dimension 0 is called a "principal system" if its inverse system consists of the derivates of a single element.

More generally, for an unmixed ideal of dimension r, Macaulay (always assuming general position for the variables) defined $I^{(r)}$ to be $I\mathbb{C}(x_{r+1},\ldots,x_n)[x_1,\ldots,x_r]$ and noted that the inverse system of this ideal (regarding elements of $\mathbb{C}(x_{r+1},\ldots,x_n)$ as constants) determines I as well. He allowed inverse functions in

$$\mathbb{C}(x_{r+1},\ldots,x_n)[[x_1^{-1},\ldots,x_r^{-1}]]$$

and showed how to derive an inverse system for an unmixed ideal I of codimension r from an inverse system for $I^{(r)}$, so that the inverse system in this new sense also determines the ideal.

By describing the intersection of primary components of each codimension in this way, Macaulay could give a finite description, through inverse systems, of an arbitrary ideal. This idea has been taken up again in a modern context in [CRHS21].

Macaulay said that an ideal I is a principal system if its inverse system consists of the derivates of a single element. Macaulay showed that ideals of the principal class are principal systems (see [Mac16, §72]), but not conversely. For the falsity of the converse he gave in [Mac16, §71], the example of the principal system $(x_1^2 - x_2^2, x_1^2 - x_3^2, x_2x_3, x_3x_1, x_1x_2)$.

Moreover, he showed that all the primary components of a principal system are principal systems, and remarked in a footnote in [Mac16, §76] that the converse does not hold. For example, a single (reduced) point is of the principal class, but the union of three general points in the plane is not a principal system. However, he noted in [Mac16, §§61, 62] that any principal system has the form $I = (f_1,\ldots,f_k) : g$, where $(f_1,\ldots,f_k)$ is of the principal class. In modern parlance, such an ideal is quasi-Gorenstein, and is Gorenstein if and only if it is also perfect. Macaulay proved that linkage preserves perfection, so I is Gorenstein if and only if the linked ideal

$$(f_1, \dots, f_c, g) = (f_1, \dots, f_c) : I$$

is perfect.

Macaulay proved several of the results on residuation from [Mac13] and added more. He used his results to generalize the *Restsatz* of Brill and Noether that was important in his early work:

Theorem 10 (Theorem of Residuation (*Tract*, §87)). *Let M be an unmixed homogeneous ideal of codimension r, and let $K = (F_2, \dots, F_r) \subset \mathbb{C}[x_1, \dots, x_n]$ be a homogeneous ideal with $r-1$ generators contained in M. Suppose that $F_1, F_1' \in M$ are forms of the same degree d, such that $J = (F_1) + K$ and $J' = (F_1') + K$ have codimension r. Set $M' = J : M$ and $M_1' = J' : M$. Suppose that for some form $F' \in M'$ of degree d the ideal $J_1 = (F') + K$ also has codimension r. Then there is a form $F \in M_1'$, again of degree d, such that $J_1' := (F') + K$ has codimension r and*

$$J_1' : M_1 = M_1.$$

To understand the significance of this confusing statement in modern terms, let X be the scheme defined by the ideal K. The hypothesis implies that X is a complete intersection in $\mathbb{P}^{n-1}$, and the conclusion is a statement about the divisors on X that can be defined by forms of degree $d := \deg F_1$. The forms F_1, F_1' define Cartier divisors D_1, D_1' containing a not-necessarily Cartier divisor E defined by M on X, while M' and M_1' define the divisors $E' := D_1 - E$ and $E_1' := D_1' - E$, respectively. The form F' defines a Cartier divisor D' containing the divisor E_1'. Since F_1, F_1', F' are forms of the same degree d, the divisors D_1, D_1', D' are linearly equivalent on X. The assertion is that there exists a form $F \in M_1'$, defining a divisor D such that $D' - E_1' = D - E'$. In the free group generated by the divisors on X this equation is

$$D' - (D_1' - E) = D - (D_1 - E); \quad \text{that is, } D = D' - D_1' + D_1.$$

Note that the expression $D' - D_1' + D_1$ contains only Cartier divisors of the same degree, so it is automatically represented by a rational function on X of degree d, in this case $F'F_1/F_1'$. The strength of Macaulay's theorem is that this rational function is equivalent modulo K to a form of degree d. Macaulay's residuation theorem is thus the statement that the linear system cut out by hypersurfaces of degree d on a complete intersection is a complete linear series, a consequence of the statement that a complete intersection is perfect.

Example 4. To see that this is nontrivial, consider the smooth rational quartic curve in $\mathbb{P}^3$ parametrized by

$$\mathbb{P}^1 \ni (s, t) \mapsto (x_0, x_1, x_2, x_3) = (s^4, s^3t, st^3, t^4).$$

On this curve the rational function x_1^2/x_0 which is linearly equivalent to a hyperplane section, is not represented by a linear form, since it pulls back to the form s^2t^2 on $\mathbb{P}^1$.

Macaulay's proof of Theorem 10 [Mac16, §86] is based on another result of modern significance [Mac16, §86]: Again, suppose that M is an unmixed ideal of codimension r, and choose r elements $F_1, \dots, F_r \subset M$ that generate an ideal also of codimension r—a complete intersection. Let $M' = (F_1, \dots, F_r) : M$ be the residual ideal. Macaulay proved the following.

Theorem 11. *The number of generators required by M' in addition to $F_1, \dots, F_r$—that is, the number of generators required by $M'/(F_1, \dots F_r)$—is independent of the choice of $F_1, \dots, F_r$.*

Denoting the ambient polynomial ring by $S = \mathbb{C}[x_1, \dots, x_n]$ in modern terms, the ring $S/(F_1, \dots, F_r)$ is Gorenstein, so the ideal

$$\mathrm{Hom}(S/M, S/(F_1, \dots, F_r)) = M'/(F_1, \dots, F_r)$$

is (up to a shift in grading) the canonical ideal of S/M, which is thus independent, up to shift, of the choice of $F_1, \dots, F_r$. This surprising fact is a pillar of the modern theory of linkage, which is a fundamental tool for the study of space curves, given power through its homological characterization in the work of Hartshorne and Rao [PR78]. It has also been generalized to residual interections of higher codimension in the works of Artin and Nagata, of Huneke, of Ulrich [Ulr94], and others.

6.5. **Perfect Ideals.** Perfect ideals were defined in [Mac13], first for homogeneous ideals and then for all ideals by considering the associated homogeneous ideal in an additional variable. In the *Tract* Macaulay's definition applies to all ideals in a polynomial ring, and uses the association of an ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ having codimension r and the ideal $I^{(r)} = I\mathbb{C}(x_{r+1}, \dots, x_n)[x_1, \dots, x_r]$, which has dimension 0 (note that the x_i are assumed to be general). However, the definition he gave is equivalent to the modern definition only in the case of homogeneous ideals.

An H-basis of the ideal I is a subset of elements $f_1, \dots, f_k \in I$ such that the highest degree term of any element $g \in I$ is a linear combination of the leading forms (the sum of the highest degree terms) of the f_i. If $f_1, \dots, f_k$ is an H-basis of I, then a maximal set of monomials independent modulo the leading forms of I is a vector space basis of S/I.

Definition 1. *The ideal I is perfect if there is an H-basis $f_1, \dots, f_k$ of I such that*

1. $(f_1, \dots, f_k)$ *is also an H-basis of $I^{(r)}$; and*
2. *the degree of each f_i is the same as the degree of f_i in the variables $x_1, \dots, x_r$ alone; that is, among the top degree terms of f_i there is a term involving only $x_1, \dots, x_r$.*

The essential point is that the monomials in the variables $x_1, \dots, x_r$ that form a vector space basis of $\mathbb{C}(x_{r+1}, \dots, x_n)[x_1, \dots, x_r]/I^{(r)}$ will then generate S/I as a module over $\mathbb{C}[x_{r+1}, \dots, x_n]$, showing that S/I is a free module over $\mathbb{C}[x_{r+1}, \dots, x_n]$, and thus that $x_{r+1}, \dots, x_n$ is a regular sequence modulo I. To see that this definition is equivalent to the one from [Mac13] when they both apply, it suffices, by induction, to show that x_1 is a nonzero divisor modulo I and that the given H-basis will retain properties (1) and (2) modulo x_1, so that $I + (x_1)$ is again a perfect ideal. Macaulay does this using properties of what he called "dialytic arrays" that he had developed over a number of pages, but both properties can also be checked directly. Conversely, if I is homogeneous and satisfies the definition of perfection from [Mac13], so that $x_{r+1}, \dots, x_n$ is a regular sequence modulo I, then S/I is a free $\mathbb{C}[x_{r+1}, \dots, x_n]$-module and any vector space basis of $\mathbb{C}(x_{r+1}, \dots, x_n)[x_1, \dots, x_r]/I^{(r)}$ is a basis of the free module. This is enough to show that the leading forms of I generate I^*. Note that the equivalence above involves a special case of the Auslander–Buchsbaum formula connecting projective dimension with depth. A final assertion in the *Tract* that is worth recording was rediscovered, in a more general form by Gaeta [Gae52] and systematized by Peskine and Szpiro [PS74]. In the short [Mac16, §92], the last substantive paragraph of the *Tract*, Macaulay reiterated a result from [Mac13]:

Theorem 12. *Suppose that $M \subset M'$ are perfect modules of codimension r. If $M_{x_{r+1}=\cdots=x_n=0}$ is a principal system, then the residual $M : M'$ is again perfect.*

7. 1927 and 1930: From the shape of a cluster to the characterization of Hilbert functions

Macaulay's next paper can be read as the completion of a journey that starts from his study of *clusters* of points defined by the intersection of two or more plane curves and passes through the extensive discussion of the modular equations and inverse systems associated to ideals in the *Tract.* In [Mac27] he completely described all possible Hilbert functions of polynomial ideals[16] and introduced a special family of ideals that has come to play an important role in the theory. Along the way, he noted that if I is a homogeneous ideal, then what is now called the "degree-lexicographic initial ideal" of I has the same Hilbert function as I, now a central point in the theory of Gröbner bases. Macaulay actually treated two cases in parallel: for a homogeneous ideal I he considered the sequence $D_0, D_1, \ldots,$ where D_ℓ is the dimension of the vector space I_ℓ of forms of degree ℓ in I; while for an inhomogeneous (he said "nonhomogeneous") ideal I, he set D_i equal to the dimension of the vector space of polynomials in I of degree $\leq \ell$. For simplicity we will treat just the case of homogeneous ideals in $S := \mathbb{C}[x_1, \ldots, x_n]$, the other case being quite similar.

In [Mac27] Macaulay said exactly what sequences of numbers D_ℓ can arise. The "shape" of a cluster defined by an ideal I could reasonably be considered to be the Hilbert function of S/I. If $\dim I_\ell = D_\ell$, then $\dim(S/I)_\ell = \dim S_\ell - D_\ell$, and Macaulay also characterized the possible Hilbert functions of rings of the form S/I in this way. Macaulay's characterization of the sequences of numbers D_ℓ has several remarkable aspects: First, given D_ℓ the possible values of $D_{\ell+1}$ are independent of the numbers $D_{\ell'}$ for $\ell' < \ell$. Thus, given D_ℓ, there will be a smallest possible $D_{\ell+1}$, say $Q(\ell, D_\ell)$, and every value from $Q(\ell, D_\ell)$ up to the dimension of S_ℓ will be possible, since one can simply add generators of degree $\ell + 1$. Macaulay gave a formula for Q, but he also identified a specific family of ideals in which all possible sequences are realized. They are now called "lex segment" ideals, and they are really the stars of the show.

Definition 2. *The monomial $x_1^{p_1}x_2^{p_2}\cdots$ is greater than the monomial $x_1^{q_1}x_2^{q_2}\cdots$ in the* lexicographic order *on monomials of degree d if, for the smallest i for which $p_i \neq q_i$, we have $p_i > q_i$.*

For example, in three variables, the ordering of the quadratic monomials, from greatest to smallest, is

$$x_1^2 > x_1x_2 > x_1x_3 > x_2^2 > x_2x_3 > x_3^2.$$

An *initial sequence* of monomials of degree d, from largest to smallest, thus starts with

$$x_1^d,\ x_1^{d-1}x_2,\ x_1^{d-1}x_2,\ \ldots.$$

Macaulay proved three fundamental results.

[16]In this paper, Macaulay switched from using the old term "modular system" to the almost modern term "polynomial ideal".

Theorem 13. *If $m_1, \dots, m_D$ is the initial sequence of the monomials of degree d in the lexicographic order, then the product of the vector space $\langle m_1, \dots, m_D \rangle$ and the vector space of linear forms $\langle x_1, \dots, x_n \rangle$ is spanned by an initial sequence of monomials of degree $d+1$ in the lexicographic order.*

An ideal J such that J_ℓ is the vector space spanned by an initial sequence of the monomials of degree ℓ in the lexicographic order for every ℓ is called a lexicographic ideal. It follows from Theorem 13 that the ideal generated by an initial sequence of the monomials of degree d in the lexicographic order is a lexicographic ideal.

Theorem 14. *Let $I \subset S$ be homogeneous ideal, and set $D = \dim I_\ell$. If $m_1, \dots, m_D$ is the initial sequence of the monomials of degree ℓ in the lexicographic order, then*

$$\dim I_{\ell+1} \geq Q(\ell, D) := \dim(m_1, \dots, m_D)_{\ell+1}.$$

For example, in three variables, since

$$(x_1^2,\ x_1x_2)_3 = \langle x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_1x_2x_3 \rangle$$

is a vector space of dimension 4, we must have $\dim I_3 \geq 4$ for *any* ideal containing two independent quadrics. Theorem 14 is the main theorem of [Mac27]; it has been analyzed, exploited, and generalized a great deal. Macaulay's proof takes about ten pages, largely of numerical formulas, and the paper is famous for the remark Macaulay added at its start: "Note: This proof of the theorem which has been assumed earlier is given only to place it on record. It is too long and complicated to provide any but the most tedious reading." (For simpler modern proofs see [BH93] or [Gre89].) From Theorems 13 and 14 it follows that the function $Q(\ell, D)$ that will serve to characterize all Hilbert functions of ideals is equal to $\dim J_{\ell+1}$ for the lexicographic ideal J generated by D monomials of degree ℓ. Macaulay's third result is an elegant computation of the function $Q(\ell, D)$ in terms of the binomial coefficients

$$(m)_n := \binom{n-1+m}{n}.$$

For example, $\dim S_\ell = \binom{n-1+\ell}{n-1} = (\ell+1)_{n-1}$. Macaulay noted that any integer D with $0 < D < \dim S_l$ can be expressed uniquely as

$$(*) \qquad D = (\ell_1)_{n-1} + (\ell_2)_{n-2} + \cdots + (\ell_r)_{n-r}$$
$$\text{with } \ell > \ell_1 \geq \ell_2 \geq \ell_r \geq \cdots \geq 1 \text{ and } n > r > 0;$$

this is now known as the *$(n-1)$-st Macaulay representation of D.*

Theorem 15. *If I is the lexicographic ideal generated in degree ℓ by D elements, then*

$$Q(\ell, D) = \dim I_{\ell+1} = (\ell_1+1)_n + (\ell_2+1)_{n-1} + \cdots + (\ell_r+1)_{n-r+1}.$$

Interesting modern extensions of this result can be found in the works of Caviglia and many others; see for example [CS18]. The related case of ideals in an exterior algebra is the subject of the Kruskal–Katona theorem, fundamental in algebraic combinatorics; see for example [GK78].

Macaulay referred to the papers of Hilbert [Hil90] and Ostrowski [Ost22] for expressions for the Hilbert function. If $I \subset S$ is a homogeneous ideal, then Hilbert proved that there is a polynomial $\chi(\ell)$ such that $\dim(S/I)_\ell = \chi(\ell)$ for $\ell \gg 0$. Macaulay quoted the form

$$\chi(\ell) = a\binom{\ell}{n-r-1} + b\binom{\ell}{n-r-2} + \cdots + k$$

and Ostrowski's form

$$\chi(\ell) = a(\ell+1)_{n-r-1} + b'(\ell+1)_{n-r-2} + \cdots + k',$$

and remarked that a is the *degree* (Macaulay called it the *order*) of I while r is the *codimension* (which Macaulay called the *rank*) of I and $n-r-1$ is the dimension of the projective variety corresponding to I. From Theorems 13 and 14 it follows at once that for any homogeneous ideal I there is a lexicographic ideal J with the same Hilbert function as I. By Hilbert's basis theorem, the degrees of the generators of J are bounded by some number ℓ_0. Writing $D := \dim J_{\ell_0}$ in the form $(*)$ above, we see that for $\delta = \ell - \ell_0 > 0$ we have

$$\dim J_\ell = D = (\ell_1+\delta)_{n-1+\delta} + (\ell_2+\delta)_{n-2+\delta} + \cdots + (\ell_r+\delta)_{n-r+\delta}.$$

From this, Macaulay deduced a third form for the function $\chi(\ell)$, which he felt is "the simplest form in which to leave $\chi(\ell)$, and shows its restrictions." This is:

$$\chi(\ell) = (\ell_1)_{n-r} - \big((\ell_1+\delta)_{n-1+\delta} + (\ell_2+\delta)_{n-2+\delta} + \cdots + (\ell_r+\delta)_{n-r+\delta}\big).$$

In the last section of the paper, Macaulay took up some special cases: Echoing material from [Mac13], he remarked that the generating function for the Hilbert function of a perfect ideal has a special form, and he computed the Hilbert function of the residual of a perfect ideal with respect to an ideal of the principal class (that is, a *linked* ideal). He uses the ideas from §70 of the *Tract* to describe the symmetry of the Hilbert functions of zero-dimensional homogeneous ideals that are principal systems. He wrote:

> It may have been observed that we have only found the conditions which govern the terms of the D series in the two cases of the general ideal and a perfect ideal and some special cases of the latter. We have not found them for the general unmixed ideal, primary ideal, the ideal with no multiple spread, and prime ideal. Each of these cases is more difficult to solve than the previous one, and I doubt whether the solution can be found for any of them, since there seems to be no law governing the discontinuities which occur.

However, in a note added later he conjectured a form for the Hilbert function of a smooth projective variety X,

$$\chi(\ell) = (\ell+1)_m + p_0(\ell)_m - p_1(\ell)_{m-1} + p_2(\ell)_{m-2} - \cdots + (-1)^m p_m,$$

where p_i is the arithmetic genus of the intersection X_i of X with a plane of codimension i. (The conjecture is correct: it follows from Bertini's theorem that the general plane sections of X in projective space are again smooth, so that the ith difference of the Hilbert function of X differs from the Hilbert function of X_i in only finitely many degrees.)

Macaulay published his next paper [Mac30] in the *Mathematical Gazette*, a magazine perhaps analogous to the *American Mathematical Monthly*. It is didactic in nature, and gives an exposition of some properties of the Macaulay representation $(*)$

above, focusing on various inequalities for the numerical value of $Q(\ell, A) - Q(\ell, B)$, given the value of $A - B$. It is pure numerics—no ideals are mentioned—and gives one some appreciation of the remark of J. E. Littlewood, one of Macaulay's pupils at St. Paul's, that Macaulay's book on Geometical Conics was "very stiff" [Bak38, p. 359].

8. 1934: Appreciating Emmy Noether and summing up

Macaulay's career is bracketed by the two famous Noethers, Max and Emmy. At the beginning, Macaulay's work centered around Max Noether's Fundamental Theorem and the problem of residuation, and it featured a deeper understanding of the latter. Macaulay's last paper [Mac34] is the first paper in English to describe Emmy Noether's work on ideal theory. B. L. van der Waerden's enormously influential two-volume *Moderne Algebra* ([van30] and [vdW67], respectively) was an exposition of the ideas of Emmy Noether and Emil Artin, among others, on the theory of ideals in commutative and noncommutative rings. Macaulay's article [Mac34], published just three years later, begins with a description of this new theory, which we believe to be its first mention in English. In §4 Macaulay specialized to the case of ideals in $S := \mathbb{C}[x_1, \ldots, x_n]$ (which he referred to throughout as "pol. ideals"). He also mentioned the work of Wolfgang Gröbner, without journal attribution, and, as mentioned above, he quoted a proof by Krull of Lasker's Satz XXVII, that he said "has not been published, as far as I am aware" [Mac34, p. 36]. This suggests considerable contact with the revolution in algebra going on in Germany, and even leads one to wonder whether he was in direct contact with Krull.

In [Mac34, §5] Macaulay turned to inverse systems, saying on page 11 that he can use the "broadened outlook of modern algebra" to give a simpler presentation than that in the *Tract*. He now presented the inverse system explicitly as an "H-ring-module" (that is, a graded S-module in the modern sense) and gave many examples. Among other advances, Macaulay could now give the modern description of what he called "principal systems" (which made their appearance in the *Tract*, §60), the ideals for which the inverse system is generated, in a suitable sense, by a single element: they are the ideals I with no embedded components that are also generically irreducible in the sense that, if I_1 is a primary component of I of codimension r, then $I_1^{(r)} := I\mathbb{C}(x_{r+1}, \ldots, x_n)[x_1, \ldots, x_r]$ (where the x_i are chosen generally), is not the intersection of two strictly larger ideals. The next section, §6, concerns the theory of perfect ideals. He defined these with a more modern version of the definition given in the *Tract*: he said that a homogeneous ideal I of codimension r is perfect if $IS/(x_{r+1}, \ldots, x_n)$ again has codimension r (which, he remarked, is always true after a general linear change of variables) and "has the same number of linearly independent elements up to any degree as $I^{(r)}$". This implies that the Hilbert function of $S/(I + (x_{r+1}, \ldots, x_n))$ as a vector space over $\mathbb{C}$ is equal to that of $\mathbb{C}(x_{r+1}, \ldots, x_n)[x_1, \ldots, x_r]/(I\mathbb{C}(x_{r+1}, \ldots, x_n)[x_1, \ldots, x_r])$ as a vector space over $\mathbb{C}(x_{r+1}, \ldots, x_n)$. In modern terms, this is equivalent to the statement that S/I is flat (in this case, even free) over the ring $\mathbb{C}[x_1, \ldots, x_r]$. Once again he defined a nonhomogeneous ideal to be perfect if its homogenization is perfect, and gave many examples. Here he pointed out that the dth power of the ideal generated by $f_1, \ldots, f_r \in S$ can be written as the ideal generated by the $d \times d$

minors of the $d \times (d+r-1)$ matrix

$$\begin{pmatrix} f_1 & f_2 & \cdots & f_{r-1} & f_r & 0 & \cdots & \cdots & 0 \\ 0 & f_1 & f_2 & \cdots & f_{r-1} & f_r & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \cdots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & f_1 & f_2 & \cdots & f_{r-1} & f_r \end{pmatrix}$$

with d rows and r nonzero diagonals. If $f_1, \ldots, f_r$ are homogeneous, then they form a regular sequence (and thus generate what Macaulay called "an ideal of the principal class") if the codimension of $(f_1, \ldots, f_r)$ is r. Macaulay had proved in [Mac13] that the ideal of maximal minors of a homogeneous $d \times (d+r-1)$ matrix is perfect whenever it has codimension r. He now deduced the result (proven differently in [Mac13]) that every power of an ideal that is a principal system is again perfect. In the final section, §7, Macaulay named for the first time another concept that first appeared in [Mac13, §71]: he said that a homogeneous ideal I is *superperfect* if it is perfect of codimension r and $S/(I + (x_{r+1}, \ldots, x_n))$, where n is the ambient dimension, is a principal system (as always, with sufficiently general choice of variables x_i). In the homogeneous case this is equivalent to the modern condition that S/I is a Gorenstein ring. (As with the definition of perfection, he defined superperfection by the condition that the associated homogeneous ideal is perfect, a less general condition than the modern one.) Superperfect ideals represent a far-reaching generalization of the notion of 1-set points, which, at the beginning of his career, was perhaps Macaulay's main contribution to the development of Brill and Noether's theory of plane curves.

9. Conclusion

Starting from the theory of plane curves—essentially a theory of one or two polynomials in two or three variables—Macaulay propelled commutative algebra toward the modern treatment of polynomial ideals with arbitrary generators and arbitrary numbers of variables, building on work of Kronecker, König, and Lasker. Though most of his work was written in an archaic style, he was the first to bring Emmy Noether's theory of commutative rings to Britain, and his work was much appreciated by her and her school.

Macaulay's work before 1900 had to do with the problems of incorporating the Riemann–Roch theorem into the theory of algebraic plane curves. In particular, he classified "clusters" of points as "t-set points" by the dimension t of the socles of their local rings (see Example 3) and showed that one could do residuation (only) with respect to 1-set points; and he understood that the residual of a t-set point in a 1-set point could be a $t-1, t$ or $(t+1)$-set point. These results are forerunners of the modern theory of Gorenstein ideals and linkage. His work in this period culminated in his generalized Riemann–Roch theorem, now widely known as the Cayley–Bacharach theorem.

In this period Macaulay and others struggled with the question of how to describe the "shape" of a cluster of points, represented by the singular point of a curve or by a nontransverse intersection of curves. Macaulay's characterization of Hilbert functions by means of lexicographic ideals in [Mac27] gives a satisfying answer to this question.

Most important was the understanding of *perfect* ideals, treated in [Mac13] and [Mac16]. One obstruction to the full proof of Noether's Fundamental Theorem was the lack of a theory that included embedded components of ideals, and the resulting impossibility of reasoning that the intersection of just two curves would never have such a component. Emanuel Lasker's discovery of the primary decomposition theorem for polynomial ideals [Las05] paved the way for such an understanding, and Lasker showed that complete intersections never have embedded components. Macaulay went much further. He isolated the property of unmixedness (not having embedded components) that had only been implicit in the work of Lasker, and identified the much larger class of perfect ideals as the natural class sharing key properties of complete intersections.

In his last paper [Mac34], Macaulay codified the extension to all dimensions of his early solution of the problems of residuation for point groups in the plane by introducing the "superperfect" ideals. Though the notion of perfect ideals dramatically influenced the development of modern commutative algebra, Macaulay's superperfect ideals were forgotten, and rediscovered in the guise of Gorenstein rings in work of Serre and Bass only in the second half of the twentieth century; see [Bas63]. The current paper is perhaps the first modern work to detail this connection.

Thus Macaulay's major achievements after 1905 can be seen as directly motivated by the difficulties encountered in his work on plane curves, giving a remarkable unity to his life's research work.

About the authors

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