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Coarse geometry of topological groups, by C. Rosendal, Cambridge Tracts in Math., Vol. 223, Cambridge University Press, Cambridge, 2022, ix+297 pp., ISBN 978-1-108-84247-1

Geometric group theory is an area of mathematics which views finitely generated discrete (or, more generally, compactly generated locally compact) groups as geometric objects. The metric structure may come from a space on which the group acts in an appropriate geometric way or from the group itself by considering the action on its Cayley graph. In many cases, the theory allows one to deduce algebraic information about the group from its geometric properties. For example, an early and famous result of the theory due to Gromov states that a finitely generated group of *polynomial growth* (i.e., such that the cardinality of the balls in its Cayley can be bounded by a polynomial of the radius) must have a nilpotent subgroup of finite index.

As discrete groups have no interesting local structure, what plays a role in the theory is their *large scale geometry*—that is, what happens at large distances. A useful equivalence relation that helps in forgetting the irrelevant local structure of a metric space is that of *quasi-isometry*: two metric spaces X and Y are *quasi-isometric* if there exists a map $f\colon X \to Y$ and a constant K such that

$$\frac{1}{K}d_X(x_1,x_2) - K \leq d_Y(f(x_1),f(x_2)) \leq Kd_X(x_1,x_2) + K \quad \text{for all } x_1,x_2 \in X,$$

and for every $y \in Y$, $d_Y(y, f(X)) < K$. For example, the group $\mathbf{Z}$ of integers is quasi-isometric to the reals with the map f simply given by the inclusion.

Every group G with a generating set S satisfying $S = S^{-1}$ can be equipped with a *word metric* d_S given by

$$d_S(g,h) = \min\{n \in \mathbf{N} : h \in gS^n\},$$

which is the same as the graph metric on the Cayley graph of (G, S). If the group G is locally compact and S is compact, then every other compact generating subset S' of G is contained in some finite power of S, so the metric spaces (G, d_S) and $(G, d_{S'})$ are quasi-isometric. These metrics behave nicely with respect to the group structure—they are invariant under left multiplication—but not so much with respect to the topology of G, as they are, in general, not continuous. However, on a locally compact group, one can always find a metric which is left-invariant, *compatible* (i.e., generates the topology), and quasi-isometric to d_S. Moreover, all of these metrics have the property that the bounded sets are precisely the precompact ones.

It is much less clear whether such a canonical coarse structure exists on topological groups when one drops the assumption of local compactness. This is the first question that the book addresses and the author gives a very satisfactory positive answer. The main subject thereafter is studying this structure both from a theoretical perspective and through many concrete examples. One of the most important motivating examples is that of Banach spaces, which do, of course, have a canonical coarse structure given by the norm. The theoretical framework developed in the book is a common generalization of that of geometric group theory of locally compact groups and the coarse geometry of Banach spaces, and it has drawn inspiration from both.

Perhaps at this point one could stop and ask: but are there any interesting groups (from a geometric perspective) which are not locally compact? Some of the classical sources of groups in geometry are the fundamental groups of topological spaces, the isometry groups of metric spaces, and the automorphism groups of various complexes (for example, trees), and subgroups thereof. Fundamental groups are finitely generated if the space is well behaved (and, in any case, do not carry a topological group structure) but for groups in the other two classes to be locally compact, one needs to impose some local conditions: for example, for the metric space to be *proper* (i.e., such that all balls are compact) or for the complex to be locally finite. So a first natural class of examples for the theory is obtained by dropping these conditions: studying the isometry groups of some interesting nonproper metric spaces (for example, the universal Urysohn space or the infinite-dimensional hyperbolic space) or the automorphism groups of nonlocally finite complexes (for example, the infinite-branching tree). Two other classes of examples are homeomorphism groups (of manifolds) and the so called *big mapping class groups*: the mapping class groups of surfaces, which are not finitely generated. (The *mapping class group* of a surface is the quotient of the group of orientation-preserving homeomorphisms fixing the boundary by the path-component of the identity.) Then, of course, there are also Banach spaces and their affine isometry groups. It turns out that all of these contain interesting geometric information.

Even though much of the theory works for rather general topological groups, most interesting examples are *Polish* (i.e., separable and completely metrizable) and, as this somewhat simplifies the exposition, I will restrict myself to these in the rest of the review.

As already mentioned, the first foundational question addressed in the book is the definition of a coarse geometric structure on a topological group. The abstract framework is that of *coarse geometry*, developed by J. Roe, which can be viewed as a dual notion to uniform spaces in topology: instead of a filter of entourages which specify the pairs of points which are close (of distance less than ϵ for some small positive ϵ if the uniformity metrizable), it is given by an ideal of entourages which specify the pairs of points which are boundedly apart (of distance less than α for some large positive α). Similarly to uniform structures, topological groups admit canonical left and right coarse structures. Perhaps the most intuitive way to specify them is to describe the *coarsely bounded* sets, which would be the sets of finite diameter in the metric setting. In the case of a topological group, there is a simple characterization: a subset A of a Polish group G is *coarsely bounded* if for every neighborhood $V \ni 1_G$, there exists a finite $F \subseteq G$ and $k \in \mathbf{N}$ such that $A \subseteq (FV)^k$. Equivalently, for every continuous action of G by isometries on a metric space X and every $x_0 \in X$, the set $A \cdot x_0$ is bounded. Unsurprisingly, the notion coincides with the well-studied ones in the context of locally compact groups and Banach spaces: a subset of a locally compact group is coarsely bounded iff it is precompact, and a subset of a Banach space is coarsely bounded iff it is bounded in norm. This also illustrates well how the definition is a common generalization of these two classical situations: in the first case, one does not need k and in the second, one does not need F.

For the uniform structures on a group, there is a classical metrization theorem due to Birkhoff and Kakutani: as long as a topological group has a countable basis at the identity, it admits a left-invariant compatible metric, which defines the left uniformity on the group. The situation with the coarse structure is a little more delicate as can be seen from the following simple example. Equip $G = \mathbf{Z}^{\mathbf{N}}$ with the product topology, taking $\mathbf{Z}$ to be discrete. Then G has isometric actions on $\mathbf{Z}$ that factor through each of the coordinates, so any coarsely bounded subset of G must have finite projections in every coordinate, i.e., must be precompact. On the other hand, if the coarse structure on G is metrizable, then G is the union of the balls of radius n centered at 1_G for $n \in \mathbf{N}$ and each such ball is coarsely bounded, so it is precompact. By the Baire category theorem, one of them must be somewhere dense, but compact sets in G have empty interior, a contradiction. It turns out that this is essentially the only obstruction and the coarse structure of a Polish group is metrizable iff it admits a coarsely bounded neighborhood of the identity. Moreover, for these groups, which are called *locally bounded*, one can find a left-invariant metric, which defines simultaneously the topology of G and the coarse structure.

Even for discrete groups, the quasi-isometry type of the group (using the word metric) is well defined only if the group is finitely generated. So it is to be expected that in the general case, one needs a similar condition. And indeed, if the group G is locally bounded and it is generated by a coarsely bounded set (which happens, for example, if G has no proper open subgroups), then among all left-invariant metrics on G there is a maximal one (up to a multiplicative and an additive constant), defining a canonical quasi-isometry type, which is also the one defined by the word metric.

One of the reasons uniformities on a group are important for the theory is that they allow one to define a notion of precompactness. A subset $A \subseteq G$ is precompact with respect to the left uniformity iff it can be covered by finitely many left translates of every neighborhood of 1_G iff $\overline{A}$ is compact, so this is nothing new. However, there is another uniformity, which yields a more interesting notion of precompactness. Entourages of the *Roelcke uniformity* of G are of the form

$$\{(g, v_1 g v_2) : g \in G, v_1, v_2 \in V\},$$

where V ranges over open neighborhoods of 1_G. Then a subset $A \subseteq G$ is *Roelcke precompact* if for every open $V \ni 1_G$, there exists a finite $F \subseteq G$ such that $A \subseteq VFV$. It is clear from this definition that Roelcke precompact sets are coarsely bounded. Many interesting groups, such as the symmetric group of an infinite set (and "large" subgroups thereof), the orthogonal group of an infinite-dimensional Hilbert space, or the homeomorphism groups of the reals and the circle, are Roelcke precompact and they behave in some respects like compact groups. However, they are somewhat trivial from a geometric perspective because they are coarsely equivalent to a point. More interesting are the *locally Roelcke precompact Polish groups*, that is, the groups admitting an identity neighborhood which is Roelcke precompact. Nonlocally compact examples include the automorphism group of the infinitely branching tree, the affine isometry group of a Hilbert space, the isometry group of the infinite-dimensional hyperbolic space (which was studied by Duchesne), and the isometry group of the Urysohn metric space. To all such groups, the theory applies nicely and the coarsely bounded sets are precisely the Roelcke precompact sets. Of these examples, a special role is played by the last one. The

Urysohn metric space $\mathbf{U}$ is the unique Polish metric space which is *universal* (i.e., it embeds every finite metric space) and *homogeneous* (i.e., every isometry between finite subspaces extends to an isometry of $\mathbf{U}$). As was proved by Uspenskij, the isometry group $\mathrm{Iso}(\mathbf{U})$ also has a universality property: every Polish group G embeds isomorphically into it. If G is locally bounded, this embedding can be made coarsely proper and if, further, G is generated by a coarsely bounded set, then it can be made quasi-isometric. (Note that while for locally compact groups, every isomorphic embedding is automatically proper, this is no longer the case for general Polish groups, so this result contains interesting information.)

The deepest results in the book concern the groups of *bounded geometry*, i.e., these that are coarsely equivalent to a proper (so locally compact) metric space. A more intrinsic definition is the following: a group G has *bounded geometry* if there exists a coarsely bounded $A \subseteq G$ such that every other coarsely bounded subset of G is covered by finitely many left translates of A (so this property can be viewed as local compactness on a large scale). Such groups (that are not locally compact) are not so easy to find and basically, the only known construction is combining a coarsely bounded group with a locally compact one. A prominent example throughout the book is given by the group

$$\mathrm{Homeo}_{\mathbf{Z}}(\mathbf{R}) = \{g \in \mathrm{Homeo}^+(\mathbf{R}) : g(x+n) = g(x) + n \text{ for all } x \in \mathbf{R}, n \in \mathbf{Z}\}$$

consisting of all orientation-preserving homeomorphisms of the reals commuting with integral translations. This group can be viewed as the central extension

$$1 \to \mathbf{Z} \to \mathrm{Homeo}_{\mathbf{Z}}(\mathbf{R}) \to \mathrm{Homeo}^+(S^1),$$

and as $\mathrm{Homeo}^+(S^1)$ is Roelcke precompact, one can show that the inclusion $\mathbf{Z} \to \mathrm{Homeo}_{\mathbf{Z}}(\mathbf{R})$ is a quasi-isometry. Groups of bounded geometry resemble locally compact groups the most and it is possible (often with significant effort and new ideas) to generalize some well-known results from that setting. Two examples are Gromov's theorem that the quasi-isometry of two groups is equivalent to the existence of a *topological coupling* (i.e., commuting actions on a locally compact space satisfying certain hypotheses) and the fact that every locally compact group admits a coarsely proper affine isometric action on a reflexive Banach space. For the second one, one also needs to assume amenability for the result to hold for groups of bounded geometry: indeed, some assumption is necessary because $\mathrm{Homeo}_{\mathbf{Z}}(\mathbf{R})$ admits no nontrivial linear representations on reflexive spaces whatsoever.

Many important examples of Polish groups come from model theory: these are the automorphism groups of countable structures, which can be viewed as closed subgroups of the infinite symmetric group $\mathrm{Sym}(\mathbf{N})$. There are well-known correspondences between model-theoretic properties of the structure and properties of its automorphism group. One such is between model-theoretic *stability* and representations of the automorphism groups on reflexive spaces. This also holds in the geometric setting: under appropriate hypotheses, the automorphism group of a stable structure admits a coarsely proper affine isometric representation on a reflexive Banach space. Also many different boundedness results are proved.

The final chapter of the book is concerned with *Zappa–Szép products*: groups G that can be represented as an internal product $G = AB$ with A and B closed subgroups of G with trivial intersection. This generalizes the well-known notion of a semidirect product (which also requires that one of A and B is normal). Apart from some results concerning the coarse structure of such groups, the chapter also contains a nontrivial general topological theorem: if G is Polish, the map $A \times B \to G$, $(a, b) \mapsto ab$ is a homeomorphism.

The idea of endowing Polish groups with a geometric structure is relatively new and was mostly developed by the author, along with his students and collaborators. It originated with the observations and results that many groups of interest, such as the Roelcke precompact groups and the homeomorphism groups of spheres, are coarsely bounded (or have *property (OB)* in the original terminology), and it was later developed in a full-fledged theory. The book is written as a research monograph and contains mostly original results, due to the author. Great care is taken with exposition and proofs are spelled out in full detail. The book does not have a lot of prerequisites apart from some basic knowledge of topological groups, and it is accessible to graduate students or non-specialists interested in the subject. As it is a research monograph rather than a textbook, exercises are not included; however, it is certainly possible to teach parts of it in a topics graduate course on Polish groups or geometric group theory.

The theory of Polish groups was born in the beginning of the twentieth century and since then they and their actions have played an important role in descriptive set theory, dynamical systems, and model theory. In view of the research presented in this monograph, now Polish groups can also be considered as geometric objects, and this new facet of the theory will undoubtedly lead to interactions with yet other branches of mathematics. The clear exposition and the numerous open questions that are discussed make the book an excellent entry point to research in the field.

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