BOOK REVIEWS

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Invitation to nonlinear algebra, by Mateusz Michałek and Bernd Sturmfels, Graduate Studies in Mathematics, No. 211, American Mathematical Society, Providence, RI, 2021, xiii+226 pp., ISBN 978-1-4704-5367-1

It is well known that many traditional applications of mathematics use tools from linear algebra. It is less known that classical and recent tools from nonlinear algebra are giving the theoretical basis for applications and computations in quite different domains, including optimization, statistics, computational biology, computer vision, signal processing, and complexity theory. This book is a somewhat special invitation to explore these tools. Our first reaction when browsing it was, "This is the book I wish I had read when I was a graduate student."

1. Nonlinear algebra

The first interesting thing about the book is in its title, "nonlinear algebra", sounding a bit different from "non-linear". As health extends beyond the mere absence of illness, here *nonlinear* is to be understood as a proper subject, which surpasses the mere negation of linearity. Even more since linearity appears as the simple case of degree one in the nonlinear algebra world.

The term nonlinear algebra (spelled "non-linear") was first introduced in the setting of theoretical physics in the book [8]. The SIAM focus group on Applications of Algebraic Geometry, initiated around 10 years ago, holds a biennial conference. The last one, held in July 2023 in Eindhoven, The Netherlands, featured diverse plenary lectures and 128 sessions of Minisymposia on different aspects of nonlinear algebra [16] with the participation of a vibrant community of young researchers. The maturity of the topic led to the creation of the SIAM Journal on Applied Algebra and Geometry in 2016, which publishes a variety of emerging applications using algebra, geometry, and topology tools.

Nonlinear algebra's focus is on computation and applications, and the theoretical results that need to be developed accordingly. Michałek and Sturmfels explain that this name is not just a rebranding of algebraic geometry but that it is intended to capture this focus, and to be more friendly to applied mathematicians, questioning the historic boundaries between *pure* and *applied* mathematics. We summarize in the following sections the different topics addressed in their book.

2. From algebra to geometry and back

Basic mathematical objects as spheres or eigenvalues of matrices are described by polynomial equations. Also, mathematical models in many domains are expressed as sets of solutions to systems of polynomial equations. The advent of personal computers and implementations of algorithms to compute with multivariate polynomials opened up the possibility of explicit computations in algebra and geometry which were outside the previous range of possibilities.

The first observation when dealing with polynomials over a field, e.g., with coefficients in $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, in a finite field, in the field of rational functions over a field, etc., is that if two polynomials f_1, f_2 have a common zero p, then any linear combination of them with polynomial coefficients $g_1 f_1 + g_2 f_2$ also vanishes at p. We are then led to work with polynomial *ideals*, which are nonempty subsets of the ring of polynomials $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ with coefficients in the field K, which are closed under taking polynomial linear combinations. By the celebrated *Hilbert basis theorem*, any ideal $I \subset K[\mathbf{x}]$ has a finite number of generators. This means that there exist $f_1, \ldots, f_r \in I$ such that

$$I = \{g_1 f_1 + \cdots + g_r f_r : g_1, \ldots, g_r \in K[\mathbf{x}]\}.$$

Note however that there is no simple translation of the concept of linear independence: for any pair of polynomials f_1, f_2 we get the equality $f_2 f_1 + (-f_1) f_2 = 0$. The main tool for dealing with multivariate polynomial ideals are *Gröbner bases* associated to any total well ordering of the set of monomials compatible with multiplication (known as a *term order*). They were introduced by B. Buchberger in his 1965 thesis written under the direction of W. Gröbner. We quote from the book under review:

> Gröbner bases for ideals are fundamental to nonlinear algebra, just like Gaussian elimination for matrices is fundamental to linear algebra.

The pioneering books [6, 7] written by D. Cox, J. Little, and D. O'Shea and first published in 1992 and 1998, respectively, showed that nonlinear algebra could be made accessible not just to mathematicians who are not experts in the area, but also to users of mathematics in engineering and computer science. Free and open-source computer algebra systems for polynomial computations based on Gröbner basis computations that started being developed in the 1980s, are now widely used and still in active development, as Macaulay2 [11] or Singular, now embedded in [17]. Also, several commercial software programs offer good implementations. The inherent complexity of most nonlinear algebraic computations has also led to the development of software for polynomial system solving based on *numerical algebraic geometry*, using well tuned algorithms for homotopy continuation.

The subsets of K^n that occur as zeros of polynomial equations in n variables are called (affine) algebraic varieties. If we endow K^n with the Zariski topology, algebraic varieties are closed; this can be similarly done for any commutative ring with unity. The translation between algebra and geometry is based on another important theorem of Hilbert known as the *Nullstellensatz*. It says that when K is algebraically closed, the polynomials g that vanish on the common zeros of all the polynomials in an ideal I (which coincides with the zeros of all the polynomials in any finite set of generators of I) are precisely those g for which a power g^m lies in I. To understand why powers occur, just think of the ideal generated in the polynomial ring in one variable by a polynomial f with multiple roots.

The notion of multiplicity has been extended to polynomial ideals as well as the notion of primes and factorizations. Coefficients in $\mathbb{Q}$ allow for exact symbolic computation, while general coefficients in $\mathbb{R}$ or $\mathbb{C}$ are approximated and manipulated via numerical computations. Computing roots of polynomials and factorization is not algorithmic in general, but there are several interesting implemented algorithms that give important information. For instance, given a polynomial ideal in $\mathbb{Q}[\mathbf{x}]$ it is possible to symbolically compute the dimension of the algebraic variety V it defines in $\mathbb{C}^n$. Such a variety is the union of a finite number of *irreducible* algebraic varieties and each irreducible variety is the closure of a differentiable manifold. The algebraic definition of the dimension of V equals the maximal dimension of these manifolds and corresponds to the maximum number of local free variables over the variety. Algebraically, this can be done using the tools of Hilbert series and Hilbert functions for an associated monomial ideal.

Ideals I in $\mathbb{C}[\mathbf{x}]$ can be interpreted as systems of linear partial differential equations (PDEs) with constant coefficients. When I has finitely many complex zeros (i.e., I has dimension 0), there is a translation between the multiplicities of I at its zeros (the *primary decomposition* of I) and the shape of the holomorphic solutions to the associated differential system. These solution spaces vary in a nice way with parameter changes. When the dimension of I is positive, the precise relation between the primary decomposition of I and the solution space of the associated system of PDEs is an important result in analysis known as the *Ehrenpreis fundamental principle*.

The standard compactification of the affine space K^n is the projective space $\mathbb{P}^n_K$, which can be defined by gluing $n+1$ affine patches or by identifying all points in the same line through the origin in $K^{n+1} \setminus 0$. There is a richer and many times simpler theory of *projective algebraic varieties*. Indeed, affine varieties can be embedded in projective space via the affine charts, and their closures are projective varieties.

The study of the topology of algebraic varieties in low dimension over the reals poses challenging questions and leads to insights that are also useful for understanding the higher-dimensional case. For instance, the real projective plane $\mathbb{P}^2_{\mathbb{R}}$ is a surface that cannot be embedded homeomorphically in $\mathbb{R}^3$, so it is not possible to produce a good picture. A smooth curve C in $\mathbb{P}^2_{\mathbb{C}}$ is a orientable surface and its genus g is computed in terms of the degree d of its (irreducible) defining equation. In the real projective plane $\mathbb{P}^2_{\mathbb{R}}$, the curve C has at most $g+1$ connected components that disconnect the space when d is even, and there is one component with connected complement when d is odd.

Elimination of variables corresponds to geometric projections. The computation of Gröbner bases corresponding to *lexicographic* term orders provide an effective way to eliminate variables in the multivariate case. We can use elimination to compute over the complex numbers the closure of the image of a variety under a polynomial or rational map, and in particular, the closure of the projection of a variety. For instance, the projection onto the (a,b,c,d)-space of the variety $V_1 = \{(a,b,c,d,x,y) : ax+by=0, cx+dy=0\}$ is the whole space since for any choice of coefficients, setting $x=y=0$ gives a solution. But if we consider for instance the variety $V_2 = \{(a,b,c,d,x,y,z) : ax+by = cx+dy = xyz-1 = 0\}$, then the closure of the image is now the variety $D = \{(a,b,c,d) : ad-bc=0\}$ cut out by the determinant. Note that for instance the point $(0,1,0,1)$ lies in D but it does not belong to the image of V_1 under the projection. Similarly, *resultants* and *discriminants* (as the *hyperdeterminants*) can be computed via elimination of variables. Another interesting application of elimination is to find the conditions under which a given partial matrix can be completed with a rank restriction.

Over an algebraically closed field, images of *projective* algebraic varieties are closed. In the affine case or over the reals, we have the following general results. A subset S of K^n is said to be *constructible* if it equals a finite union of differences of varieties. In particular, any algebraic variety is constructible. The *Chevalley theorem* states that the image of a constructible set under a polynomial mapping is a constructible set. When $K = R$, S is said to be *semialgebraic* if it can be described as a finite union of the solutions sets of finite systems of polynomial inequalities (either strict or not). In particular, real algebraic varieties are semialgebraic. It is easy to see that every constructible set is semialgebraic, but the converse is not true. The *Tarski–Seidenberg theorem* states that the image of a real algebraic variety under a polynomial mapping is a semialgebraic set.

The *weak form of the Nullstellensatz* on an algebraically closed field K, states that if a collection of polynomials $f_1, \ldots, f_r \in K[x_1, \ldots, x_n]$ has no common zero in K^n, there exists an identity $g_1 f_1 + \cdots + g_r f_r = 1$ with polynomial multipliers g_i (the other implication trivially holds). In fact, this weak form is equivalent to the general form of the Nullstellensatz via the *Rabinowitsch trick*: if f vanishes on the common zeros of $f_1, \ldots, f_r$, then the system $fy - 1 = f_1 = \cdots = f_r = 0$ in one more variable y does not have any solution. It is then enough to write 1 as a polynomial linear combination of these $r + 1$ polynomials and then substitute $y = 1/f$.

The Nullstellensatz is not true over $\mathbb{R}$: consider for instance the polynomial $x^2 + y^2 + 1 \in \mathbb{R}[x, y]$. But there are real versions. The starting point is that a real polynomial which is a sum of squares must be nonnegative, and a natural question is whether the converse holds. Hilbert showed in 1893 that the answer is negative if one asks for squares of polynomials. He proved that no counterexamples exist in one variable or in degree 4 in two variables but his proof of the negative result was not constructive. Motzkin proposed in 1965 the explicit counterexample of degree 6 in two variables $M = x^4y^2 + x^2y^4 + 1 - 3x^2y^2$; see Figure 1. Artin showed in 1927

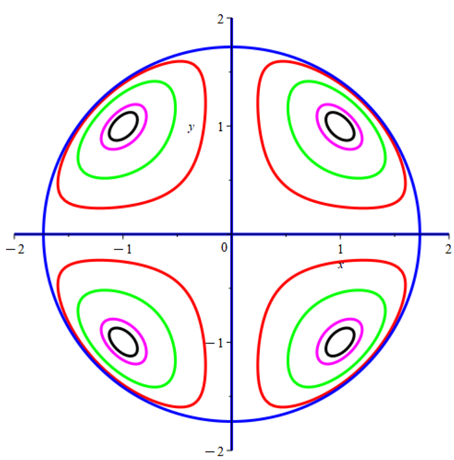

FIGURE 1. Real plots of the level curves of the Motzkin polynomial $M = c$, for $c = 1, \frac{1}{2}, \frac{1}{8}, \frac{1}{20}$.

that the answer is positive if one asks for squares of rational functions; indeed,

$$M = \frac{x^2y^2(x^2+y^2+1)(x^2+y^2-2)^2+(x^2-y^2)^2}{(x^2+y^2)^2}.$$

Distributing the three terms of the factor (x^2+y^2+1), we see that the right-hand side is a sum of four squares of rational functions. The real zeros of the Motzkin polynomial are the four points $(x, y) = (\pm 1, \pm 1)$; they are the singular points of the complex curve $M = 0$, that is, the points were M and both partial derivatives vanish. Artin's result is derived from the weak form of the *Positivstellensatz*: if an ideal in $\mathbb{R}[\mathbf{x}]$ has no real points, there exist polynomials $p_1, \ldots, p_s$ such that $1 + \sum_{i=1}^{s} p_i^2 \in I$, i.e., -1 is a sum of squares modulo I. The strong form of the Positivstellensatz can be seen as a generalization of the *Farkas lemma* holding for polynomials of degree 1. The nice article [2] substantially extends Hilbert's celebrated characterization of equality between nonnegative forms and sums of squares, giving geometric insight into the different cases via the relation with projective varieties of minimal degree.

3. Nonlinear algebra and combinatorics

The fruitful interaction of algebra and geometry with combinatorics occurs for instance in tropical geometry, in the theory of toric varieties and in the theory of matroids.

The tropical semiring $\mathbb{R} \cup \{\infty\}$ is endowed with the two nonstandard operations

$$u \oplus v = \min(u, v), \qquad u \odot v = u + v.$$

In other words, the tropical sum is the minimum and the tropical product is the usual sum. The idea to perform products with sums has been historically successful with the tool of *logarithms*. The tropical semiring $\mathbb{R} \cup \{\infty\}$ arises as the codomain of a real valuation of a field K, which is a function $\mathrm{val}\colon K \to \mathbb{R} \cup \{\infty\}$ satisfying $\mathrm{val}(a) \odot \mathrm{val}(b) = \mathrm{val}(ab)$, $\mathrm{val}(a) \oplus \mathrm{val}(b) \leq \mathrm{val}(a + b)$, and an additional property saying $\mathrm{val}(a) = \infty$ if and only if $a = 0$. A field with a valuation is an ultrametric space where the metric is defined in terms of the norm $|a| = e^{-\mathrm{val}(a)}$ for any nonzero $a \in K^* = K\setminus\{0\}$ and $|0| = 0$. Thus, one can use analytical and topological methods to study K.

Given any field, we can define the trivial valuation as $\mathrm{val}(a) = 0$ for any $a \in K^*$ and $\mathrm{val}(0) = \infty$. An important example with a nontrivial valuation is the field $K = \mathbb{C}\{\{t\}\}$ of Puiseux series with complex coefficients and rational exponents with a common denominator. The valuation of a series $c \in K$ is the smallest exponent a of a term $c_a t^a$ that appears in the series expansion of c (i.e., $c_a \neq 0$). Puiseux series were classically considered, as they provide local parametrizations of complex curves in the plane around a singular point. A nontrivial classical result is that the field of Puiseux series is algebraically closed. Another valued field with interest in number theory is the algebraic closure of the p-adic numbers, for any prime number p.

Given a *Laurent polynomial* $f \in K[\mathbf{x}^{\pm 1}] = K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ (with integer exponents not necessarily nonnegative) over a field K with a valuation, its tropicalization $\mathrm{trop}(f)$ is the polynomial over the tropical semiring obtained by replacing each coefficient of f by its valuation and the classical operations of multiplication and addition by their tropical counterparts. We will denote the tropical variables by $\mathbf{u} = (u_1, \ldots, u_n)$. Thus, if $f = \sum_{i=1}^{m} c_i \mathbf{x}^{a_i}$, then $\mathrm{trop}(f) = \bigoplus_{i=1}^{m} \mathrm{val}(c_i) \odot \mathbf{u}^{\odot a_i}$, which equals the minimum of the linear forms $\mathrm{val}(c_i) + \langle \mathbf{u}, a_i \rangle$ for $i = 1 \ldots, m$. If this minimum is attained at least *twice*, $\mathbf{u}$ is said to be a *zero* of $\mathrm{trop}(f)$. It is straightforward to check that if a point $\mathbf{x} \in (K^*)^n$ satisfies $f(\mathbf{x}) = 0$, then $\mathbf{u} = (\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n))$ is a zero of $\mathrm{trop}(f)$.[1] *Kapranov's theorem* asserts in particular that the converse is also true for $\mathbf{u} \in \mathbb{Q}^n$ when K is an algebraically closed field with $\mathrm{val}(K^*) = \mathbb{Q}$, as the field of Puiseux series. A similar result holds for ideals, known as the *fundamental theorem of tropical algebraic geometry.*

If $f \in \mathbb{C}[\mathbf{x}^{\pm 1}]$ is a Laurent polynomial with complex coefficients, we can consider the natural embedding $\mathbb{C} \to \mathbb{C}\{\{t\}\}$ and the previous result holds, considering the valuations of the roots of f in the Puiseux series field. If f is not the zero polynomial, its *Newton polytope* $N(f)$ is the convex hull in $\mathbb{R}^n$ of the set of exponents $\{a_i : c_i \neq 0, i = 1, \ldots, m\}$. It is a *lattice polytope*, a polyhedral convex body with integer vertices. The zeros of $\mathrm{trop}(f)$ consist of the codimension 1 cones in the *inner normal fan* of $N(f)$. When f has coefficients over a field with nontrivial valuation, we also get a (displaced) polyhedral object. We encourage the reader to look for a nice tropical picture on the Web. Tropicalization of algebraic varieties and ideals carry important combinatorial information on them, but one can also consider tropical objects and varieties not coming from the *algebraic world.* This is a rich subject. We refer to the book [14].

It is possible to consider vectors and matrices over the tropical semiring. Tropical linear algebra has many applications; indeed, this was one of the sources of the theory. A first application is the computation of the length of the shortest path in a weighted direct graph. This is achieved by computing tropical powers of the adjacency matrix of the graph. The tropical determinant expresses the assignment problem in *dynamic programming.* When A is a square matrix such that the directed graph with adjacency matrix A is strongly connected, A has a single tropical eigenvalue which coincides with the minimum (normalized) length of a directed cycle in the graph.

Toric varieties are special algebraic varieties with interesting relations with lattice points in polyhedra. They have dense parametrizations and allow for effective computations. Even if they are very special, a famous quote of William Fulton explains,

> toric varieties have provided a remarkably fertile testing ground for general theories in algebraic geometry.

Affine toric varieties are associated to rational polyhedral cones. Projective spaces $\mathbb{P}^n_K$ (and also *Segre* and *Veronese* varieties) are examples of compact toric varieties; they contain a dense torus consisting of the projective points with nonzero coordinates, which is isomorphic to $(K^*)^n$. This is a group that acts by coordinate-wise multiplication. Affine toric varieties $X = X_A$ are rationally parametrized by monomials with exponents in a finite subset $A = \{a_1, \ldots, a_m\}$ of $\mathbb{Z}^n$. This means that X is the closure of the image of the map $\mathbf{t} \mapsto (\mathbf{t}^{a_1}, \ldots, \mathbf{t}^{a_m})$, with $t \in (K^*)^n$. For any $v, v' \in \mathbb{Z}^m_{\geq 0}$ with $\sum_{i=1}^m (v_i - v'_i) a_i = 0$, the binomial $\mathbf{x}^v - \mathbf{x}^{v'}$ vanishes on X_A, and such binomials generate the ideal of all the polynomials vanishing on X_A. Denote by P the convex hull of A. The closure $\overline{X}_A$ in projective space of X_A equals a union of toric varieties associated to faces of P of all dimensions. If $K = \mathbb{C}$, we can

[1]There is an equivalent formulation of the zero set of a tropical polynomial in a *Gröbner-like fashion.*

also consider the real nonnegative points $\overline{X}_{A\geq 0}$. This semialgebraic set is homeomorphic to P under the *algebraic moment map* sending (the class of) a point $\mathbf{x}$ to the linear combination $\frac{1}{\sum_{i=1}^{m}|x_i|}\sum_{i=1}^{m}|x_i|a_i$.

In many applications, the affine or projective varieties we need to consider are indeed toric varieties. This happens for instance in the realm of chemical reaction networks and in particular, when dealing with many enzymatic networks in biochemistry [5, Chapter 5]. But also, many times the occurrence of toric structures can be hidden, and it is only visible after a suitable change of coordinates, as it happens when dealing with *complex balanced chemical reaction networks* (also called *toric dynamical systems* for this reason), in the *maximum likelihood estimation* of certain Gaussian models, or in certain *group-based models in phylogenetics*.

Matroids were independently introduced in the 1930s by T. Nakasawa and H. Whitney to abstract the property of linear dependence. They occur prominently in the work of J. Huh (see for instance the gentle introduction [12]). *Independent subsets* in a matroid abstract not only linearly independent subsets in a vector space but also acyclic subsets of a graph and algebraically independent subsets of a field extension. Matroids have many equivalent definitions and can be studied via their associated *matroid basis polytopes*. They are related to the geometry of special subvarieties of the *Grassmanians* and to tropical geometry. There is also an important notion of *oriented matroids* that generalizes positivity properties in real vector spaces, and that is associated to the existence of positive solutions of real polynomial systems [15].

The transition from linear algebra to nonlinear algebra has a counterpart in the transition from *linear programming* to *semidefinite programming*. Linear programming deals with the optimization of linear functions subject to linear constraints. Thus, the feasible region is a polyhedron and the optimal solutions are on a face. A symmetric matrix is called *positive definite* if all its eigenvalues are nonnegative. A *spectrahedron* is a closed convex set defined by the intersection of the cone of positive semidefinite matrices with a linear space. Semidefinite programming is the computational problem of minimizing a linear function over a spectrahedron. Given a polynomial $f \in \mathbb{R}[\mathbf{x}]$ of even degree, computing the global minimum of f on $\mathbb{R}^n$, or equivalently, maximizing c such that $f - c$ is nonnegative on $\mathbb{R}^n$, is very hard in general. Instead, maximizing c such that $f - c$ is a sum of squares gives a lower bound for the maximum, and it is a much easier question of semidefinite programming. We refer to [1] for the basic ingredients of this area.

4. Group actions on tensors and everywhere

According to the Erlangen Program by F. Klein, a quantity is geometric if it is invariant under the action of an underlying group of transformations. With the words of Michałek and Sturmfels, "in short, geometry is invariant theory" (see [19]). Moreover, group actions have become more and more important in all mathematics. One of the main themes of the book is to explore the utility of group actions in nonlinear algebra. Assume a group G acts over an affine variety X with coordinate ring $K[X]$. In *geometric invariant theory*, the quotient space $X//G$ has the coordinate ring given, by definition, by the subring $K[X]^G$ of polynomial invariant by this action. For a finite group of matrices, the Hilbert series of $K[X]^G$ can be computed by the Molien formula. We do not reproduce it here; instead, we quote again from the book under review:

> It says we can count invariants by averaging the reciprocals of the characteristic polynomials of all matrices in the group.

The paradise of group actions are homogeneous varieties, where the group acts transitively, in other words the variety consist of a single orbit. On a homogeneous variety the neighborhood of a point looks like the neighborhood of any other point. In particular homogeneous varieties are smooth. The archetype of homogeneous varieties is the *Grassmannian* $G(k,n)$, another nonlinear object parametrizing all linear subspaces of a fixed dimension k in an n-dimensional vector space. Grassmannians are ubiquitous in algebraic geometry since by functoriality many constructions on $G(k,n)$ can be transferred to any algebraic scheme. Any subspace of dimension k in $\mathbb{C}^n$ is generated by the rows of a $k\times n$ matrix. The vector of maximal minors of this matrix does not depend (up to multiplicative constant) on the generators but only on the subspace. It defines an embedding of $G(k,n)$ into the projective space $\mathbb{P}^{\binom{n}{k}-1}$ by the vector containing all its maximal minors, which is called the *Plücker embedding*. It is a nontrivial fact that the equations defining the Grassmannian in this projective space are quadrics, called Plücker quadrics. Up to $GL(k)$-action on the left, the open part of $G(k,n)$ given by matrices such that the left submatrix corresponding to the first k columns is invertible, can be represented by

$$M = \left[I|A\right].$$

Any maximal minor of M corresponds (up to sign) to a minor, of some size, of A. Any Laplace expansion of $\det(A)$ gives a quadratic relation between the minors of A, which, homogenized, gives rise to a Plücker quadric. Enumerative questions on linear spaces are at the basis of the *Schubert calculus*, which can be understood in modern terms by tensor products of GL-representations, described by *Young diagrams* and by *Pieri's rule*.

It is time now to speak about *tensors* and their geometry. The notion of tensor is in principle a generalization of the notion of a matrix, indeed in the influential and popular book [10], tensors are represented as multidimensional matrices, say of format $n_1\times n_2\times\cdots\times n_d$ where the case $d=2$ corresponds to classical matrices of format $n_1\times n_2$. The $3\times 3\times 3$ case corresponds to Rubik's cube, but the first new case is the format $2\times 2\times 2$ which deserves a special attention. The *hyperdeterminant* of such tensor was explicitly computed by Cayley (see [10]), it is a degree 4 polynomial in the eight entries x_{ijk} $(0\le i,j,k\le 1)$ of the tensor, that vanishes at the coefficients of a cubic surface in K^3 (K any algebraically closed field) with equation $\sum_{0\le i,j,k\le 1} x_{ijk}z_1^i z_2^j z_3^k$ when it is singular. It is the equation of the dual variety of the *Segre variety* $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$.

Basic attributes of matrices, such as eigenvectors and rank, can also be defined for tensors. Why are tensors more difficult than matrices? The main reason relies on the group action which performs linear change of coordinates. We explain this fact in the hypercubic format. The space of square matrices $\mathbb{C}^n\times\mathbb{C}^n$ has dimension n^2 and it is endowed with the action of the group $GL(n)\times GL(n)$ which has the larger dimension $2n^2$. This action has only finitely many orbits, classified by the rank of the matrix. This means, as is well known, that every matrix is equivalent to a canonical diagonal form with only 1, 0 appearing on the diagonal. The space of tensors $(\mathbb{C}^n)^{\otimes d}$ has dimension n^d, and it is endowed with the action of the group $GL(n)^d$ which has dimension dn^2. This smaller dimension does not allow, except for a few cases, to have finitely many orbits, and even to have a dense orbit. Hence tensors must have much finer invariants than the rank. In recent years there has been a lot of advances regarding tensors and their rank. These advances include applications in phylogenetics, algebraic statistics, signal processing, quantum information, convex algebraic geometry, and combinatorial algebraic geometry. A readable introduction to tensors focusing on signal processing and other applications can be found in the article by P. Comon [4]; a standard textbook is [13] by J. Landsberg; B. Sturmfels wrote a nice survey [22]; for spectral theory the standard reference has become the book by L. Qi and Z. Luo [18].

We come now to the notion of *tensor rank*. Every symmetric tensor in $\mathrm{Sym}^d\mathbb{C}^n$ can be identified with a homogeneous polynomial of degree d in n variables. The symmetric tensors which are powers of a linear form have (symmetric) rank 1 and correspond to the Veronese variety. A symmetric tensor $T \in \mathrm{Sym}^d\mathbb{C}^n$ has symmetric rank r if T is the sum of r symmetric tensors of rank 1 and r is minimal. The interesting phenomenon is that symmetric tensors of symmetric rank $\leq r$, for $r \geq 2$ make a nonclosed variety. The first example is given by tensors in $\mathrm{Sym}^3\mathbb{C}^2$, where tensors of rank ≤ 2 are the complement of the orbit $GL(2)\cdot x^2y$. This orbit consists of tensors of rank 3, thanks to the decomposition

$$6x^2y = (x+y)^3 - (x-y)^3 - 2y^3.$$

The three cubes on the right-hand side show show that the rank is 3, indeed x^2y cannot be expressed by only two cubes. This phenomenon marks a difference between matrices (where the locus of matrices of rank $\leq r$ is closed, given by the vanishing of $(r+1)$-minors) and general tensors. Note however that x^2y has symmetric rank 3 but it can be approximated by tensors of rank 2 by the limit

$$x^2y = \lim_{\epsilon\to 0} \frac{1}{3\epsilon}\left[(x+\epsilon y)^3 - x^3\right].$$

We say in this case that x^2y has *symmetric border rank* 2. The symmetric border rank is less or equal than the symmetric rank; the previous example shows it can be strictly less. This example with the limit also shows that x^2y lies on a tangent line to the Veronese variety $v_3(\mathbb{P}^1)$ of rank 1 tensors at the point x^3. These notions can be generalized to the nonsymmetric case. The *decomposable tensors* $v_1 \otimes \cdots \otimes v_d \in V_1 \otimes \cdots \otimes V_d$ are not symmetric for general v_i, have rank 1, and correspond to the Segre variety. Similar examples with tangent lines to the Segre variety can be constructed, again the *border rank* of a tensor can be strictly smaller than its rank.

The *flattenings* of a tensor $T \in V_1 \otimes V_2 \otimes V_3$ are defined as the linear maps $\hat{T}\colon V_i^\vee \to V_j \otimes V_k$ with $\{i,j,k\} = \{1,2,3\}$. This means that the three-dimensional matrix T has been flattened to the two-dimensional matrix $\hat{T}$, consisting of the two-dimensional slices of T arranged side by side; see Figure 2. Moreover, this

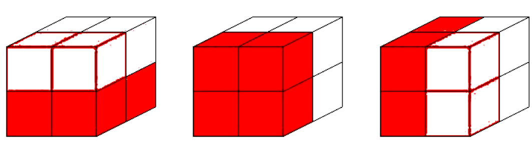

FIGURE 2. The 2×2 slices of a $2 \times 2 \times 2$ tensor can be arranged to give the 4×2 flattenings .

map behaves well under group action. It is elementary to check that the rank, and even the border rank of T, is bounded by the matrix rank of $\hat{T}$. *Young flattenings* generalize this classical flattening. This allows us to get finer estimates for the rank and border rank of tensors. Let M_n be the space of $n \times n$ matrices. One striking application of Young flattenings concerns the rank of the matrix multiplication tensor corresponding to the map $M_n \otimes M_n \to M_n$ which sends the decomposable tensor $A \otimes B \in M_n \otimes M_n$ to the matrix product AB. This rank governs the complexity of matrix multiplication algorithm for large size matrices, one of the open challenges in complexity theory. The reader is invited to read [13] for more advanced applications.

The spectral theory of matrices finds a nice interpretation in the realm of dynamical systems. We briefly recall the dynamical interpretation of the singular pairs of a matrix. Any $m \times n$ matrix B can be identified with the bilinear form $B(x, y) = y^T Bx$ where $x \in K^n$, $y \in K^m$. The gradient map $\nabla B = (B_x, B_y) = (y^T B, Bx)$ can be seen as a self-map on the product of projective spaces

$$\nabla B \colon \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \dashrightarrow \mathbb{P}^{n-1} \times \mathbb{P}^{m-1}.$$

The gradient map is rational (so not regular) in the sense that it is not defined at pairs (x, y) such that $y^t B = 0$ or $Bx = 0$. Such points are called base points. More interestingly, the fixed points of the gradient map ∇B satisfy the equations $Bx = \lambda y$, $B^T y = \lambda x$. In other words, the singular pairs of B corresponding to $\lambda \neq 0$ are exactly the fixed points of ∇B. This nice dynamical interpretation can be generalized for any tensor. In the simpler case of a symmetric tensor $T \in \mathrm{Sym}^d \mathbb{C}^n$, the gradient map gives a nonlinear self-map $\nabla T \colon \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$. Now the eigenvectors of T are fixed points ($\lambda \neq 0$) and base points ($\lambda = 0$) of ∇T. For general tensors the same construction defines singular tuples. The numbers of singular tuples of a general tensor have been computed in [9].

5. The structure of the book

The book by Michałek and Sturmfels started as notes for a general lecture series (called *Ringvorlesung* in German) on nonlinear algebra, which is also the name of the research group that started at MPI Leipzig in early 2017. The 13 chapters are chosen to cover 13 weeks of an introductory course, although this seems quite ambitious. Indeed the authors warn us in the short but incisive preface:

> Our presentation is structured into 13 chapters, one for each week in a semester. Many of the chapters are rather ambitious in that they promise a first introduction to an area of mathematics that would normally be covered in a full-year course. But what we offer is really just an invitation.

We definitely agree with this.

The book fulfills its role as an invitation to the realm of nonlinear algebra, not only for students but also for working mathematicians in other areas, although more advanced arguments naturally require further reading. By browsing this book one can taste the flavour of a growing and developing topic. The strategic choice of the 13 chapters is important in itself. The text is interactive and invites the readers to google some key words to get further information and to experience for themselves several paradigmatic examples with a computer algebra system. Also, several interesting applications are sketched or exemplified.

We refer to the book for definitions, proofs, and references of the mathematical objects we mention in this review. The reader might also take a look at the previous books by Sturmfels [20, 21]. We have not included references to the recent papers in the area because there are simply too many, but the reader could find in the article [3] over two hundred references on applications of nonlinear algebra to polynomial optimization, partial differential equations, algebraic statistics, integrable systems, configuration spaces of frameworks, biochemical reaction networks, algebraic vision, and tensor decompositions. Further references can be found in the book [5].[2]

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