

## SPECTRAL THEORY AND SELF-SIMILAR BLOWUP IN WAVE EQUATIONS

ROLAND DONNINGER

ABSTRACT. This is an expository article that describes the spectral-theoretic aspects in the study of the stability of self-similar blowup for nonlinear wave equations. The linearization near a self-similar solution leads to a genuinely nonself-adjoint operator which is difficult to analyze. The main goal of this article is to provide an accessible account of the only known method that is capable of providing sufficient spectral information to complete the stability analysis. The exposition is based on a mini course given at the *Summer School on Geometric Dispersive PDEs* in Obergurgl, Austria, in September 2022.

### 1. INTRODUCTION

Nonlinear wave equations play a fundamental role in many branches of the natural sciences and mathematics. Probably the most famous examples in physics are the Einstein equation of general relativity and the Yang–Mills equations of particle physics. What all of these fundamental equations have in common is the fact that they are *energy-supercritical* (in the case of Yang–Mills in spatial dimensions larger than four). This means that the known conserved quantities (most notably the energy) are not strong enough to control the evolution. As a result, the mathematical understanding of large-data evolutions is still embarrassingly poor. In many cases, however, there exist self-similar solutions and one may learn something about the general large-data behavior by looking at perturbations of these large but special solutions. This approach is promising because it allows one to employ perturbative techniques in a large-data regime that is otherwise inaccessible to rigorous mathematical analysis. Such a perturbative treatment involves a number of interesting spectral-theoretic aspects that are at the center of this article.

**1.1. Wave maps.** For the purpose of this exposition we will not discuss nonlinear wave equations in any kind of generality but rather focus on a particular example: the classical *SU(2)-sigma model* from particle physics, also known as the *wave maps equation*, which constitutes the simplest and prototypical example of a *geometric wave equation*. The methods we discuss, however, have a much broader scope and we mention applications to other problems in the end. In order to introduce the model, we consider maps  $U : \mathbb{R}^{1,3} \rightarrow \mathbb{S}^3 \subset \mathbb{R}^4$ , where  $\mathbb{R}^{1,3}$  denotes the  $(1+3)$ -dimensional Minkowski space. Then  $U$  is called a wave map if it satisfies the partial differential equation

$$(1.1) \quad \partial^\mu \partial_\mu U + \langle \partial_\mu U, \partial^\mu U \rangle_{\mathbb{R}^4} U = 0.$$

---

This work was supported by the Austrian Science Fund FWF, Project P 34560: “Stable blowup in supercritical wave equations”.

Here, we employ standard relativistic notation with Einstein’s summation convention in force<sup>1</sup> and  $\langle \cdot, \cdot \rangle_{\mathbb{R}^4}$  denotes the Euclidean inner product on  $\mathbb{R}^4$ . The wave maps equation arises as the Euler–Lagrange equation of the action functional

$$(1.2) \quad U \mapsto \int_{\mathbb{R}^{1,3}} \langle \partial^\mu U, \partial_\mu U \rangle_{\mathbb{R}^4}$$

under the constraint that  $U(t, x) \in \mathbb{S}^3$  for all  $(t, x) \in \mathbb{R}^{1,3}$ . Note that without the constraint, the Euler–Lagrange equation associated to equation (1.2) is the standard free wave equation  $\partial^\mu \partial_\mu U = 0$ . In this sense, wave maps are natural generalizations of solutions to the wave equation when the unknown takes values in the sphere. In place of Minkowski space and the three-sphere, one may also consider more general manifolds by adapting the functional (1.2) accordingly. This shows that the wave maps action is a rich source for interesting and natural geometric wave equations. In this exposition, for the sake of concreteness, we restrict ourselves to maps from  $\mathbb{R}^{1,3}$  to  $\mathbb{S}^3$ . We remark in passing that in more traditional notation, equation (1.1) would read

$$\begin{aligned}
 & -\partial_t^2 U(t, x) + \Delta_x U(t, x) \\
 & = \left( \langle \partial_t U(t, x), \partial_t U(t, x) \rangle_{\mathbb{R}^4} - \sum_{j=1}^3 \langle \partial_{x_j} U(t, x), \partial_{x_j} U(t, x) \rangle_{\mathbb{R}^4} \right) U(t, x)
 \end{aligned}$$

but in this form, the underlying geometric structure is severely obscured.

**1.2. Corotational wave maps and singularity formation.** The most basic question concerns the existence of smooth solutions to equation (1.1). For the sake of simplicity, we further restrict our attention to the special class of *corotational maps* which are of the form

$$U(t, x) = \begin{pmatrix} \sin(u(t, |x|)) \frac{x}{|x|} \\ \cos(u(t, |x|)) \end{pmatrix}$$

for an auxiliary function  $u : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ . This ansatz turns out to be compatible with the wave maps equation, i.e., when plugging it in, we obtain the single semilinear radial wave equation

$$(1.3) \quad \left( \partial_t^2 - \partial_r^2 - \frac{2}{r} \partial_r \right) u(t, r) + \frac{\sin(2u(t, r))}{r^2} = 0.$$

The principal goal is to construct global solutions and since equation (1.3) is a wave equation, the natural mathematical setting to approach this question is to study the *Cauchy problem*, i.e., we prescribe *initial data*  $u(0, \cdot)$  and  $\partial_0 u(0, \cdot)$  and try to construct a solution to equation (1.3) with these data. However,

$$u^T(t, r) := 2 \arctan\left(\frac{r}{T-t}\right)$$

for any  $T \in \mathbb{R}$  solves equation (1.3) on  $\mathbb{R} \times [0, \infty) \setminus \{(T, 0)\}$ , as one checks by a direct computation. At  $(t, r) = (T, 0)$ ,  $u^T$  exhibits a gradient blowup and hence, it is impossible to construct global smooth solutions for arbitrary data. Consequently,

---

<sup>1</sup>That is to say, we number the slots of a function on  $\mathbb{R}^{1,3}$  from 0 to 3 where the 0th slot holds the time variable. The partial derivative with respect to the  $\mu$ th slot is denoted by  $\partial_\mu$  and we write  $\partial^0 := -\partial_0$  as well as  $\partial^j := \partial_j$  for  $j \in \{1, 2, 3\}$ . Furthermore, indices that come in pairs of subscripts and superscripts get summed over implicitly. Greek (spacetime) indices run from 0 to 3 and latin (spatial) indices run from 1 to 3.

the goal is to understand the nature of this breakdown (or “loss of smoothness” or “singularity formation” or “blowup”) and its relevance for “generic” evolutions. More precisely, the question is whether  $u^T$  can tell us something about more general large-data evolutions, even though it is just one particular solution. In other words, we are interested in stability properties of  $u^T$ , i.e., we would like to understand *all* solutions that are close to  $u^T$ . We remark that  $u^T$  is a *self-similar solution*, i.e., it depends on the ratio  $\frac{r}{T-t}$  only. The existence of self-similar solutions to equation (1.3) was first proved in [34] and the explicit example  $u^T$  was found in [37]; see [4] for higher dimensions. In fact, there are many more self-similar solutions to equation (1.3) (see [2]) but they are all linearly unstable and hence less important for studying generic evolutions.

## 2. THE MODE STABILITY PROBLEM

If the self-similar solution  $u^T$  has any relevance for generic large-data evolutions, it certainly must be stable under perturbations of the initial data. Thus, an important mathematical goal is to prove (or disprove) the stability of  $u^T$ . The most elementary form of stability is *mode stability*. The formulation of the mode stability problem can be given purely on the level of the differential equation and requires no operator-theoretic framework.

**2.1. Similarity coordinates.** In order to introduce the mode stability problem, we start with the wave maps equation (1.3) and switch to *similarity coordinates*

$$(2.1) \quad \tau = -\log(T - t) + \log T, \quad \rho = \frac{r}{T - t},$$

or

$$t = T - Te^{-\tau}, \quad r = Te^{-\tau} \rho,$$

where  $T > 0$  is a parameter. Then  $u$  satisfies equation (1.3) if and only if  $v_T(\tau, \rho) := u(T - Te^{-\tau}, Te^{-\tau} \rho)$  satisfies

$$(2.2) \quad \left[ \partial_\tau^2 + 2\rho \partial_\tau \partial_\rho + \partial_\tau - (1 - \rho^2) \partial_\rho^2 + \left( 2\rho - \frac{2}{\rho} \right) \partial_\rho \right] v_T(\tau, \rho) + \frac{\sin(2v_T(\tau, \rho))}{\rho^2} = 0.$$

Observe the remarkable fact that equation (2.2) is an autonomous equation, i.e., its coefficients do not depend on  $\tau$ . This is, in fact, a decisive feature of the similarity coordinates (2.1). Furthermore, the parameter  $T$  does not show up in equation (2.2). To begin with, we will consider equation (2.2) in the coordinate range  $\tau \geq 0$  and  $\rho \in [0, 1]$ , which corresponds to the backward lightcone of the point  $(T, 0)$  in the “physical” coordinates  $(t, r)$ .

The blowup solution  $u^{T'}(t, r) = 2 \arctan(\frac{r}{T'-t})$  transforms into

$$v_T^{T'}(\tau, \rho) := u^{T'}(T - Te^{-\tau}, Te^{-\tau} \rho) = 2 \arctan \left( \frac{\rho}{1 + (\frac{T'}{T} - 1)e^\tau} \right).$$

We would like to understand the stability of the family  $\{v_T^{T'} : T' > 0\}$ . First, let us point out that  $v_T^T$  is independent of  $\tau$  whereas nearby members of the family move away from  $v_T^T$  as  $\tau$  increases. Indeed, if  $T' < T$ ,  $v_T^{T'}(\tau, \cdot)$  develops a gradient blowup as  $\tau \rightarrow \tau_*$ , where  $\tau_*$  is determined by  $(\frac{T'}{T} - 1)e^{\tau_*} = -1$ . On the other hand, if  $T' > T$ ,  $v_T^{T'}(\tau, \rho) \rightarrow 0$  as  $\tau \rightarrow \infty$ . By these observations, it is expected that the  $\tau$ -independent solution  $v_T^T$  is unstable because a generic perturbation will

push it towards a nearby member of the family. However, such a “push” can be compensated by adapting  $T$ . Thus, the instability is “artificial” and caused by the free parameter  $T$  in the definition of the similarity coordinates or, on a more fundamental level, by the time-translation invariance of the wave maps equation. In other words, stability of the blowup means that for any given (small) initial perturbation of  $u^1$ , say, there exists a  $T$  close to 1 that makes the evolution in similarity coordinates with parameter  $T$  converge to  $v_T^T$ . This is very natural in view of the expectation that a perturbation of a blowup solution will, in general, change the blowup time.

**2.2. Mode solutions.** The most elementary stability analysis consists of looking for *mode solutions*. This means that one plugs in the ansatz

$$v_T(\tau, \rho) = v_T^T(\rho) + e^{\lambda\tau} f(\rho), \quad \lambda \in \mathbb{C}$$

into equation (2.2) and linearizes in  $f$ . This yields the “spectral problem” (2.3)

$$-(1 - \rho^2)f''(\rho) - \frac{2}{\rho}f'(\rho) + 2(\lambda + 1)\rho f'(\rho) + \frac{2 \cos(2v_T^T(\rho))}{\rho^2}f(\rho) + \lambda(\lambda + 1)f(\rho) = 0.$$

Clearly, if there are “admissible” mode solutions with  $\text{Re } \lambda > 0$ , we expect the solution  $v_T^T$  to be unstable. What exactly “admissible” in this context means can only be answered once one has set up the functional analytic framework to study the wave maps evolution. For now we will restrict ourselves to smooth solutions and we will see later that this is the correct class of functions. Furthermore, observe that equation (2.3) has singular points at  $\rho = 0$  and  $\rho = 1$  and, therefore, it is expected that only for special values of  $\lambda$  there will be nontrivial solutions in  $C^\infty([0, 1])$ . Another important fact is that equation (2.3) does not constitute a standard eigenvalue problem because the spectral parameter  $\lambda$  appears in the coefficient of the derivative  $f'$  as well. This is easily traced back to the fact that the wave maps equation is second-order in time. Consequently, this issue is not present in analogous parabolic problems where the corresponding spectral analysis is therefore much simpler. Of course, the first-order term can always be removed but the corresponding transformation depends on  $\lambda$  itself. As a consequence, it turns out that equation (2.3) is *not* a self-adjoint Sturm–Liouville problem in disguise where standard methods from mathematical physics would apply. We discuss this in more detail below.

We have already argued that we expect an “artificial” instability of  $v_T^T$ . So how does this instability show up in the context of the spectral problem equation (2.3)? To see this, we differentiate the equation

$$\left[ \partial_\tau^2 + 2\rho\partial_\tau\partial_\rho + \partial_\tau - (1 - \rho^2)\partial_\rho^2 + \left(2\rho - \frac{2}{\rho}\right)\partial_\rho \right] v_T^{T'}(\tau, \rho) + \frac{\sin(2v_T^T(\tau, \rho))}{\rho^2} = 0$$

with respect to  $T'$  and evaluate the result at  $T' = T$ . This yields

$$\left[ \partial_\tau^2 + 2\rho\partial_\tau\partial_\rho + \partial_\tau - (1 - \rho^2)\partial_\rho^2 + \left(2\rho - \frac{2}{\rho}\right)\partial_\rho \right] v_*(\tau, \rho) + \frac{2 \cos(2v_T^T(\rho))}{\rho^2} v_*(\tau, \rho) = 0$$

with

$$v_*(\tau, \rho) := \partial_{T'} v_T^{T'}(\tau, \rho)|_{T=T'} = -\frac{2}{T} e^\tau \frac{\rho}{1 + \rho^2}.$$

Observe that  $v_*$  is a mode solution. Consequently, the function  $\rho \mapsto \frac{\rho}{1+\rho^2}$  solves equation (2.3) with  $\lambda = 1$  and this is the mode solution that reflects the expected “artificial” instability. This observation naturally leads to the following definition.

**Definition 2.1.** We say that the blowup solution  $u^T$  is *mode stable*<sup>2</sup> if the existence of a nontrivial  $f \in C^\infty([0, 1])$  that satisfies equation (2.3) necessarily implies that  $\text{Re } \lambda < 0$  or  $\lambda = 1$ .

In what follows, we somewhat imprecisely call  $\lambda \in \mathbb{C}$  an *eigenvalue* of equation (2.3) if equation (2.3) has a nontrivial solution in  $C^\infty([0, 1])$ . Accordingly, we call such a solution an *eigenfunction* of equation (2.3).

### 3. SOLUTION OF THE MODE STABILITY PROBLEM

In this section, which is at the heart of the present exposition, we describe an approach to the mode stability problem that was developed in Irfan Glogić’s PhD thesis [27] and first published in [9, 10], building on earlier work [8, 15, 22] and ideas in [2, 3, 5]. So far, it is the only known method that can rigorously deal with spectral problems like equation (2.3).

**Theorem 3.1.** *The blowup solution  $u^T$  is mode stable.*

The proof of mode stability proceeds by the following main steps.

- We use Frobenius’s method to determine the local behavior of solutions to equation (2.3) near the singular points  $\rho = 0$  and  $\rho = 1$ .
- By a factorization procedure inspired by supersymmetric quantum mechanics we “remove” the eigenvalue  $\lambda = 1$ . More precisely, we derive a “supersymmetric problem”, similar to equation (2.3), that has the same eigenvalues as equation (2.3) except for  $\lambda = 1$ .
- We prove that the supersymmetric problem has no eigenvalues in the closed complex right half-plane. To this end, we derive a recurrence relation for the coefficients of the power series of the admissible solution near  $\rho = 0$  and prove that the series necessarily diverges at  $\rho = 1$  if  $\text{Re } \lambda \geq 0$ . This requires the interplay of techniques from the theory of difference equations and complex analysis.

**3.1. Fuchsian classification.** To begin with, we would like to understand better which problem we are actually facing. The term in equation (2.3) involving the cosine turns out to be a rational function. Indeed, we have

$$2 \cos(2v_T^T(\rho)) = 2 \cos(4 \arctan(\rho)) = 2 \frac{1 - 6\rho^2 + \rho^4}{(1 + \rho^2)^2}$$

and thus, equation (2.3) has the (regular) singular points  $0, \pm 1, \pm i, \infty$ . By switching to the independent variable  $\rho^2$ , the number of singular points can be reduced to four:  $0, \pm 1$ , and  $\infty$ . This means that equation (2.3) is a Fuchsian differential equation of Heun type. The normal form for a Heun equation reads

$$g''(z) + \left[ \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right] g'(z) + \frac{\alpha\beta z - q}{z(z-1)(z-a)} g(z) = 0$$

---

<sup>2</sup>The experienced reader might think ahead and be worried about spectral multiplicities. It turns out that this is never an issue in the class of problems we consider here and, therefore, Definition 2.1 is the “correct” one. At this point we cannot even discuss multiplicities because we do not yet have a proper operator-theoretic framework.

where  $\alpha, \beta, \gamma, \delta, \epsilon, a, q \in \mathbb{C}$ . Around each of the singular points there exist two linearly independent local solutions. The interesting question then is how local solutions around different singular points are related to each other. This is known as the *connection problem* and, unfortunately for Heun equations, this problem is widely open. If we had only three regular singular points we would be dealing with a *hypergeometric differential equation* for which the connection problem was solved in the 19th century. This indicates that the spectral problem we are dealing with is potentially hard.

**3.2. Frobenius analysis.** Now we turn to a more quantitative analysis and first recall Frobenius' theory for Fuchsian equations of second order. These are equations over the complex numbers of the form

$$(3.1) \quad f''(z) + p(z)f'(z) + q(z)f(z) = 0$$

where  $p$  and  $q$  are given functions and  $f$  is the unknown. In the following, we write  $\mathbb{D}_R := \{z \in \mathbb{C} : |z| < R\}$ .

**Theorem 3.2.** *Let  $R > 0$  and let  $p, q : \mathbb{D}_R \setminus \{0\} \rightarrow \mathbb{C}$  be holomorphic. Suppose that the limits*

$$p_0 := \lim_{z \rightarrow 0} [zp(z)], \quad q_0 := \lim_{z \rightarrow 0} [z^2q(z)]$$

*exist and let  $s_{\pm} \in \mathbb{C}$  satisfy  $P(s_{\pm}) = 0$ , where*

$$P(s) := s(s - 1) + p_0s + q_0$$

*is the indicial polynomial. Let  $\operatorname{Re} s_+ \geq \operatorname{Re} s_-$ . Then there exists a holomorphic function  $h_+ : \mathbb{D}_R \rightarrow \mathbb{C}$  with  $h_+(0) = 1$  and such that  $f : \mathbb{D}_R \setminus (-\infty, 0] \rightarrow \mathbb{C}$ , given by  $f(z) = z^{s_+}h_+(z)$ , satisfies equation (3.1). Furthermore, if  $s_+ - s_- \notin \mathbb{N}_0$ , there exists a holomorphic function  $h_- : \mathbb{D}_R \rightarrow \mathbb{C}$  with  $h_-(0) = 1$  and such that  $f(z) = z^{s_-}h_-(z)$  is another solution of equation (3.1) on  $\mathbb{D}_R \setminus (-\infty, 0]$ . Finally, if  $s_+ - s_- \in \mathbb{N}_0$ , there exist  $c \in \mathbb{C}$  and a holomorphic function  $h_- : \mathbb{D}_R \rightarrow \mathbb{C}$  with  $h_-(0) = 1$  such that*

$$f(z) = z^{s_-}h_-(z) + cz^{s_+}h_+(z) \log z$$

*is another solution of equation (3.1) on  $\mathbb{D}_R \setminus (-\infty, 0]$ .*

*Idea of proof.* The idea is to plug in a generalized power series ansatz  $z^\sigma \sum_{k=0}^\infty a_k z^k$  and to determine  $\sigma$  and the coefficients  $(a_k)_{k \in \mathbb{N}_0}$  by comparing powers of  $z$ . The convergence of the corresponding series is then shown by a simple induction. The second solution can be obtained by the reduction of order ansatz. We remark in passing that even in the case  $s_+ - s_- \in \mathbb{N}_0$ , the log term may be absent but this depends on the fine structure and needs to be analyzed on a case-by-case basis. We omit the details of the proof because Theorem 3.2 is a classical result that can be found in many textbooks; see, e.g., [36] for a modern account. □

Slightly rearranged, equation (2.3) reads

$$(3.2) \quad f''(\rho) + 2 \frac{1 - (\lambda + 1)\rho^2}{\rho(1 - \rho^2)} f'(\rho) - \left[ V(\rho) + \frac{\lambda(\lambda + 1)}{1 - \rho^2} \right] f(\rho) = 0$$

with

$$V(\rho) := 2 \frac{1 - 6\rho^2 + \rho^4}{\rho^2(1 - \rho^2)(1 + \rho^2)^2}$$

and the indicial polynomial at  $\rho = 0$  reads  $s(s - 1) + 2s - 2$  with zeros 1 and  $-2$ . As expected, there is only one smooth solution around  $\rho = 0$  and it behaves like  $\rho$ . At  $\rho = 1$ , the indicial polynomial is given by  $s(s - 1) + \lambda s = 0$  with zeros 0 and  $1 - \lambda$ . Again, there is only one smooth solution around  $\rho = 1$  if  $\text{Re } \lambda \geq 0$  (the cases  $\lambda \in \{0, 1\}$  require some extra care). Thus, our goal is to show that the local solution that is smooth around  $\rho = 0$  is necessarily nonsmooth at  $\rho = 1$  if  $\text{Re } \lambda \geq 0$  (and  $\lambda \neq 1$ ).

**3.3. Supersymmetric removal.** The case  $\lambda = 1$  is special and we already know that this is an eigenvalue. In order to proceed, it is necessary to “remove” it. This can be achieved by a factorization procedure that has its origin in supersymmetric quantum mechanics (hence the name). In our setting, the procedure is as follows. First, we introduce an auxiliary function  $g$  by  $f(\rho) = p(\rho)g(\rho)$ , where we choose  $p$  in such a way that the resulting equation for  $g$  has no first-order derivative. Indeed, inserting the above ansatz into equation (3.2) yields the condition

$$p'(\rho) = -\frac{1 - (\lambda + 1)\rho^2}{\rho(1 - \rho^2)}p(\rho)$$

which is satisfied, e.g., by  $p(\rho) = \rho^{-1}(1 - \rho^2)^{-\frac{\lambda}{2}}$ . Plugging the ansatz

$$f(\rho) = \rho^{-1}(1 - \rho^2)^{-\frac{\lambda}{2}}g(\rho)$$

into equation (3.2) yields

$$(3.3) \quad g''(\rho) - V(\rho)g(\rho) = \frac{\lambda(\lambda - 2)}{(1 - \rho^2)^2}g(\rho).$$

Recall that the function  $\rho \mapsto \frac{\rho}{1+\rho^2}$  solves equation (3.2) with  $\lambda = 1$ . Thus,

$$g_0(\rho) := (1 - \rho^2)^{\frac{1}{2}} \frac{\rho^2}{1 + \rho^2}$$

satisfies

$$g_0''(\rho) - V(\rho)g_0(\rho) = -\frac{1}{(1 - \rho^2)^2}g_0(\rho).$$

Motivated by this, we rewrite equation (3.3) as

$$(3.4) \quad g''(\rho) + \left[ \frac{1}{(1 - \rho^2)^2} - V(\rho) \right] g(\rho) = \frac{(\lambda - 1)^2}{(1 - \rho^2)^2}g(\rho).$$

This resembles a spectral problem for a Schrödinger operator with a ground state  $g_0$ .

At this point we digress and reiterate that our mode stability problem *cannot* be reduced to studying the spectrum of the self-adjoint realization of the Schrödinger operator in equation (3.4). The reason is that an admissible eigenfunction of equation (2.3) transforms into a solution of equation (3.4) that behaves like  $(1 - \rho)^{\frac{\lambda}{2}}$  near  $\rho = 1$ . However, if  $\text{Re } \lambda \leq 1$ , this function is not in  $L^2_w(0, 1)$  with weight  $w(\rho) = \frac{1}{(1 - \rho^2)^2}$ , which is the natural Hilbert space for equation (3.4). As a consequence, eigenvalues  $\lambda$  with  $\text{Re } \lambda \leq 1$  are “invisible” in the “self-adjoint picture” of equation (3.4).

Nevertheless, we can employ the factorization procedure from supersymmetric quantum mechanics, as this is in fact a pure ODE argument that has nothing to

do with operator theory. To this end, observe that  $g_0$  has no zeros in  $(0, 1)$  and we have the factorization

$$\begin{aligned} \left(\partial_\rho + \frac{g'_0(\rho)}{g_0(\rho)}\right) \left(\partial_\rho - \frac{g'_0(\rho)}{g_0(\rho)}\right) &= \partial_\rho^2 - \partial_\rho \left(\frac{g'_0(\rho)}{g_0(\rho)}\right) - \frac{g'_0(\rho)^2}{g_0(\rho)^2} \\ &= \partial_\rho^2 - \frac{g''_0(\rho)}{g_0(\rho)} \\ &= \partial_\rho^2 + \left[\frac{1}{(1-\rho^2)^2} - V(\rho)\right]. \end{aligned}$$

Consequently, equation (3.4) can be written as

$$(1-\rho^2)^2 \left(\partial_\rho + \frac{g'_0(\rho)}{g_0(\rho)}\right) \left[\left(\partial_\rho - \frac{g'_0(\rho)}{g_0(\rho)}\right) g(\rho)\right] = (\lambda-1)^2 g(\rho).$$

The trick is now to apply the operator  $\partial_\rho - \frac{g'_0(\rho)}{g_0(\rho)}$  to this equation. In terms of

$$\tilde{g}(\rho) := \left(\partial_\rho - \frac{g'_0(\rho)}{g_0(\rho)}\right) g(\rho),$$

the resulting equation reads

$$\left(\partial_\rho - \frac{g'_0(\rho)}{g_0(\rho)}\right) \left[(1-\rho^2)^2 \left(\partial_\rho + \frac{g'_0(\rho)}{g_0(\rho)}\right) \tilde{g}(\rho)\right] = (\lambda-1)^2 \tilde{g}(\rho).$$

Note that

$$\left(\partial_\rho - \frac{g'_0(\rho)}{g_0(\rho)}\right) g_0(\rho) = 0,$$

i.e., the solution that comes from the artificial instability gets annihilated by this transformation. Finally, we write  $\tilde{f}(\rho) = \rho^{-1}(1-\rho^2)^{1-\frac{\lambda}{2}} \tilde{g}(\rho)$  and the equation turns into

$$(3.5) \quad -(1-\rho^2)\tilde{f}''(\rho) - \frac{2}{\rho}\tilde{f}'(\rho) + 2(\lambda+1)\rho\tilde{f}'(\rho) + \frac{2(3-\rho^2)}{\rho^2(1+\rho^2)}\tilde{f}(\rho) + \lambda(\lambda+1)\tilde{f}(\rho) = 0,$$

which has the exact same structure as equation (2.3) but with a different “potential”. Based on the above, we have the following correspondence result.

**Lemma 3.3.** *Let  $\lambda \in \mathbb{C} \setminus \{1\}$  and suppose that there exists a nontrivial  $f \in C^\infty([0, 1])$  that satisfies equation (2.3). Then there exists a nontrivial  $\tilde{f} \in C^\infty([0, 1])$  that satisfies equation (3.5).*

*Proof.* Given  $f$ , we set

$$\tilde{f}(\rho) := \rho^{-1}(1-\rho^2)^{1-\frac{\lambda}{2}} \left(\partial_\rho - \frac{2-3\rho^2-\rho^4}{\rho(1-\rho^2)(1+\rho^2)}\right) \left[\rho(1-\rho^2)^{\frac{\lambda}{2}} f(\rho)\right]$$

and since

$$\frac{g'_0(\rho)}{g_0(\rho)} = \frac{2-3\rho^2-\rho^4}{\rho(1-\rho^2)(1+\rho^2)},$$

the above derivation shows that  $\tilde{f}$  is nontrivial (here  $\lambda \neq 1$  is used) and satisfies equation (3.5). The fact that  $\tilde{f} \in C^\infty([0, 1])$  follows by inspection because  $f(\rho)$  behaves like  $\rho$  near 0 by Frobenius’s method. □



**3.4. Transformation to standard Heun form.** equation (3.5) is again of Heun type. To see this, we first observe that the indicial polynomial of equation (3.5) at  $\rho = 0$  is  $s(s - 1) + 2s - 6$  with zeros 2 and  $-3$ . At  $\rho = 1$  we have, as with the original equation,  $s(s - 1) + \lambda s$  with zeros 0 and  $1 - \lambda$ . In order to obtain the standard Heun form, one of the indices at each of the singular points must equal zero. Thus, we introduce new variables by writing  $\tilde{f}(\rho) = \rho^2 \hat{f}(\rho^2)$ . Then  $\tilde{f}$  satisfies equation (3.5) if and only if  $\hat{f}$  satisfies the Heun equation

$$(3.6) \quad \hat{f}''(x) + \left( \frac{7}{2x} + \frac{\lambda}{x-1} \right) \hat{f}'(x) + \frac{1}{4} \frac{(\lambda+3)(\lambda+2)x + \lambda^2 + 5\lambda - 2}{x(x-1)(x+1)} \hat{f}(x) = 0.$$

The domain we are interested in is  $x \in [0, 1]$  (which corresponds to  $\rho \in [0, 1]$ ). However, as will become clear below, the fact that the singularity at  $x = -1$  has the same distance from 0 as the singularity at  $x = 1$  spoils our analysis. For this reason, we need to move it, which is possible by the Möbius transform  $x \mapsto \frac{2x}{x+1}$ , which maps 0 to 0, 1 to 1,  $-1$  to  $\infty$ , and  $\infty$  to 2. Then 1 is the only singularity within distance 1 from 0. We note that this transformation was introduced in the present context in [3]. Upon writing

$$\hat{f}(x) = \left( 2 - \frac{2x}{x+1} \right)^{1+\frac{\lambda}{2}} g \left( \frac{2x}{x+1} \right),$$

we finally arrive at the Heun equation (3.7)

$$g''(z) + \left( \frac{7}{2z} + \frac{\lambda}{z-1} + \frac{1}{2(z-2)} \right) g'(z) + \frac{1}{4} \frac{(\lambda+4)(\lambda+2)z - (\lambda^2 + 12\lambda + 12)}{z(z-1)(z-2)} g(z) = 0.$$

Tracing back the above derivation, we obtain Lemma 3.4.

**Lemma 3.4.** *Let  $\lambda \in \mathbb{C} \setminus \{1\}$  and suppose that there exists a nontrivial  $f \in C^\infty([0, 1])$  that satisfies equation (2.3). Then there exists a nontrivial  $g \in C^\infty([0, 1])$  that satisfies equation (3.7).*

**3.5. The recurrence relation.** The indicial polynomial of equation (3.7) at  $z = 0$  is  $s(s - 1) + \frac{7}{2}s$  with zeros 0 and  $-\frac{5}{2}$ . At  $z = 1$  we have  $s(s - 1) + \lambda s = 0$  with zeros 0 and  $1 - \lambda$ . Thus, a solution  $g \in C^\infty([0, 1])$  is holomorphic around both  $z = 0$  and  $z = 1$ . The “next” singularity in equation (3.7) is at  $z = 2$  and thus, a solution  $g \in C^\infty([0, 1])$  is in fact holomorphic on  $\mathbb{D}_2$ . Note that this line of reasoning would fail for equation (3.6) because of the singularity at  $x = -1$ . Since the power series representation of a function that is holomorphic on a disc converges on that very disc, we see that a solution  $g \in C^\infty([0, 1])$  of equation (3.7) can be represented by a power series centered at  $z = 0$  with radius of convergence at least 2. Thus, the idea is to insert a power series ansatz, obtain a recurrence relation for the coefficients and then prove that the radius of convergence equals 1 if  $\text{Re } \lambda \geq 0$ . The reduction of the mode stability problem to the convergence properties of the corresponding power series is from [3], which also provides convincing numerical evidence for mode stability.

Concretely, from Frobenius’s theory we know that there exists a solution

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

to equation (3.7), where the power series has radius of convergence at least 1. Thus,

$$g'(z) = \sum_{n=0}^{\infty} (n+1)a_{n+1}z^n$$

and

$$g''(z) = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}z^n.$$

By inserting this into equation (3.7), rewritten as

$$z(z-1)(z-2)g''(z) + \left[\frac{7}{2}(z-1)(z-2) + \lambda z(z-2) + \frac{1}{2}z(z-1)\right]g'(z) + \frac{1}{4}[(\lambda+4)(\lambda+2)z - (\lambda^2 + 12\lambda + 12)]g(z) = 0,$$

we obtain

$$\begin{aligned} 0 = & (z^3 - 3z^2 + 2z) \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}z^n \\ & + [(\lambda+4)z^2 - (2\lambda+11)z + 7] \sum_{n=0}^{\infty} (n+1)a_{n+1}z^n \\ & + \frac{1}{4} [(\lambda+4)(\lambda+2)z - (\lambda^2 + 12\lambda + 12)] \sum_{n=0}^{\infty} a_n z^n \end{aligned}$$

and balancing the powers of  $z$ , we find

$$\begin{aligned} 0 = & \sum_{n=-1}^{\infty} \left[7(n+2)a_{n+2} - \frac{1}{4}(\lambda^2 + 12\lambda + 12)a_{n+1}\right] z^{n+1} \\ & + \sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2} - (2\lambda+11)(n+1)a_{n+1} + \frac{1}{4}(\lambda+4)(\lambda+2)a_n\right] z^{n+1} \\ & + \sum_{n=1}^{\infty} [-3(n+1)na_{n+1} + (\lambda+4)na_n] z^{n+1} + \sum_{n=2}^{\infty} n(n-1)a_n z^{n+1}. \end{aligned}$$

By setting  $a_{-1} = 0$ , we can start all sums at  $n = -1$  and we arrive at the recurrence relation

$$(3.8) \quad a_{n+2} = A_n(\lambda)a_{n+1} + B_n(\lambda)a_n$$

for  $n \in \{-1\} \cup \mathbb{N}_0$  and with

$$\begin{aligned} A_n(\lambda) &:= \frac{12n^2 + (8\lambda + 56)n + \lambda^2 + 20\lambda + 56}{8n^2 + 52n + 72}, \\ B_n(\lambda) &:= -\frac{4n^2 + (4\lambda + 12)n + \lambda^2 + 6\lambda + 8}{8n^2 + 52n + 72}. \end{aligned}$$

In order to start the recurrence, we choose the initial condition  $a_0 = 1$ . This freedom comes from the fact that we are solving a linear differential equation with a one-parameter family of solutions.

**3.6. Properties of the coefficients.** As a first and easy observation we can now rule out the existence of polynomial solutions.

**Lemma 3.5.** *Let  $\operatorname{Re} \lambda \geq 0$  and suppose that  $g : \mathbb{C} \rightarrow \mathbb{C}$  is a polynomial that satisfies equation (3.7). Then  $g = 0$ .*

*Proof.* Since  $g$  is a polynomial, there exists an  $N \in \mathbb{N}_0$  and coefficients  $(a_n)_{n=0}^N$  such that

$$g(z) = \sum_{n=0}^N a_n z^n.$$

Furthermore, by the above, the coefficients  $a_n$  satisfy the recurrence relation equation (3.8). Now observe that  $B_n(\lambda) = 0$  if and only if  $\lambda \in \{-2(n + 1), -2(n + 2)\}$  and thus,  $B_n(\lambda) \neq 0$  for all  $n \in \mathbb{N}_0$  and all  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda \geq 0$ . This implies that

$$a_n = -\frac{A_n(\lambda)}{B_n(\lambda)} a_{n+1} + \frac{1}{B_n(\lambda)} a_{n+2}$$

for all  $n \in \mathbb{N}_0$  and since  $a_n = 0$  for all  $n > N$ , we conclude that  $a_n = 0$  for all  $n \in \mathbb{N}_0$ . Consequently,  $g = 0$ . □

Next, we turn to the asymptotic behavior of the coefficients. More precisely, we are interested in the convergence radius of the series  $\sum_{n=0}^\infty a_n z^n$  and thus, we need to understand the asymptotic behavior of the ratio  $\frac{a_{n+1}}{a_n}$ . To begin with, we fix notation.

**Definition 3.6.** For  $\lambda \in \mathbb{C}$  the sequence  $(a_n(\lambda))_{n \in \mathbb{N}_0}$  is defined recursively by  $a_{-1}(\lambda) = 0$ ,  $a_0(\lambda) = 1$ , and

$$a_{n+2}(\lambda) = A_n(\lambda) a_{n+1}(\lambda) + B_n(\lambda) a_n(\lambda)$$

for  $n \in \{-1\} \cup \mathbb{N}_0$ .

**Lemma 3.7.** *Let  $\operatorname{Re} \lambda \geq 0$ . Then we have*

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}(\lambda)}{a_n(\lambda)} \in \left\{ \frac{1}{2}, 1 \right\}.$$

*Proof.* We have

$$\lim_{n \rightarrow \infty} A_n(\lambda) = \frac{3}{2}, \quad \lim_{n \rightarrow \infty} B_n(\lambda) = -\frac{1}{2},$$

and  $s^2 - \frac{3}{2}s + \frac{1}{2} = 0$  if and only if  $s \in \{\frac{1}{2}, 1\}$ . Consequently, by Poincaré’s theorem on difference equations (Theorem A.3) we either have

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}(\lambda)}{a_n(\lambda)} \in \left\{ \frac{1}{2}, 1 \right\}$$

or there exists an  $N \in \mathbb{N}$  such that  $a_n = 0$  for all  $n \geq N$ , but the latter is ruled out by Lemma 3.5. □

If  $\lim_{n \rightarrow \infty} \frac{a_{n+1}(\lambda)}{a_n(\lambda)} = \frac{1}{2}$ , the radius of convergence of the series  $\sum_{n=0}^\infty a_n(\lambda) z^n$  equals 2 and in particular,  $\sum_{n=0}^\infty a_n(\lambda) z^n$  is a solution to equation (3.7) that belongs to  $C^\infty([0, 1])$ . This is precisely the case we want to rule out. Consequently, our goal is to show that  $\lim_{n \rightarrow \infty} \frac{a_{n+1}(\lambda)}{a_n(\lambda)} = 1$ .

Whenever  $a_n(\lambda) \neq 0$ , we write  $r_n(\lambda) := \frac{a_{n+1}(\lambda)}{a_n(\lambda)}$ . With this notation, the recurrence relation equation (3.8) reads

$$r_{n+1}(\lambda) = \frac{a_{n+2}(\lambda)}{a_{n+1}(\lambda)} = \frac{A_n(\lambda)a_{n+1}(\lambda) + B_n(\lambda)a_n(\lambda)}{a_{n+1}(\lambda)} = A_n(\lambda) + \frac{B_n(\lambda)}{r_n(\lambda)}$$

for  $n = 0, 1, 2 \dots$  and we keep in mind that this is only defined as long as  $r_n(\lambda) \neq 0$ . Furthermore, we have the initial condition

$$r_0(\lambda) = \frac{a_1(\lambda)}{a_0(\lambda)} = a_1(\lambda) = A_{-1}(\lambda)a_0(\lambda) + B_{-1}(\lambda)a_{-1}(\lambda) = A_{-1}(\lambda) = \frac{\lambda^2 + 12\lambda + 12}{28}.$$

**3.7. The quasi-solution.** Rephrased in terms of  $r_n$ , our goal is to show that  $\lim_{n \rightarrow \infty} r_n(\lambda) = 1$  if  $\operatorname{Re} \lambda \geq 0$ . The idea is now to achieve this by means of a *quasi-solution*

$$\tilde{r}_n(\lambda) := \frac{\lambda^2}{8n^2 + 33n + 28} + \frac{5\lambda}{5n + 16} + \frac{5n + 6}{5n + 13}, \quad n \in \mathbb{N},$$

which is supposed to approximate  $r_n$  well enough. More precisely, we will show that

$$\left| \frac{r_n(\lambda)}{\tilde{r}_n(\lambda)} - 1 \right| \leq \frac{1}{3}$$

for all  $n \in \mathbb{N}$  and hence, we must have  $\lim_{n \rightarrow \infty} r_n(\lambda) = 1$  because by Lemma 3.7, the only other possibility is  $\lim_{n \rightarrow \infty} r_n(\lambda) = \frac{1}{2}$  which is not compatible with the above bound as  $\lim_{n \rightarrow \infty} \tilde{r}_n(\lambda) = 1$ . In particular, this estimate implies that  $r_n(\lambda) \neq 0$  for all  $n \in \mathbb{N}$  and a posteriori we see that the recursion for  $r_n$  is defined for all  $n \in \mathbb{N}_0$ . Note also that our line of reasoning provides the necessary wiggle room for feasible estimates. Indeed, it is not necessary to prove that  $\lim_{n \rightarrow \infty} \frac{r_n(\lambda)}{\tilde{r}_n(\lambda)} = 1$  directly, which would be next to impossible.

The quasi-solution we use comes out of the blue and finding it involves a bit of art indeed. However, there are some rules of thumb. It is a natural first attempt to look for a quasi-solution that is quadratic in  $\lambda$  because both  $A_n(\lambda)$  and  $B_n(\lambda)$  are quadratic polynomials in  $\lambda$ . Then, with the help of a computer algebra system, one can look at the first few terms of the sequences  $(r_n(0))_{n \in \mathbb{N}_0}$ ,  $(\frac{1}{2}(r_n(1) - r_n(-1)))_{n \in \mathbb{N}_0}$  and  $(\frac{1}{2}(r_n(1) - 2r_n(0) + r_n(-1)))_{n \in \mathbb{N}_0}$  and fit simple rational functions in  $n$ . Sometimes some additional tweaking is necessary. Another approach is based on a careful asymptotic analysis; see the corresponding discussion in [29].

**Lemma 3.8.** *We have  $\tilde{r}_n(\lambda) \neq 0$  for all  $n \in \mathbb{N}$  and  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda \geq 0$ .*

*Proof.* Since  $\tilde{r}_n(\lambda)$  is completely explicit, the proof consists of solving a quadratic equation and is hence omitted. □

**Corollary 3.9.** *Let  $n \in \mathbb{N}$  and  $\Omega := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ . Then the functions*

$$r_1 : \overline{\Omega} \rightarrow \mathbb{C}, \quad \frac{1}{\tilde{r}_n} : \overline{\Omega} \rightarrow \mathbb{C}$$

*are continuous and holomorphic on  $\Omega$ .*

**Definition 3.10.** For  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda \geq 0$  and  $n \in \mathbb{N}_0$ , we set

$$\delta_n(\lambda) := \frac{r_n(\lambda)}{\tilde{r}_n(\lambda)} - 1$$

as well as

$$\epsilon_n(\lambda) := \frac{A_n(\lambda)\tilde{r}_n(\lambda) + B_n(\lambda)}{\tilde{r}_n(\lambda)\tilde{r}_{n+1}(\lambda)} - 1$$

and

$$C_n(\lambda) := \frac{B_n(\lambda)}{\tilde{r}_n(\lambda)\tilde{r}_{n+1}(\lambda)}.$$

Note carefully that  $\epsilon_n$  and  $C_n$  are explicit.

**Lemma 3.11.** *Let  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda \geq 0$ . Then the functions  $\delta_n$  satisfy the recurrence relation*

$$\delta_{n+1}(\lambda) = \epsilon_n(\lambda) - C_n(\lambda) \frac{\delta_n(\lambda)}{1 + \delta_n(\lambda)}$$

for all  $n = 1, 2, \dots$ , as long as  $1 + \delta_n(\lambda) \neq 0$ .

*Proof.* This follows straightforwardly by inserting the definition of  $\delta_n$  and by taking into account that  $r_n$  satisfies the recurrence relation  $r_{n+1} = A_n + \frac{B_n}{r_n}$ . □

Next, we provide quantitative bounds on the functions in play.

**Lemma 3.12.** *We have the bounds*

$$|\delta_1(it)| \leq \frac{1}{3}, \quad |\epsilon_n(it)| \leq \frac{1}{12}, \quad |C_n(it)| \leq \frac{1}{2}$$

for all  $n \in \mathbb{N}$  and all  $t \in \mathbb{R}$ .

*Proof.* These are all bounds on explicit expressions and they can be proved by elementary means. For instance, we have

$$|C_n(it)|^2 = \frac{P_n(t^2)}{Q_n(t^2)}$$

for polynomials  $P_n$  and  $Q_n$ . Consequently, the bound  $|C_n(it)| \leq \frac{1}{2}$  is equivalent to  $Q_n(t^2) - 4P_n(t^2) \geq 0$  and the latter is trivially satisfied because the polynomial  $Q_n(t^2) - 4P_n(t^2)$  turns out to have only nonnegative coefficients for all  $n \in \mathbb{N}$ . We refer to [9] for more details. □

By the Phragmén–Lindelöf principle, the bounds on the imaginary axis extend to the whole complex right half-plane.

**Lemma 3.13.** *We have the bounds*

$$|\delta_1(\lambda)| \leq \frac{1}{3}, \quad |\epsilon_n(\lambda)| \leq \frac{1}{12}, \quad |C_n(\lambda)| \leq \frac{1}{2}$$

for all  $n \in \mathbb{N}$  and all  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda \geq 0$ .

*Proof.* Let  $n \in \mathbb{N}$ . By construction and Corollary 3.9, the functions  $\delta_1, \epsilon_n, C_n$  are continuous on the closed complex right half-plane and holomorphic on the open right half-plane. Furthermore, since  $\delta_1, \epsilon_n$ , and  $C_n$  are rational functions, there exists a  $K_n > 0$  such that

$$|\delta_1(\lambda)| + |\epsilon_n(\lambda)| + |C_n(\lambda)| \leq K_n e^{|\lambda|^{\frac{1}{2}}}$$

for all  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda \geq 0$ . Consequently, Lemma 3.12 and the Phragmén–Lindelöf principle (Lemma A.2) yield the claim. □

Now we can conclude the proof of mode stability by a simple induction.

**Lemma 3.14.** *We have the bound*

$$|\delta_n(\lambda)| \leq \frac{1}{3}$$

for all  $n \in \mathbb{N}$  and all  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda \geq 0$ .

*Proof.* By Lemma 3.13, the claim holds for  $n = 1$ . Assuming that it holds for  $n$ , we find, again by Lemma 3.13,

$$|\delta_{n+1}(\lambda)| \leq |\epsilon_n(\lambda)| + |C_n(\lambda)| \frac{|\delta_n(\lambda)|}{1 - |\delta_n(\lambda)|} \leq \frac{1}{12} + \frac{1}{2} \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{3}$$

and the claim follows inductively.  $\square$

This concludes the proof of mode stability and Theorem 3.1 is established.

#### 4. FUNCTIONAL ANALYTIC SETUP

In this second part, we describe the functional analytic setup for studying the stability of the wave maps blowup. In particular, we will see how the mode stability problem embeds into a proper operator-theoretic framework where it occurs as the effective spectral equation for the nonself-adjoint operator that drives the linearized evolution near the blowup solution.

We remark that our earlier papers, e.g., [12, 22] that implemented this approach for the first time focused on the evolution in the backward lightcone of the singularity. However, it is also possible to treat the problem in the whole space with basically no additional effort. Interestingly, the challenging spectral problems are insensitive to this modification. Furthermore, for the full space problem Fourier methods become available, which simplify things considerably. In particular, the treatment of the free wave evolution in similarity coordinates can be based on the standard wave propagators and this is the approach we present here. A systematic study of blowup stability in the whole space was recently developed in [30] with a slightly different approach that does not make explicit use of the wave propagators but relies on abstract semigroup theory instead.

**4.1. Wave propagators.** To begin with, we recall the standard wave propagators. Our convention for the Fourier transform is

$$(\mathcal{F}f)(y) := \int_{\mathbb{R}^d} e^{-2\pi i y \cdot x} f(x) dx,$$

initially defined on the Schwartz space  $\mathcal{S}(\mathbb{R}^d)$  and by duality extended to the tempered distributions  $\mathcal{S}'(\mathbb{R}^d)$ .

**Definition 4.1.** We define the *scalar gradient* by

$$|\nabla|f := \mathcal{F}^{-1}(2\pi|\cdot|\mathcal{F}f)$$

for  $f \in \mathcal{S}(\mathbb{R}^d)$ .

*Remark 4.2.* Note that  $|\nabla|^2 f = -\Delta f$ .

**Definition 4.3** (Wave propagators). For  $f \in \mathcal{S}(\mathbb{R}^d)$  we set

$$\begin{aligned} \cos(t|\nabla|)f &:= \mathcal{F}^{-1}(\cos(2\pi t|\cdot|)\mathcal{F}f), \\ \frac{\sin(t|\nabla|)}{|\nabla|}f &:= \mathcal{F}^{-1}\left(\frac{\sin(2\pi t|\cdot|)}{2\pi|\cdot|}\mathcal{F}f\right). \end{aligned}$$

Recall that if  $f, g \in \mathcal{S}(\mathbb{R}^d)$ ,

$$u(t, \cdot) := \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g$$

is the unique solution of the Cauchy problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u(t, x) = 0 & \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0, x) = f(x), \quad \partial_0 u(0, x) = g(x) & \text{for } x \in \mathbb{R}^d \end{cases}.$$

Furthermore, recall the homogeneous Sobolev norms

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} := \| |\nabla|^s f \|_{L^2(\mathbb{R}^d)} = \| (2\pi|\cdot|)^s \mathcal{F}f \|_{L^2(\mathbb{R}^d)}, \quad s > -\frac{d}{2}, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

The wave propagators behave very well with respect to these norms.

**Lemma 4.4.** *Let  $s \geq 0$ . Then we have the bounds*

$$\begin{aligned} \|\cos(t|\nabla|)f\|_{\dot{H}^s(\mathbb{R}^d)} &\leq \|f\|_{\dot{H}^s(\mathbb{R}^d)} \\ \left\| \frac{\sin(t|\nabla|)}{|\nabla|}f \right\|_{\dot{H}^{s+1}(\mathbb{R}^d)} &\leq \|f\|_{\dot{H}^s(\mathbb{R}^d)} \end{aligned}$$

for all  $t \in \mathbb{R}$  and  $f \in \mathcal{S}(\mathbb{R}^d)$ .

*Proof.* The proof is just an application of Plancherel’s theorem. □

**4.2. The wave propagators in similarity coordinates.** Next, we switch to the similarity coordinates

$$\tau = -\log(T - t) + \log T, \quad \xi = \frac{x}{T - t}$$

or

$$t = T - Te^{-\tau}, \quad x = Te^{-\tau}\xi.$$

We consider the coordinate range  $\tau \geq 0$  and  $\xi \in \mathbb{R}^d$ . The solution to the wave equation in similarity coordinates is then given by the wave propagators in similarity coordinates.

**Definition 4.5.** For  $f \in \mathcal{S}(\mathbb{R}^d)$ ,  $\tau \geq 0$ , and  $T > 0$ , we set

$$\begin{aligned} [C_T(\tau)f](\xi) &:= [\cos((T - Te^{-\tau})|\nabla|)f](Te^{-\tau}\xi), \\ [S_T(\tau)f](\xi) &:= \left[ \frac{\sin((T - Te^{-\tau})|\nabla|)}{|\nabla|}f \right](Te^{-\tau}\xi). \end{aligned}$$

The crucial observation now is the fact that the wave propagators in similarity coordinates decay exponentially, provided one takes sufficiently many derivatives.

**Lemma 4.6.** *Let  $s \geq 0$ . Then we have the bounds*

$$\begin{aligned} \|C_T(\tau)f\|_{\dot{H}^s(\mathbb{R}^d)} &\leq T^{-\frac{d}{2}+s} e^{(\frac{d}{2}-s)\tau} \|f\|_{\dot{H}^s(\mathbb{R}^d)}, \\ \|S_T(\tau)f\|_{\dot{H}^{s+1}(\mathbb{R}^d)} &\leq T^{-\frac{d}{2}+s+1} e^{(\frac{d}{2}-s-1)\tau} \|f\|_{\dot{H}^s(\mathbb{R}^d)}, \end{aligned}$$

for all  $\tau \geq 0$ ,  $T > 0$ , and  $f \in \mathcal{S}(\mathbb{R}^d)$ .

*Proof.* This follows by a simple scaling argument combined with Lemma 4.4. □

**4.3. Back to the wave maps equation.** Now we return to the wave maps equation (1.3),

$$(4.1) \quad \left( \partial_t^2 - \partial_r^2 - \frac{2}{r} \partial_r \right) u(t, r) + \frac{\sin(2u(t, r))}{r^2} = 0.$$

Despite its appearance, this is not a standard radial nonlinear wave equation because of the singularity at  $r = 0$ . A Taylor expansion shows that smooth solutions of this equation must vanish at  $r = 0$ . This observation motivates the introduction of the new variable  $\tilde{v}(t, r) := \frac{u(t, r)}{r}$ . In terms of  $\tilde{v}$ , equation (4.1) reads

$$\left( \partial_t^2 - \partial_r^2 - \frac{4}{r} \partial_r \right) \tilde{v}(t, r) + \frac{\sin(2r\tilde{v}(t, r)) - 2r\tilde{v}(t, r)}{r^3} = 0.$$

This is now a proper radial semilinear wave equation with a smooth nonlinearity (observe the cancelation in the numerator) but in 5 rather than 3 spatial dimensions. Thus, it is natural to formulate the problem in terms of the function  $v(t, x) := \tilde{v}(t, |x|)$ , where  $v(t, \cdot)$  is a radial function on  $\mathbb{R}^5$ . This leads to the equation

$$(\square v)(t, x) + \frac{\sin(2|x|v(t, x)) - 2|x|v(t, x)}{|x|^3} = 0,$$

where

$$(\square v)(t, x) := (\partial_t^2 - \Delta_x)v(t, x)$$

denotes the *d'Alembertian* or *wave operator*. In terms of the function  $\tilde{w}(\tau, \xi) := v(T - Te^{-\tau}, Te^{-\tau}\xi)$ , we obtain

$$T^{-2}e^{2\tau}\tilde{\square}_{\tau, \xi}\tilde{w}(\tau, \xi) + \frac{\sin(2Te^{-\tau}|\xi|\tilde{w}(\tau, \xi)) - 2Te^{-\tau}|\xi|\tilde{w}(\tau, \xi)}{T^3e^{-3\tau}|\xi|^3} = 0,$$

where  $T^{-2}e^{2\tau}\tilde{\square}_{\tau, \xi}$  is the wave operator in similarity coordinates, i.e.,

$$T^{-2}e^{2\tau}\tilde{\square}_{\tau, \xi}\tilde{w}(\tau, \xi) = (\square v)(T - Te^{-\tau}, Te^{-\tau}\xi).$$

Explicitly, we have

$$\tilde{\square}_{\tau, \xi} = \partial_\tau^2 + 2\xi^j \partial_{\xi^j} \partial_\tau - (\delta^{jk} - \xi^j \xi^k) \partial_{\xi^j} \partial_{\xi^k} + \partial_\tau + 2\xi^j \partial_{\xi^j}.$$

Note that the coefficients of  $\tilde{\square}_{\tau, \xi}$  are independent of  $\tau$ . In order to obtain an autonomous equation, we switch to the variable  $w(\tau, \xi) := Te^{-\tau}\tilde{w}(\tau, \xi)$ . This leads to

$$(4.2) \quad e^{-\tau}\tilde{\square}_{\tau, \xi}(e^\tau w(\tau, \xi)) + \frac{\sin(2|\xi|w(\tau, \xi)) - 2|\xi|w(\tau, \xi)}{|\xi|^3} = 0.$$

Note that

$$\begin{aligned} e^{-\tau}\tilde{\square}_{\tau, \xi}(e^\tau w(\tau, \xi)) &= [\partial_\tau^2 + 2\xi^j \partial_{\xi^j} \partial_\tau - (\delta^{jk} - \xi^j \xi^k) \partial_{\xi^j} \partial_{\xi^k} + 3\partial_\tau + 4\xi^j \partial_{\xi^j} + 2] w(\tau, \xi) \end{aligned}$$

and thus, the parameter  $T$  does not occur and may be formally set to 1. Consequently, the solution of

$$e^{-\tau}\tilde{\square}_{\tau, \xi}(e^\tau w(\tau, \xi)) = 0$$

is given by

$$e^\tau w(\tau, \cdot) = C_1(\tau)w(0, \cdot) + S_1(\tau)[\partial_0 w(0, \cdot) + (\cdot)^j \partial_j w(0, \cdot) + w(0, \cdot)]$$

since  $(\partial_0 v)(0, \xi) = [e^\tau \partial_\tau + e^\tau \xi^j \partial_{\xi^j}](e^\tau w(\tau, \xi))|_{\tau=0}$ . Note carefully that we gain an additional factor of decay.



Recall that we have the static solution

$$w_*(\xi) := \frac{2}{|\xi|} \arctan(|\xi|)$$

which we want to perturb. Thus, we plug in the ansatz  $w(\tau, \xi) = w_*(\xi) + \varphi(\tau, \xi)$  and obtain the equation

$$(4.3) \quad e^{-\tau} \tilde{\square}_{\tau, \xi} (e^\tau \varphi(\tau, \xi)) + \frac{2|\xi| \cos(2|\xi|w_*(\xi))\varphi(\tau, \xi) - 2|\xi|\varphi(\tau, \xi)}{|\xi|^3} + N(\varphi(\tau, \xi), \xi) = 0,$$

where

$$N(y, \xi) := \frac{\sin(2|\xi|(w_*(\xi) + y)) - \sin(2|\xi|w_*(\xi)) - 2|\xi| \cos(2|\xi|w_*(\xi))y}{|\xi|^3}.$$

Note that  $N(y, \xi)$  is quadratic in  $y$  and smooth in  $\xi$ .

**4.4. Semigroup formulation.** By introducing the variable

$$\Phi(\tau)(\xi) = \begin{pmatrix} \varphi(\tau, \xi) \\ (\partial_\tau + \xi^j \partial_{\xi^j} + 1)\varphi(\tau, \xi) \end{pmatrix},$$

equation (4.3) can be written as the first-order system

$$(4.4) \quad \partial_\tau \Phi(\tau) = \widehat{\mathbf{L}}_0 \Phi(\tau) + \mathbf{L}' \Phi(\tau) + \mathbf{N}(\Phi(\tau)),$$

with the *formal* differential operator

$$\widehat{\mathbf{L}}_0 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} := \begin{pmatrix} -\Lambda f_1 - f_1 + f_2 \\ \Delta f_1 - \Lambda f_2 - 2f_2 \end{pmatrix}, \quad (\Lambda f)(\xi) := \xi^j \partial_{\xi^j} f(\xi),$$

and

$$\left[ \mathbf{L}' \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right] (\xi) := \begin{pmatrix} 0 \\ -\frac{2 \cos(2|\xi|w_*(\xi)) - 2}{|\xi|^2} f_1(\xi) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{16}{(1+|\xi|^2)^2} f_1(\xi) \end{pmatrix},$$

as well as

$$\mathbf{N} \left( \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right) (\xi) := \begin{pmatrix} 0 \\ -N(f_1(\xi), \xi) \end{pmatrix}.$$

By construction, the solution of the Cauchy problem  $\partial_\tau \Phi(\tau) = \widehat{\mathbf{L}}_0 \Phi(\tau)$ ,  $\Phi(0) = \mathbf{f} = (f_1, f_2) \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ , is given by

$$\begin{aligned} \Phi(\tau)(\xi) &= \begin{pmatrix} e^{-\tau} [C_1(\tau)f_1](\xi) + e^{-\tau} [S_1(\tau)f_2](\xi) \\ (\partial_\tau + \xi^j \partial_{\xi^j} + 1)[e^{-\tau} C_1(\tau)f_1](\xi) + (\partial_\tau + \xi^j \partial_{\xi^j} + 1)[e^{-\tau} S_1(\tau)f_2](\xi) \end{pmatrix} \\ &=: [\mathbf{S}_0(\tau)\mathbf{f}](\xi). \end{aligned}$$

Note that

$$\begin{aligned} \partial_\tau [C_1(\tau)f](\xi) &= \partial_\tau [\cos((1 - e^{-\tau})|\nabla|)f] (e^{-\tau}\xi) \\ &= -e^{-\tau} [\sin((1 - e^{-\tau})|\nabla|)|\nabla|f] (e^{-\tau}\xi) \\ &\quad - e^{-\tau} \xi^j \partial_j [\cos((1 - e^{-\tau})|\nabla|)f] (e^{-\tau}\xi) \\ &= e^{-\tau} [S_1(\tau)\Delta f] (\xi) - \xi^j \partial_{\xi^j} [C_1(\tau)f] (\xi) \end{aligned}$$

and thus,

$$(\partial_\tau + \xi^j \partial_{\xi^j} + 1) [e^{-\tau} C_1(\tau)f] (\xi) = e^{-2\tau} [S_1(\tau)\Delta f] (\xi).$$

Analogously,

$$(\partial_\tau + \xi^j \partial_{\xi^j} + 1) [e^{-\tau} S_1(\tau)f] (\xi) = e^{-2\tau} [C_1(\tau)f](\xi)$$

and this yields the representation

$$\mathbf{S}_0(\tau)\mathbf{f} = \begin{pmatrix} e^{-\tau}C_1(\tau)f_1 + e^{-\tau}S_1(\tau)f_2 \\ e^{-2\tau}S_1(\tau)\Delta f_1 + e^{-2\tau}C_1(\tau)f_2 \end{pmatrix}.$$

Consequently, by Lemma 4.6, we obtain the bound

$$\|\mathbf{S}_0(\tau)\mathbf{f}\|_{\dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d)} \lesssim e^{(\frac{d}{2}-1-s)\tau} \|\mathbf{f}\|_{\dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-1}(\mathbb{R}^d)}$$

for all  $\mathbf{f} \in \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ ,  $\tau \geq 0$ , and any  $s \geq 0$ . In particular,  $\mathbf{S}_0$  extends to a semigroup on  $\mathcal{H} := (\dot{H}^2(\mathbb{R}^5) \times \dot{H}^1(\mathbb{R}^5)) \cap (\dot{H}^4(\mathbb{R}^5) \times \dot{H}^3(\mathbb{R}^5))$  with operator norm satisfying

$$\|\mathbf{S}_0(\tau)\|_{\mathcal{H}} \lesssim e^{-\frac{1}{2}\tau}$$

for all  $\tau \geq 0$ . Furthermore, it is a simple exercise to show that the map  $\tau \mapsto \mathbf{S}_0(\tau)\mathbf{f} : [0, \infty) \rightarrow \mathcal{H}$  is continuous for any  $\mathbf{f} \in \mathcal{H}$  and the whole abstract machinery of semigroup theory applies. In view of the nonlinear problem, it is also crucial that we have an  $L^\infty$ -embedding of *intersection Sobolev spaces* which implies that  $\mathcal{H}$  is a Banach algebra.

**Lemma 4.7.** *Let  $0 \leq s < \frac{d}{2} < t$  and  $H := \dot{H}^s(\mathbb{R}^d) \cap \dot{H}^t(\mathbb{R}^d)$ . Then we have the Sobolev embedding  $H \hookrightarrow C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$  and*

$$\|fg\|_H \lesssim \|f\|_H \|g\|_H$$

for all  $f, g \in \mathcal{S}(\mathbb{R}^d)$ .

*Proof.* Left as an exercise. □

**4.5. Spectral analysis of the generator.** By construction,

$$\partial_\tau \mathbf{S}_0(\tau)\mathbf{f} = \widehat{\mathbf{L}}_0 \mathbf{S}_0(\tau)\mathbf{f}$$

for all  $\mathbf{f} \in \mathcal{S}(\mathbb{R}^5) \times \mathcal{S}(\mathbb{R}^5) =: \mathcal{D}(\widehat{\mathbf{L}}_0)$  and thus, the generator of  $\mathbf{S}_0$  is the closure  $\mathbf{L}_0 : \mathcal{D}(\mathbf{L}_0) \subset \mathcal{H} \rightarrow \mathcal{H}$  of the operator  $\widehat{\mathbf{L}}_0 : \mathcal{D}(\widehat{\mathbf{L}}_0) \subset \mathcal{H} \rightarrow \mathcal{H}$ . Since  $\mathbf{L}_0$  is an abstract object, we need the following auxiliary result in order to get our hands on the spectral problem for  $\mathbf{L}_0$  and its perturbations.

**Lemma 4.8.** *Let  $\mathbf{f} \in \mathcal{D}(\mathbf{L}_0)$  and set  $\mathbf{g} := (g_1, g_2) := \mathbf{L}_0\mathbf{f}$ . Then  $\mathbf{f} = (f_1, f_2)$  satisfies*

$$\begin{aligned} g_1(\xi) &= -\xi^j \partial_{\xi^j} f_1(\xi) - f_1(\xi) + f_2(\xi), \\ g_2(\xi) &= \Delta f_1(\xi) - \xi^j \partial_{\xi^j} f_2(\xi) - 2f_2(\xi), \end{aligned}$$

in the sense of distributions.

*Proof.* Let  $\mathbf{f} \in \mathcal{D}(\mathbf{L}_0)$ . By definition of the closure, there exists a sequence  $(\mathbf{f}_n)_{n \in \mathbb{N}} \subset \mathcal{D}(\widehat{\mathbf{L}}_0) = \mathcal{S}(\mathbb{R}^5) \times \mathcal{S}(\mathbb{R}^5)$  such that  $\lim_{n \rightarrow \infty} \|\mathbf{f}_n - \mathbf{f}\|_{\mathcal{H}} = 0$  and  $\lim_{n \rightarrow \infty} \|\mathbf{g}_n - \mathbf{g}\|_{\mathcal{H}} = 0$ , where  $\mathbf{g}_n := \widehat{\mathbf{L}}_0 \mathbf{f}_n$ . By definition,

$$\begin{aligned} g_{1n}(\xi) &= -\xi^j \partial_{\xi^j} f_{1n}(\xi) - f_{1n}(\xi) + f_{2n}(\xi), \\ g_{2n}(\xi) &= \Delta f_{1n}(\xi) - \xi^j \partial_{\xi^j} f_{2n}(\xi) - 2f_{2n}(\xi), \end{aligned}$$

where  $\mathbf{f}_n = (f_{1n}, f_{2n})$  and  $\mathbf{g}_n = (g_{1n}, g_{2n})$ . Consequently, by testing these equations with a function in  $C_c^\infty(\mathbb{R}^5)$ , integrating by parts, and taking the limit  $n \rightarrow \infty$ , the claim follows thanks to the Sobolev embedding  $\mathcal{H} \hookrightarrow L^\infty(\mathbb{R}^5) \times L^\infty(\mathbb{R}^5)$ . □

Next, we turn to the full linear operator  $\mathbf{L} := \mathbf{L}_0 + \mathbf{L}'$  with  $\mathcal{D}(\mathbf{L}) := \mathcal{D}(\mathbf{L}_0)$  as it occurs in equation (4.4). Here, we have a nice compactness property.

**Lemma 4.9.** *The operator  $\mathbf{L}' : \mathcal{H} \rightarrow \mathcal{H}$  is compact.*

*Proof.* By definition, we have

$$\mathbf{L}' \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ Vf_1 \end{pmatrix}, \quad V(\xi) := \frac{16}{(1 + |\xi|^2)^2}.$$

Let  $(\mathbf{f}_n)_{n \in \mathbb{N}}$  be a bounded sequence in  $\mathcal{H}$ , where we write  $\mathbf{f}_n = (f_{1n}, f_{2n})$ . Furthermore, let  $\chi : \mathbb{R}^5 \rightarrow [0, 1]$  be a smooth cutoff that satisfies  $\chi(\xi) = 1$  if  $|\xi| \leq 1$  and  $\chi(\xi) = 0$  if  $|\xi| \geq 2$ . By Lemma 4.7, we have

$$\begin{aligned} \|\mathbf{L}'\mathbf{f}_n\|_{\mathcal{H}} &= \|Vf_{1n}\|_{\dot{H}^3(\mathbb{R}^5) \cap \dot{H}^1(\mathbb{R}^5)} \\ &\lesssim \|\chi_k Vf_{1n}\|_{H^3(\mathbb{R}^5)} + \|(1 - \chi_k)Vf_{1n}\|_{\dot{H}^1(\mathbb{R}^5) \cap \dot{H}^3(\mathbb{R}^5)} \\ &\lesssim \|f_{1n}\|_{H^3(\mathbb{B}_{2k}^5)} + \|(1 - \chi_k)Vf_{1n}\|_{\dot{H}^1(\mathbb{R}^5)} + \|(1 - \chi_k)Vf_{1n}\|_{\dot{H}^2(\mathbb{R}^5) \cap \dot{H}^4(\mathbb{R}^5)} \end{aligned}$$

for all  $n, k \in \mathbb{N}$ , where  $\chi_k(\xi) := \chi(\frac{\xi}{k})$ . Now we employ Hardy’s inequality (see, e.g., [32, p. 243, Theorem 9.5]) and the decay of  $V$  to obtain the bound

$$\begin{aligned} \|(1 - \chi_k)Vf_{1n}\|_{\dot{H}^1(\mathbb{R}^5)} &\lesssim \|\nabla[(1 - \chi_k)V]f_{1n}\|_{L^2(\mathbb{R}^5)} + \|(1 - \chi_k)V|\nabla|f_{1n}\|_{L^2(\mathbb{R}^5)} \\ &\lesssim k^{-1} \|\cdot\|^{-2} f_{1n}\|_{L^2(\mathbb{R}^5)} + k^{-1} \|\cdot\|^{-1} |\nabla|f_{1n}\|_{L^2(\mathbb{R}^5)} \\ &\lesssim k^{-1} \|f_{1n}\|_{\dot{H}^2(\mathbb{R}^5)}. \end{aligned}$$

Combined with the above, this leads to the estimate

$$\|\mathbf{L}'\mathbf{f}_n\|_{\mathcal{H}} \lesssim \|\mathbf{f}_n\|_{H^3(\mathbb{B}_{2k}^5) \times H^2(\mathbb{B}_{2k}^5)} + k^{-1} \|\mathbf{f}_n\|_{\mathcal{H}} \lesssim \|\mathbf{f}_n\|_{H^3(\mathbb{B}_{2k}^5) \times H^2(\mathbb{B}_{2k}^5)} + k^{-1}$$

for all  $n, k \in \mathbb{N}$ . By the Sobolev embedding  $\mathcal{H} \hookrightarrow L^\infty(\mathbb{R}^5) \times L^\infty(\mathbb{R}^5)$  and Hölder’s inequality, we obtain

$$\mathcal{H} \hookrightarrow H_{loc}^4(\mathbb{R}^5) \times H_{loc}^3(\mathbb{R}^5) \hookrightarrow H_{loc}^3(\mathbb{R}^5) \times H_{loc}^2(\mathbb{R}^5)$$

and by the compactness of the latter embedding, there exists, for each  $k \in \mathbb{N}$ , a subsequence  $(\mathbf{f}_{k,n})_{n \in \mathbb{N}}$  of  $(\mathbf{f}_n)_{n \in \mathbb{N}}$  that converges in  $H^3(\mathbb{B}_{2k}^5) \times H^2(\mathbb{B}_{2k}^5)$  and such that  $(\mathbf{f}_{k+1,n})_{n \in \mathbb{N}}$  is a subsequence of  $(\mathbf{f}_{k,n})_{n \in \mathbb{N}}$ . In particular, there exists an  $N(k) \in \mathbb{N}$  such that

$$\|\mathbf{f}_{k,m} - \mathbf{f}_{k,n}\|_{H^3(\mathbb{B}_{2k}^5) \times H^2(\mathbb{B}_{2k}^5)} \leq k^{-1}$$

for all  $m, n \geq N(k)$ . Clearly, we may choose  $k \mapsto N(k) : \mathbb{N} \rightarrow \mathbb{N}$  to be monotonically increasing. We set  $\mathbf{g}_k := \mathbf{f}_{k,N(k)}$  for  $k \in \mathbb{N}$ . Then  $(\mathbf{g}_k)_{k \in \mathbb{N}}$  is a subsequence of  $(\mathbf{f}_n)_{n \in \mathbb{N}}$  and we have

$$\begin{aligned} \|\mathbf{L}'\mathbf{g}_{k+\ell} - \mathbf{L}'\mathbf{g}_k\|_{\mathcal{H}} &\lesssim \|\mathbf{g}_{k+\ell} - \mathbf{g}_k\|_{H^3(\mathbb{B}_{2k}^5) \times H^2(\mathbb{B}_{2k}^5)} + k^{-1} \\ &= \|\mathbf{f}_{k+\ell,N(k+\ell)} - \mathbf{f}_{k,N(k)}\|_{H^3(\mathbb{B}_{2k}^5) \times H^2(\mathbb{B}_{2k}^5)} + k^{-1} \\ &\lesssim k^{-1} \end{aligned}$$

for all  $k, \ell \in \mathbb{N}$  because  $(\mathbf{f}_{k+\ell,n})_{n \in \mathbb{N}}$  is a subsequence of  $(\mathbf{f}_{k,n})_{n \in \mathbb{N}}$ . Consequently,  $(\mathbf{L}'\mathbf{g}_k)_{k \in \mathbb{N}}$  is a Cauchy sequence in  $\mathcal{H}$  and this proves the claim. □

As a consequence of the bound  $\|\mathbf{S}_0(\tau)\|_{\mathcal{H}} \lesssim e^{-\frac{1}{2}\tau}$ , we see that the *free resolvent*

$$\mathbf{R}_{\mathbf{L}_0}(\lambda) := (\lambda\mathbf{I} - \mathbf{L}_0)^{-1} = \int_0^\infty e^{-\lambda\tau} \mathbf{S}_0(\tau) d\tau$$

exists provided that  $\operatorname{Re} \lambda > -\frac{1}{2}$ . By the *Birman–Schwinger principle*, i.e., the identity

$$\lambda \mathbf{I} - \mathbf{L} = [\mathbf{I} - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)](\lambda \mathbf{I} - \mathbf{L}_0),$$

it follows that  $\lambda \mathbf{I} - \mathbf{L}$  is bounded invertible for  $\operatorname{Re} \lambda > -\frac{1}{2}$  if and only if  $\mathbf{I} - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)$  is bounded invertible. This observation leads to the important result that possible spectral points  $\lambda$  of  $\mathbf{L}$  with  $\operatorname{Re} \lambda \geq -\frac{1}{4}$ , say, are confined to a compact region. In order to formulate the exact statement, we define

$$\Gamma_R := \{z \in \mathbb{C} : \operatorname{Re} z > -\frac{1}{4}, |z| > R\}$$

for  $R > 0$ .

**Lemma 4.10.** *There exists an  $R > 0$  such that  $\sigma(\mathbf{L}) \cap \Gamma_R = \emptyset$ . Furthermore, we have*

$$\sup_{\lambda \in \Gamma_R} \|(\lambda \mathbf{I} - \mathbf{L})^{-1}\|_{\mathcal{H}} < \infty.$$

*Proof.* Let  $\operatorname{Re} \lambda \geq -\frac{1}{4}$ ,  $\mathbf{g} \in \mathcal{H}$ , and set  $\mathbf{f} := \mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{g}$ . Then  $\mathbf{f} \in \mathcal{D}(\mathbf{L}_0)$  and  $(\lambda \mathbf{I} - \mathbf{L}_0)\mathbf{f} = \mathbf{g}$ . By Lemma 4.8, we have

$$g_1(\xi) = \xi^j \partial_{\xi^j} f_1(\xi) + (\lambda + 1)f_1(\xi) - f_2(\xi)$$

in the sense of distributions, where  $\mathbf{g} = (g_1, g_2)$  and  $\mathbf{f} = (f_1, f_2)$ . Consequently,

$$f_1(\xi) = \frac{1}{\lambda + 1} [-\xi^j \partial_{\xi^j} f_1(\xi) + f_2(\xi) + g_1(\xi)]$$

and Hardy’s inequality yields

$$\begin{aligned} \|\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{g}\|_{\mathcal{H}} &= \|Vf_1\|_{\dot{H}^1(\mathbb{R}^5) \cap \dot{H}^3(\mathbb{R}^5)} \\ &\lesssim \frac{1}{|\lambda + 1|} (\|\mathbf{f}\|_{\mathcal{H}} + \|\mathbf{g}\|_{\mathcal{H}}) \\ &= \frac{1}{|\lambda + 1|} (\|\mathbf{R}_{\mathbf{L}_0}(\lambda)\mathbf{g}\|_{\mathcal{H}} + \|\mathbf{g}\|_{\mathcal{H}}) \\ &\lesssim \frac{1}{|\lambda + 1|} \|\mathbf{g}\|_{\mathcal{H}} \end{aligned}$$

for all  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda \geq -\frac{1}{4}$ , where we have exploited the decay of the potential  $V(\xi) = \frac{16}{(1+|\xi|^2)^2}$  as in the proof of Lemma 4.9. Thus, if  $R > 0$  is chosen large enough, we obtain  $\|\mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)\|_{\mathcal{H}} \leq \frac{1}{2}$  for all  $\lambda \in \Gamma_R$  and the existence of  $[\mathbf{I} - \mathbf{L}'\mathbf{R}_{\mathbf{L}_0}(\lambda)]^{-1}$  follows by a Neumann series argument. By the Birman–Schwinger principle, this implies the claim.  $\square$

**4.6. Connection to mode stability.** Since  $\mathbf{L}'$  is compact, it follows from Lemma 4.10 and the *analytic Fredholm theorem* (see, e.g., [35, p. 194, Theorem 3.14.3]) that  $\lambda \mathbf{I} - \mathbf{L}$  is bounded invertible for all  $\lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda \geq -\frac{1}{4}$  except for a finite number of eigenvalues, each with finite algebraic multiplicity.

In order to locate these eigenvalues, we need to solve the equation  $(\lambda \mathbf{I} - \mathbf{L})\mathbf{f} = \mathbf{0}$ . Suppose there exists a (nontrivial) solution  $\mathbf{f} = (f_1, f_2) \in \mathcal{D}(\mathbf{L}) = \mathcal{D}(\mathbf{L}_0)$ . By Lemma 4.8, we see that

$$\begin{aligned} \xi^j \partial_{\xi^j} f_1(\xi) + (\lambda + 1)f_1(\xi) - f_2(\xi) &= 0, \\ -\Delta f_1(\xi) + \xi^j \partial_{\xi^j} f_2(\xi) + (\lambda + 2)f_2(\xi) - \frac{16}{(1 + |\xi|^2)^2} f_1(\xi) &= 0, \end{aligned}$$

in the sense of distributions and by inserting the first equation into the second one, we find

$$(4.5) \quad -(\delta^{jk} - \xi^j \xi^k) \partial_{\xi^j} \partial_{\xi^k} f_1(\xi) + 2(\lambda + 2) \xi^j \partial_{\xi^j} f_1(\xi) + (\lambda + 1)(\lambda + 2) f_1(\xi) - \frac{16}{(1 + |\xi|^2)^2} f_1(\xi) = 0.$$

Furthermore, by Sobolev embedding and Hölder’s inequality, we have  $\dot{H}^2(\mathbb{R}^5) \cap \dot{H}^4(\mathbb{R}^5) \subset L^\infty(\mathbb{R}^5) \subset L^2_{\text{loc}}(\mathbb{R}^5)$  and thus,  $f_1 \in H^4_{\text{loc}}(\mathbb{R}^5)$ . Consequently, by elliptic regularity, we conclude that  $f_1 \in C^\infty(\mathbb{R}^5 \setminus \mathbb{S}^4)$  and  $f_1$  satisfies equation (4.5) on  $\mathbb{R}^5 \setminus \mathbb{S}^4$  in the sense of classical derivatives. Recall that  $f_1$  is radial and thus, in terms of the auxiliary function  $\widehat{f}_1 \in C^\infty(\mathbb{R} \setminus \{-1, 1\})$ , given by  $\widehat{f}_1(\rho) := \rho f_1(\rho e_1)$ , equation (4.5) reads

$$-(1 - \rho^2) \widehat{f}_1''(\rho) - \frac{2}{\rho} \widehat{f}_1'(\rho) + 2(\lambda + 1) \rho \widehat{f}_1'(\rho) + \lambda(\lambda + 1) \widehat{f}_1(\rho) + 2 \frac{1 - 6\rho^2 + \rho^4}{\rho^2(1 + \rho^2)^2} \widehat{f}_1(\rho) = 0.$$

Consequently, since  $\widehat{f}_1 \in C^\infty(\mathbb{R} \setminus \{-1, 1\}) \cap H^4((\frac{1}{2}, \frac{3}{2}))$ , it follows by Frobenius’s method that  $\widehat{f}_1 \in C^\infty(\mathbb{R})$  and  $\widehat{f}_1$  is a nontrivial solution in  $C^\infty([0, 1])$  of equation (2.3). This is the connection to the mode stability problem that gives the latter a proper functional analytic interpretation.

**Proposition 4.11.** *For the spectrum  $\sigma(\mathbf{L})$  of  $\mathbf{L}$  we have*

$$\sigma(\mathbf{L}) \subset \{z \in \mathbb{C} : \text{Re } z < 0\} \cup \{1\}$$

and 1 is an eigenvalue of  $\mathbf{L}$ .

*Proof.* The statement about the spectrum follows from the above derivation in conjunction with Lemma 4.10 and mode stability (Theorem 3.1). The eigenvalue 1 stems from time translation symmetry as explained in the discussion of mode stability. □

**4.7. Control of the linearized evolution.** We define the Riesz or spectral projection associated to the eigenvalue 1 by

$$\mathbf{P} := \frac{1}{2\pi i} \int_\gamma (z\mathbf{I} - \mathbf{L})^{-1} dz,$$

where  $\gamma : [0, 1] \rightarrow \mathbb{C}$  is given by  $\gamma(t) := 1 + \frac{1}{2} e^{2\pi i t}$ .

**Lemma 4.12.** *The eigenvalue 1 of  $\mathbf{L}$  is simple, i.e.,  $\mathbf{P}$  has rank 1.*

*Proof.* By the analytic Fredholm theorem (see above),  $\mathbf{P}$  has finite rank. Consequently,  $\mathbf{I} - \mathbf{L}$  restricts to a finite-dimensional operator on  $\text{rg } \mathbf{P}$  and from linear algebra we infer that the part of  $\mathbf{I} - \mathbf{L}$  in  $\text{rg } \mathbf{P}$  is nilpotent. The fact that the eigenvalue 1 has algebraic multiplicity exactly equal to 1 is then proved by ODE methods, by showing that the equation  $(\mathbf{I} - \mathbf{L})\mathbf{f} = \mathbf{f}_*$ , where  $\mathbf{f}_*$  is the eigenfunction associated to the eigenvalue 1, has no solution. This is an exercise with the variation of parameters formula that we leave to the interested reader; see, e.g., [22, Lemma 4.20], for the precise argument. □

From abstract semigroup theory we can now obtain a sufficiently detailed understanding of the linearized evolution generated by  $\mathbf{L}$ .

**Lemma 4.13.** *The operator  $\mathbf{L}$  generates a strongly continuous semigroup  $\mathbf{S}$  on  $\mathcal{H}$ . Furthermore, there exists an  $\epsilon > 0$  and a  $C > 0$  such that*

$$\begin{aligned} \|\mathbf{S}(\tau)(\mathbf{I} - \mathbf{P})\mathbf{f}\|_{\mathcal{H}} &\leq Ce^{-\epsilon\tau}\|(\mathbf{I} - \mathbf{P})\mathbf{f}\|_{\mathcal{H}}, \\ \mathbf{S}(\tau)\mathbf{P}\mathbf{f} &= e^{\tau}\mathbf{P}\mathbf{f}, \end{aligned}$$

for all  $\tau \geq 0$  and  $\mathbf{f} \in \mathcal{H}$ .

*Proof.* The operator  $\mathbf{L}$  differs from the semigroup generator  $\mathbf{L}_0$  by the bounded operator  $\mathbf{L}'$  and hence generates a semigroup  $\mathbf{S}$  itself. The statement about the evolution on the unstable subspace,  $\mathbf{S}(\tau)\mathbf{P}\mathbf{f} = e^{\tau}\mathbf{P}\mathbf{f}$ , is a direct consequence of the fact that the range of  $\mathbf{P}$  is one-dimensional and hence spanned by the eigenfunction of  $\mathbf{L}$  associated to the eigenvalue 1. For the evolution on the stable subspace, we note that Lemma 4.10 and Proposition 4.11 imply that

$$\sup_{\operatorname{Re} \lambda > 0} \|(\lambda\mathbf{I} - \mathbf{L})^{-1}(\mathbf{I} - \mathbf{P})\|_{\mathcal{H}} < \infty$$

and the claimed growth bound follows from the Gearhart–Prüß–Greiner–Theorem; see, e.g., [26, p. 302, Theorem 1.11].  $\square$

**4.8. The nonlinear problem.** We finally sketch how to proceed with the nonlinear stability. In Duhamel form, the equation we would like to solve reads

$$\Phi(\tau) = \mathbf{S}(\tau)\mathbf{f} + \int_0^{\tau} \mathbf{S}(\tau - \tau')\mathbf{N}(\Phi(\tau'))d\tau'.$$

Typically, such an equation is solved by a fixed point argument. However, in the present form this is not possible due to the exponential growth of the semigroup on  $\operatorname{rg} \mathbf{P}$ . Thus, we borrow an idea from dynamical systems theory known as the *Lyapunov–Perron method* and consider instead the equation

$$(4.6) \quad \Phi(\tau) = \mathbf{S}(\tau)[\mathbf{f} - \mathbf{C}(\mathbf{f}, \Phi)] + \int_0^{\tau} \mathbf{S}(\tau - \tau')\mathbf{N}(\Phi(\tau'))d\tau',$$

where

$$\mathbf{C}(\mathbf{f}, \Phi) := \mathbf{P}\mathbf{f} + \mathbf{P} \int_0^{\infty} e^{-\tau'}\mathbf{N}(\Phi(\tau'))d\tau'$$

is a correction term that stabilizes the evolution. Formally, this term is obtained by applying the spectral projection  $\mathbf{P}$  to the original equation. Consequently, the subtraction of  $\mathbf{C}(\mathbf{f}, \Phi)$  corrects the initial data along the one-dimensional subspace  $\operatorname{rg} \mathbf{P}$  on which the linearized evolution grows exponentially. Note, however, that there is a nonlinear self-interaction, i.e., the correction term depends on the solution itself and is not known in advance as would be the case for a linear problem. Nonetheless, by a routine fixed point argument utilizing the Banach algebra property of  $\mathcal{H}$ , we can show that equation (4.6) has a solution  $\Phi \in C([0, \infty), \mathcal{H})$  for any small data  $\mathbf{f}$ . Finally, by realizing that the data we want to describe depend on  $T$ , we see, e.g., by the intermediate value theorem, that there always exists a  $T$  that makes the correction term vanish. By translating back to the original variables, we finally arrive at the following result on the stability of the wave maps blowup. Recall the wave maps equation in corotational symmetry reduction,

$$(4.7) \quad (\square v)(t, x) + \frac{\sin(2|x|v(t, x)) - 2|x|v(t, x)}{|x|^3} = 0,$$

and the self-similar blowup solution

$$v_*^T(t, x) := (T - t)^{-1} w_* \left( \frac{x}{T - t} \right), \quad w_*(\xi) = \frac{2}{|\xi|} \arctan(|\xi|).$$

**Theorem 4.14** (Nonlinear asymptotic stability of wave maps blowup). *There exist constants  $M, \delta_0 > 0$  such that the following holds. Let  $\delta \in [0, \delta_0]$  and suppose that  $f, g \in C^\infty(\mathbb{R}^5)$  are radial and satisfy*

$$\|f - v_*^1(0, \cdot)\|_{\dot{H}^2(\mathbb{R}^5) \cap \dot{H}^4(\mathbb{R}^5)} + \|g - \partial_0 v_*^1(0, \cdot)\|_{\dot{H}^1(\mathbb{R}^5) \cap \dot{H}^3(\mathbb{R}^5)} \leq \frac{\delta}{M}.$$

*Then there exist a  $T \in [1 - \delta, 1 + \delta]$  and a unique solution  $v \in C^\infty([0, T] \times \mathbb{R}^5)$  of equation (4.7) that satisfies  $(v(0, \cdot), \partial_0 v(0, \cdot)) = (f, g)$ . Furthermore, we have the decomposition*

$$v(t, x) = (T - t)^{-1} \left[ w_* \left( \frac{x}{T - t} \right) + \varepsilon \left( t, \frac{x}{T - t} \right) \right]$$

where

$$\|\varepsilon(t, \cdot)\|_{\dot{H}^2(\mathbb{R}^5) \cap \dot{H}^4(\mathbb{R}^5)} + \|(T - t)\partial_t \varepsilon(t, \cdot) + \Lambda \varepsilon(t, \cdot) + \varepsilon(t, \cdot)\|_{\dot{H}^1(\mathbb{R}^5) \cap \dot{H}^3(\mathbb{R}^5)} \rightarrow 0$$

as  $t \rightarrow T^-$ .

*Remark 4.15.* Analogous results are known in all supercritical dimensions; see [6, 30]. The stability problem outside of corotational symmetry is still open, though. However, there are stability results on self-similar blowup without symmetry assumptions for the simpler wave equation with a power nonlinearity [11, 20, 29, 33].

### 5. CONCLUSION

Understanding large-data solutions of supercritical evolution equations remains one of the great challenges in contemporary analysis. The only rigorous methods we have depend on the existence of special solutions that are sufficiently well known. In many cases, self-similar solutions play this role and provide an entrance point to the rigorous study of large-data regimes because they open up the possibility of perturbative treatments. However, the understanding of the linearized evolution close to a self-similar solution is very challenging and requires knowledge of the spectrum of the corresponding linear operator that is genuinely nonself-adjoint in case of wave equations. This is the point where the analysis typically fails because there are no general methods to treat these spectral problems. In this exposition we presented the only known method so far that is capable of extracting the necessary spectral information in a number of nontrivial cases in a rigorous way. It consists of a “hard part” that proves the mode stability and a “soft part” that embeds the mode stability problem into a proper spectral-theoretic framework for the generator of the linearized evolution. Once the linearized evolution is understood, the treatment of the full nonlinear problem is routine. The approach was successfully applied to some of the most important models such as wave maps [6, 9, 12, 16, 22, 30], Yang–Mills fields [10, 13, 28, 31], and wave equations with power nonlinearities [11, 19–21, 29]. In addition, extensions to more general coordinate systems [1, 7, 17, 33] and weaker topologies [14, 18, 23, 24, 38] were considered.

Despite this recent success, a lot remains to be done. The presented method relies strongly on the fine properties of the perturbed solution and is probably hard to implement if the solution is not known in closed form. Consequently, it would

be very desirable to develop more conceptual methods that provide easy-to-verify criteria for mode stability. Whether this is possible at all remains to be seen. A conceptual breakthrough in this area would constitute a major step forward in modern PDE analysis and open a whole new spectrum of problems that could be rigorously dealt with. In any case, we hope that this exposition provides an accessible account to the current method that may be useful for researchers that are confronted with similar problems in their work.

APPENDIX A. BACKGROUND MATERIAL

**A.1. The Phragmén–Lindelöf principle.** The Phragmén–Lindelöf principle is an extension of the maximum principle to unbounded domains. There are many different versions and we present the simplest one that is sufficient for our purposes. First, recall the fundamental maximum principle from complex analysis.

**Lemma A.1** (Maximum principle). *Let  $\Omega \subset \mathbb{C}$  be open, connected, and bounded. Suppose that  $f : \overline{\Omega} \rightarrow \mathbb{C}$  is continuous and that  $f|_{\Omega} : \Omega \rightarrow \mathbb{C}$  is holomorphic. Then*

$$|f(z)| \leq \max_{\zeta \in \partial\Omega} |f(\zeta)|$$

for all  $z \in \Omega$ .

The maximum principle shows that if we want to control a holomorphic function on a bounded domain  $\Omega$ , it is enough to control it on the boundary. The assumption of boundedness is crucial here. However, under a mild growth condition, the maximum principle extends to unbounded domains and in this situation it goes by the name of Phragmén–Lindelöf. There are many different versions of this principle. We use a very basic one that allows us to bound a function on the complex right half-plane by its values on the imaginary axis.

**Lemma A.2** (Phragmén–Lindelöf principle). *Let  $\Omega := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$  and suppose that  $f : \overline{\Omega} \rightarrow \mathbb{C}$  is continuous and that  $f|_{\Omega} : \Omega \rightarrow \mathbb{C}$  is holomorphic. Let  $M > 0$ . If*

- (1)  $|f(it)| \leq M$  for all  $t \in \mathbb{R}$  and
- (2) there exists a  $C \geq 0$  such that  $|f(z)| \leq Ce^{|z|^{\frac{1}{2}}}$  for all  $z \in \Omega$

then

$$|f(z)| \leq M$$

for all  $z \in \Omega$ .

*Proof.* The proof is very simple and plays the situation back to the standard maximum principle. First, we note that the function  $z \mapsto z^{\frac{3}{4}} : \overline{\Omega} \rightarrow \mathbb{C}$  is continuous and holomorphic on  $\Omega$ . Then, for  $\epsilon > 0$ , we define an auxiliary function  $f_{\epsilon} : \overline{\Omega} \rightarrow \mathbb{C}$  by  $f_{\epsilon}(z) := e^{-\epsilon z^{\frac{3}{4}}} f(z)$ . Again,  $f_{\epsilon}$  is continuous and holomorphic on  $\Omega$ . Furthermore,

$$|f_{\epsilon}(z)| = e^{-\epsilon \operatorname{Re} z^{\frac{3}{4}}} |f(z)| = e^{-\epsilon |z|^{\frac{3}{4}} \cos(\frac{3}{4} \arg z)} |f(z)|$$

for all  $z \in \overline{\Omega}$  and thus,

$$|f_{\epsilon}(it)| = e^{-\epsilon |t|^{\frac{3}{4}} \cos(\frac{3}{4} \frac{\pi}{2})} |f(it)| \leq |f(it)| \leq M$$

for all  $t \in \mathbb{R}$  and  $\epsilon > 0$  because  $\eta := \cos(\frac{3}{4} \frac{\pi}{2}) > 0$ . Next, we have the bound

$$|f_{\epsilon}(z)| \leq e^{-\epsilon \eta |z|^{\frac{3}{4}}} |f(z)| \leq Ce^{-\epsilon \eta |z|^{\frac{3}{4}} + |z|^{\frac{1}{2}}} = Ce^{-|z|^{\frac{3}{4}}(\epsilon \eta - |z|^{-\frac{1}{4}})} \rightarrow 0$$



as  $|z| \rightarrow \infty$  and thus,  $|f_\epsilon(z)| \leq M$  if  $|z|$  is sufficiently large. For  $R > 0$  we define the domain

$$\Omega_R := \{z \in \mathbb{C} : |z| < R\} \cap \Omega.$$

By the above,  $f_\epsilon$  is holomorphic on  $\Omega_R$ , continuous on  $\overline{\Omega}_R$ , and there exists an  $R_\epsilon > 0$  such that  $|f_\epsilon(z)| \leq M$  for all  $z \in \partial\Omega_R$ , provided that  $R \geq R_\epsilon$ . Consequently, by the maximum principle,  $|f_\epsilon(z)| \leq M$  for all  $z \in \Omega_R$  and since this argument works for any  $R \geq R_\epsilon$ , we see that, in fact,  $|f_\epsilon(z)| \leq M$  for all  $z \in \Omega$ . This yields

$$|f(z)| \leq e^{\epsilon|z|^{\frac{3}{4}}} |f_\epsilon(z)| \leq M e^{\epsilon|z|^{\frac{3}{4}}}$$

for any  $z \in \Omega$  and any  $\epsilon > 0$  and upon letting  $\epsilon \rightarrow 0$ , we obtain the desired bound. □

**A.2. Asymptotics of difference equations.** Another important building block in the proof of mode stability is the asymptotic behavior of solutions to difference equations.

**Theorem A.3** (Poincaré). *Let  $p, q : \mathbb{N} \rightarrow \mathbb{C}$  and suppose that*

$$p_\infty := \lim_{n \rightarrow \infty} p(n), \quad q_\infty := \lim_{n \rightarrow \infty} q(n)$$

*exist. Assume further that there exist  $z_1, z_2 \in \mathbb{C}$  with  $|z_1| > |z_2|$  and such that*

$$z_j^2 + p_\infty z_j + q_\infty = 0, \quad j \in \{1, 2\}.$$

*Let  $a : \mathbb{N} \rightarrow \mathbb{C}$  satisfy*

$$(A.1) \quad a(n+2) + p(n)a(n+1) + q(n)a(n) = 0$$

*for all  $n \in \mathbb{N}$ . Then either there exists an  $n_0 \in \mathbb{N}$  such that  $a(n) = 0$  for all  $n \geq n_0$  or we have*

$$\lim_{n \rightarrow \infty} \frac{a(n+1)}{a(n)} \in \{z_1, z_2\}.$$

*Idea of proof.* In order to understand what is going on, we consider the *limiting equation*

$$(A.2) \quad a(n+2) + p_\infty a(n+1) + q_\infty a(n) = 0.$$

Then it follows that the functions  $n \mapsto z_j^n$  for  $j \in \{1, 2\}$  solve this equation simply because

$$z_j^{n+2} + p_\infty z_j^{n+1} + q_\infty z_j^n = z_j^n (z_j^2 + p_\infty z_j + q_\infty) = 0.$$

Consequently, the general solution of equation (A.2) is given by  $a(n) = \alpha_1 z_1^n + \alpha_2 z_2^n$ , where  $\alpha_j \in \mathbb{C}$  can be chosen arbitrarily. Thus, if  $\alpha_1 \neq 0$ , we can write

$$a(n) = \alpha_1 z_1^n \left[ 1 + \frac{\alpha_2}{\alpha_1} \left( \frac{z_2}{z_1} \right)^n \right]$$

and  $\lim_{n \rightarrow \infty} \frac{a(n+1)}{a(n)} = z_1$  follows immediately because  $|\frac{z_2}{z_1}| < 1$  by assumption. On the other hand, if  $\alpha_1 = 0$ , we obviously have  $\lim_{n \rightarrow \infty} \frac{a(n+1)}{a(n)} = z_2$ . Thus, since  $p(n)$  and  $q(n)$  get arbitrarily close to  $p_\infty$  and  $q_\infty$  for large  $n$ , the proof consists of showing that the above logic is stable under a suitable perturbation argument; see, e.g., [25]. □

## REFERENCES

- [1] Paweł Biernat, Roland Donninger, and Birgit Schörkhuber, *Hyperboloidal similarity coordinates and a globally stable blowup profile for supercritical wave maps*, Int. Math. Res. Not. IMRN **21** (2021), 16530–16591, DOI 10.1093/imrn/rnz286. MR4338226
- [2] Piotr Bizoń, *Equivariant self-similar wave maps from Minkowski spacetime into 3-sphere*, Comm. Math. Phys. **215** (2000), no. 1, 45–56, DOI 10.1007/s002200000291. MR1799875
- [3] Piotr Bizoń, *An unusual eigenvalue problem*, Acta Phys. Polon. B **36** (2005), no. 1, 5–15. MR2125332
- [4] Piotr Bizoń and Paweł Biernat, *Generic self-similar blowup for equivariant wave maps and Yang–Mills fields in higher dimensions*, Comm. Math. Phys. **338** (2015), no. 3, 1443–1450, DOI 10.1007/s00220-015-2404-y. MR3355819
- [5] Piotr Bizoń, Tadeusz Chmaj, and Zbislav Tabor, *Dispersion and collapse of wave maps*, Nonlinearity **13** (2000), no. 4, 1411–1423, DOI 10.1088/0951-7715/13/4/323. MR1767966
- [6] Athanasios Chatzikaleas, Roland Donninger, and Irfan Glogić, *On blowup of co-rotational wave maps in odd space dimensions*, J. Differential Equations **263** (2017), no. 8, 5090–5119, DOI 10.1016/j.jde.2017.06.011. MR3680948
- [7] Po-Ning Chen, Roland Donninger, Irfan Glogić, Michael McNulty, and Birgit Schörkhuber, *Co-dimension one stable blowup for the quadratic wave equation beyond the light cone*, Comm. Math. Phys. **405** (2024), no. 2, Paper No. 34, 46, DOI 10.1007/s00220-023-04888-2. MR4700297
- [8] O. Costin, R. Donninger, and X. Xia, *A proof for the mode stability of a self-similar wave map*, Nonlinearity **29** (2016), no. 8, 2451–2473, DOI 10.1088/0951-7715/29/8/2451. MR3538419
- [9] Ovidiu Costin, Roland Donninger, and Irfan Glogić, *Mode stability of self-similar wave maps in higher dimensions*, Comm. Math. Phys. **351** (2017), no. 3, 959–972, DOI 10.1007/s00220-016-2776-7. MR3623242
- [10] Ovidiu Costin, Roland Donninger, Irfan Glogić, and Min Huang, *On the stability of self-similar solutions to nonlinear wave equations*, Comm. Math. Phys. **343** (2016), no. 1, 299–310, DOI 10.1007/s00220-016-2588-9. MR3475668
- [11] Elek Csobo, Irfan Glogić, and Birgit Schörkhuber, *On blowup for the supercritical quadratic wave equation*, Anal. PDE **17** (2024), no. 2, 617–680, DOI 10.2140/apde.2024.17.617. MR4713110
- [12] Roland Donninger, *On stable self-similar blowup for equivariant wave maps*, Comm. Pure Appl. Math. **64** (2011), no. 8, 1095–1147, DOI 10.1002/cpa.20366. MR2839272
- [13] Roland Donninger, *Stable self-similar blowup in energy supercritical Yang–Mills theory*, Math. Z. **278** (2014), no. 3-4, 1005–1032, DOI 10.1007/s00209-014-1344-0. MR3278903
- [14] Roland Donninger, *Strichartz estimates in similarity coordinates and stable blowup for the critical wave equation*, Duke Math. J. **166** (2017), no. 9, 1627–1683, DOI 10.1215/00127094-0000009X. MR3662440
- [15] Roland Donninger and Peter C. Aichelburg, *On the mode stability of a self-similar wave map*, J. Math. Phys. **49** (2008), no. 4, 043515, 9, DOI 10.1063/1.2908159. MR2412310
- [16] Roland Donninger and Irfan Glogić, *On the existence and stability of blowup for wave maps into a negatively curved target*, Anal. PDE **12** (2019), no. 2, 389–416, DOI 10.2140/apde.2019.12.389. MR3861895
- [17] Roland Donninger and Matthias Ostermann, *A globally stable self-similar blowup profile in energy supercritical Yang–Mills theory*, Comm. Partial Differential Equations **48** (2023), no. 9, 1148–1213, DOI 10.1080/03605302.2023.2263208. MR4661000
- [18] Roland Donninger and Ziping Rao, *Blowup stability at optimal regularity for the critical wave equation*, Adv. Math. **370** (2020), 107219, 81, DOI 10.1016/j.aim.2020.107219. MR4101905
- [19] Roland Donninger and Birgit Schörkhuber, *Stable blow up dynamics for energy supercritical wave equations*, Trans. Amer. Math. Soc. **366** (2014), no. 4, 2167–2189, DOI 10.1090/S0002-9947-2013-06038-2. MR3152726
- [20] Roland Donninger and Birgit Schörkhuber, *On blowup in supercritical wave equations*, Comm. Math. Phys. **346** (2016), no. 3, 907–943, DOI 10.1007/s00220-016-2610-2. MR3537340
- [21] Roland Donninger and Birgit Schörkhuber, *Stable blowup for wave equations in odd space dimensions*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **34** (2017), no. 5, 1181–1213, DOI 10.1016/j.anihpc.2016.09.005. MR3742520

- [22] Roland Donn timer, Birgit Schörkhuber, and Peter C. Aichelburg, *On stable self-similar blowup for equivariant wave maps: the linearized problem*, Ann. Henri Poincaré **13** (2012), no. 1, 103–144, DOI 10.1007/s00023-011-0125-0. MR2881965
- [23] Roland Donn timer and David Wallauch, *Optimal blowup stability for supercritical wave maps*, Adv. Math. **433** (2023), Paper No. 109291, 86, DOI 10.1016/j.aim.2023.109291. MR4640202
- [24] Roland Donn timer and David Wallauch, *Optimal blowup stability for three-dimensional wave maps*, Preprint, arXiv:2212.08374, 2023.
- [25] Saber Elaydi, *An introduction to difference equations*, 3rd ed., Undergraduate Texts in Mathematics, Springer, New York, 2005. MR2128146
- [26] Klaus-Jochen Engel and Rainer Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. MR1721989
- [27] Irfan Glogić, *On the Existence and Stability of Self-Similar Blowup in Nonlinear Wave Equations*, ProQuest LLC, Ann Arbor, MI, 2018. Thesis (Ph.D.)—The Ohio State University. MR4198863
- [28] Irfan Glogić, *Stable blowup for the supercritical hyperbolic Yang–Mills equations*, Adv. Math. **408** (2022), Paper No. 108633, 52, DOI 10.1016/j.aim.2022.108633. MR4469070
- [29] Irfan Glogić and Birgit Schörkhuber, *Co-dimension one stable blowup for the supercritical cubic wave equation*, Adv. Math. **390** (2021), Paper No. 107930, 79, DOI 10.1016/j.aim.2021.107930. MR4292964
- [30] Irfan Glogić, *Globally stable blowup profile for supercritical wave maps in all dimensions*, Preprint, arXiv:2207.06952, 2022.
- [31] Irfan Glogić, *Global-in-space stability of singularity formation for Yang–Mills fields in higher dimensions*, Preprint, arXiv:2305.10312, 2023.
- [32] Camil Muscalu and Wilhelm Schlag, *Classical and multilinear harmonic analysis. Vol. I*, Cambridge Studies in Advanced Mathematics, vol. 137, Cambridge University Press, Cambridge, 2013. MR3052498
- [33] Matthias Ostermann, *Stable blowup for focusing semilinear wave equations in all dimensions*, Preprint, arXiv:2304.08187, 2023.
- [34] Jalal Shatah, *Weak solutions and development of singularities of the SU(2)  $\sigma$ -model*, Comm. Pure Appl. Math. **41** (1988), no. 4, 459–469, DOI 10.1002/cpa.3160410405. MR933231
- [35] Barry Simon, *Operator theory*, A Comprehensive Course in Analysis, Part 4, American Mathematical Society, Providence, RI, 2015, DOI 10.1090/simon/004. MR3364494
- [36] Gerald Teschl, *Ordinary differential equations and dynamical systems*, Graduate Studies in Mathematics, vol. 140, American Mathematical Society, Providence, RI, 2012, DOI 10.1090/gsm/140. MR2961944
- [37] Neil Turok and David Spergel. *Global texture and the microwave background*. Phys. Rev. Lett., 64:2736–2739, Jun 1990.
- [38] David Wallauch, *Strichartz estimates and blowup stability for energy critical nonlinear wave equations*, Trans. Amer. Math. Soc. **376** (2023), no. 6, 4321–4360, DOI 10.1090/tran/8879. MR4586813

UNIVERSITÄT WIEN, FAKULTÄT FÜR MATHEMATIK OSKAR-MORGENSTERN-PLATZ 1, 1090 VIENNA, AUSTRIA

*Email address:* roland.donn timer@univie.ac.at