BOOK REVIEWS

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 61, Number 4, October 2024, Pages 687–691
https://doi.org/10.1090/bull/1844
Article electronically published on August 15, 2024

Complex algebraic threefolds, by M. Kawakita, Cambridge Stud. Adv. Math., Vol. 209, Cambridge University Press, Cambridge, 2024, xii+490 pp., ISBN 978-1-108-84423-9

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Integrals involving square roots of polynomials were studied by Newton, Leibnitz, Euler, Abel, and Jacobi. More generally, these authors considered integrands of the form $\phi(x)dx$ where $\phi(x)$ satisfies some polynomial equation $F\big(x, \phi(x)\big) \equiv 0$.

Riemann observed that instead of dealing with a multivalued integrand, one should consider the algebraic curve

$$C(F) := \big\{(x,y)\colon F(x,y) = 0\big\} \subset \mathbb{C}^2,$$

and change the multivalued $\phi(x)dx$ to the single-valued ydx on $C(F)$. Riemann added some points at infinity to make $C(F)$ compact, and—following Newton—ironed out the singularities, resulting in a compact complex manifold $R(F)$, now called a *Riemann surface.*[1] Riemann surfaces admit constant curvature metrics, resulting in 3 distinct types:

(1) Positively curved—there is only one, the Riemann sphere $\mathbb{CP}^1$;
(2) Flat—called elliptic curves;
(3) Negatively curved—this is the largest class.

This trichotomy still guides our approach to algebraic varieties in higher dimensions, but we have to be flexible about what we mean by curvature. There are deep results about the existence of Kähler–Einstein metrics, but in many cases we understand only the integral of the Ricci curvature on algebraic curves $C \subset X$, equivalently, the degree of the *first Chern class* of X on C. So I say that "X is positively (resp., negatively) curved" when all these integrals/degrees are positive (resp., negative).

As a further source of confusion, algebraic geometers prefer to work with the cotangent bundle, which switches negative to positive. The *canonical class*, usually denoted by K_X, is the negative of the first Chern class. In this review I will stick to the first Chern class, which is better known outside algebraic geometry.

With these caveats, there are three "pure" types of algebraic varieties.

[1]A Riemann surface has complex dimension 1, so it is also called an algebraic curve. An algebraic surface is locally like $\mathbb{C}^2$.

(1) Positively curved, called *Fano* varieties.
(2) Flat, called *Calabi–Yau* varieties.
(3) Negatively curved, called *general type.*

Taking the product of varieties of different types gives new examples. It turns out that the right way to think about it is to add to the list two "mongrel" types.

(4) Semi-negatively curved. For these there is a unique morphism $X \to I(X)$ such that the curvature is flat exactly along the fibers, and $I(X)$ is negatively curved in a precise sense; clarified by Iitaka's conjectures [Iit72].
(5) Mori fiber spaces. Here there is a morphism $X \to M(X)$ such that the curvature is positive along the fibers. The morphism is not always unique, as we will discuss later. Note that if $M(X)$ is a point, then X is positively curved, and so (1) becomes a special case of (5).

Starting around 1890, the Italian school of Castelnuovo, Enriques, and Severi understood how to fit each algebraic surface into this framework. Their approach proceeded in two stages.

Contraction steps. Given a smooth surface S, find a curve $S \supset C \cong \mathbb{CP}^1$ such that the first Chern class of S has degree 1 on C. Then prove that there is a morphism $\pi : S \to S'$ such that $\pi(C)$ is a point, and π is an isomorphism on $S \setminus C$.

Here S' is again a smooth surface, and the second Betti number drops by 1, so this step can be repeated only finitely many times. At the end we get a morphism $S \to S^{\mathrm{m}in}$ from S to a *minimal model* of S.

Structure of minimal models. Prove that $S^{\mathrm{m}in}$ is in one of the Cases 1–5, and it is unique in Cases 2–4.

The program was completed and brought up to contemporary standards of rigor by Kodaira in the sixties; see [Bea96] for a teatment. Higher dimensional situations were considered occasionally, but examples of Hironaka and Ueno seemed to show that the Italian school's approach would not work in dimensions 3 and up.

Then birational geometry radically changed around 1980.

Reid observed that for all known examples there was a mildly singular candidate for $X^{\mathrm{m}in}$, and classified the resulting—now called *terminal*—singularities in [Rei80]. These singularities lead to technical complications, but do not seem to change the general features of the theory.

Mori developed a plan to go from X to $X^{\mathrm{m}in}$ using more complicated Contraction steps [Mor82]. The program was completed with contributions from Kawamata, Kollár, Miyaoka, Reid, and Shokurov, culminating in Mori's "Flip paper" [Mor88].

Since then the theory has developed in two closely related but technically quite distinct directions.

Higher dimensions. Much of the program was generalized to all dimensions. The major breakthrough was the work of Hacon and McKernan [HM07], with subsequent contributions from Birkar, Cascini, Corti, Shokurov, and Xu. Roughly speaking, the program is currently known to work when we are expected to end up in Cases 1, 3, or 5. [CL12, CL13] give an especially short treatment of the negatively curved case.

The higher dimensional proofs are quite indirect. Hence, although we know that $X^{\mathrm{m}in}$ exists, it is almost impossible to use the process itself to get any information about its structure.

Threefolds. In dimension 3 we expect to have a complete description of the required Contraction steps. After the introductory Chapters 1–2, the main aim of Kawakita's book is to present the classification in Chapters 3–5, and then apply the results to other questions of birational geometry in Chapters 6–9.

For surfaces, we always contract a curve to a point. For 3-folds, there are three types of contractions, each requiring different techniques. Chapter 3 deals with the cases when a surface is contracted to a point. This is mostly the work of Kawakita, with many subtle examples. Chapter 4 studies the cases when a surface is contracted to a curve, based mostly on the work of Tziolas. This turns out to be quite intricate, and we do not yet have a full description. Chapter 5 discusses the hardest case, called a *flip,* when a curve in X is replaced by another curve. The existence of flips was established by Mori [Mor88], with a full classification in [KM92].

Readers reaching the end of Chapter 5 may wonder why all this work was necessary. Indeed, in many applications it is enough to know that these Contraction steps exist. Their finer structure is not important.

However, there are several questions where knowing more about these steps is crucial. One of the first such instances was the study of the topology of the real points of 3-folds, solving a conjecture of Nash [Kol99]. Kawakita's book focuses on the most important application: understanding the nonuniqueness of $X^{\mathrm{m}in}$ in Cases 1 and 5.

For surfaces, this direction was started by Noether around 1870. For 3-folds, pioneering work was done by Fano in the 1920s, and a plan was developed by Sarkisov and Corti [Cor95]. Starting with a variety X', the Sarkisov–Corti program requires knowing all possible Contraction steps $X \to X'$ that end with X'. The answer turned out to be unexpectedly complicated, even when X' is smooth. The general description of this program is given in Chapter 6, while Chapters 7–9 deal with $X \to M(X)$ of Case 5, where $M(X)$ in turn has dimension 2, 1, and 0.

A distinctive feature and a great strength of the book is the wealth of simple yet enlightening examples that illustrate even the most exotic aspects of the theory. They are a most valuable resource for testing questions and conjectures.

I strongly recommend the book to anyone who wants to delve deeper into the study of 3-folds. The papers describing the steps of Mori's program are long and difficult, as are the ones about birational maps between 3-folds or their plurigenera. In each of these areas the author has chosen basic results and special cases that can be explained in a chapter, yet give a true introduction to the main difficulties of the general theory. For the steps of Mori's program and the plurigenera, Kawakita gives the first textbook treatments that go beyond the elementary results. Anyone wanting to read the full proofs should start with this book and ponder the many examples presented here.

However, Kawakita's treatment is not so well suited for casual readers, who just want a quick look to learn the state of the art, or to find convenient references. They may find the book frustrating or even misleading.

Good editorial guidance could have made this book much more useful to all readers, with modest effort from the editor and the author. I am looking forward to a revised 2nd edition, addressing the following issues.

The chapter introductions should be expanded to contain precise statments and an overview of the main theorems to come. Readers familiar with Mori's program will know what to expect in each chapter, but the uninitiated find little guidance.

There are many statements called Theorems in each chapter, and it is not easy to pick out which are the main ones. Compounding the problem, in many cases, reading a Theorem and a few lines before and after its statement, it is not clear whether a complete proof is given later, or whether substantial parts will be explained in detail, or if this result is to be used without any hint of proof.

For example, the introduction to Chapter 3 ends with the statement "we nearly conclude that the divisorial contraction is a certain weighted blow-up." I believe that here "nearly" means the the claim is true (this is Theorem 3.5.12), but not all details of the proof are given. However, on p. 156 there is a "Proof of Theorem 3.5.12," which seems to contradict my interpretation of "nearly".

As another example, Section 5.5 starts with saying "We shall establish the general elephant[2] conjecture" and then states Theorem 5.5.1. On the next page there is a discussion that only some of the cases of Theorem 5.5.1 will be proved. Then on p. 244 there is a "Proof of Theorem 5.5.1" which, as its last sentence says, covers only case (vii). So while a careful reader will know what is proved and what is left to references, a quick perusal may leave the wrong impression.

Related books and surveys. For the introductory materials, Chapters 1–3 of [KM98] is a good start, and [Rei87] is excellent for 3-dimensional terminal singularities. Looking at [KSC04] before reading Chapter 6 may also be helpful. [CR00] and [Kol19] treat results related to Chapter 9. For flips and abundance, there are more details in [Kol92], though it is somewhat dated.

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[2] This is one of Reid's many additions to the vocabulary of birational geometry.

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János Kollár
Department of Mathematics,
Princeton University
Email address: kollar@math.princeton.edu