ASYMPTOTIC VANISHING OF SYZYGIES OF ALGEBRAIC VARIETIES

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ABSTRACT. The purpose of this paper is to prove Ein–Lazarsfeld's conjecture on asymptotic vanishing of syzygies of algebraic varieties. This result, together with Ein–Lazarsfeld's asymptotic nonvanishing theorem, describes the overall picture of asymptotic behaviors of the minimal free resolutions of the graded section rings of line bundles on a projective variety as the positivity of the line bundles grows. Previously, Raicu reduced the problem to the case of products of three projective spaces, and we resolve this case here.

1. INTRODUCTION

Throughout the paper, we work over an algebraically closed field $\Bbbk$ of arbitrary characteristic. Let X be a projective variety of dimension n, and L be a very ample line bundle on X which gives rise to an embedding

$$X \subseteq \mathbb{P}H^0(X, L) = \mathbb{P}^r,$$

where $r = h^0(X, L) - 1$. Denote by S the homogeneous coordinate ring of $\mathbb{P}^r$. Fix a coherent sheaf B on X, and let

$$R = R(X, B; L) := \bigoplus_{m \in \mathbb{Z}} H^0(X, B \otimes L^m)$$

be the graded section S-module of B associated to L. By the Hilbert syzygy theorem, R has a minimal free resolution

$$0 \longleftarrow R \longleftarrow E_0 \longleftarrow E_1 \longleftarrow \cdots \longleftarrow E_r \longleftarrow 0$$

where

$$E_p = \bigoplus_q K_{p,q}(X, B; L) \otimes_{\Bbbk} S(-p-q).$$

The *Koszul cohomology* group $K_{p,q}(X, B; L)$ is the space of *p-th syzygies of weight q*. When $B = \mathcal{O}_X$, we set $K_{p,q}(X, L) := K_{p,q}(X, \mathcal{O}_X; L)$. After the pioneering work of Green [21, 22], there has been a considerable amount of work to understand vanishing and nonvanishing of $K_{p,q}(X, B; L)$.

We say that L satisfies the *property N_k* if $K_{0,1}(X, L) = 0$ and $K_{p,q}(X, L) = 0$ for $0 \leq p \leq k$ and $q \geq 2$. The property N_0 means that $X \subseteq \mathbb{P}^r$ is projectively normal, and the property N_1 means that the defining ideal of X in $\mathbb{P}^r$ is generated by quadratic polynomials. Thus the property N_k provides a natural framework to generalize classical results on defining equations of algebraic varieties to the results on their syzygies. Along this line, Green proved that if X is a smooth projective complex curve of genus

g and $\deg L \geq 2g + 1 + k$, then L satisfies the property N_k (see [21, Theorem (4.a.1)]). Green's celebrated theorem has stimulated further work in this direction, and several analogous statements for higher dimensional algebraic varieties have been established, e.g. [7,8,11,16,20,22,27]. On the other hand, Green–Lazarsfeld [23,24] and Ottaviani–Paoletti [26] called attention to the failure of the property N_k. The main result of [26] asserts that

$$K_{p,2}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \neq 0 \quad \text{for } 3d - 2 \leq p \leq r_d - 2,$$

where $r_d = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) - 1$. In particular, $\mathcal{O}_{\mathbb{P}^2}(d)$ does not satisfy the property N_{3d-2}. As $r_d \approx d^2/2$, the property N_k for $\mathcal{O}_{\mathbb{P}^2}(d)$ describes only a small fraction of the syzygies of the d-th Veronese embedding of $\mathbb{P}^2$. Eisenbud–Green–Hulek–Popescu observed in [18, Proposition 3.4] that a similar phenomenon occurs for other smooth projective surface, and Ein–Lazarsfeld proved in [12, Theorem A] that this always happens for all smooth projective varieties.

It is an interesting problem to describe the overall asymptotic behaviors of $K_{p,q}(X, B; L)$ as the positivity of L grows. This type of question was first suggested by Green [21, Problem 5.13] and also considered by Ein–Lazarsfeld [11, Problem 4.4]. To set the stage for asymptotic syzygies of algebraic varieties, assume that X is smooth and B is a line bundle, and let

$$L_d := \mathcal{O}_X(dA + P) \quad \text{for an integer } d \geq 1,$$

where A is an ample divisor and P is an arbitrary divisor on X. We suppose that d is sufficiently large, so in particular, L_d is very ample. Put $r_d := h^0(X, L_d) - 1$. Elementary considerations of Castelnuovo–Mumford regularity show that

$$K_{p,q}(X, B; L_d) = 0 \quad \text{for } q \geq n + 2.$$

When $q = 0$ or $n + 1$, [12, Proposition 5.1 and Corollary 5.2] state that

$$K_{p,0}(X, B; L_d) \neq 0 \Longleftrightarrow 0 \leq p \leq h^0(B) - 1;$$
$$K_{p,n+1}(X, B; L_d) \neq 0 \Longleftrightarrow r_d - n - h^0(X, K_X - B) + 1 \leq p \leq r_d - n.$$

The main issue is then to study vanishing and nonvanishing of $K_{p,q}(X, B; L_d)$ for $1 \leq q \leq n$. Motivated by the nonvanishing results of [18, 26], Ein–Lazarsfeld established the *asymptotic nonvanishing theorem* in [12, Theorem 4.1]: For each $1 \leq q \leq n$, there are constants $C_1, C_2 > 0$ such that if d is sufficiently large, then

$$K_{p,q}(X, B; L_d) \neq 0 \quad \text{for } C_1 d^{q-1} \leq p \leq r_d - C_2 d^{n-1}.$$

If furthermore $H^i(X, B) = 0$ for $1 \leq i \leq n - 1$, then

$$K_{p,q}(X, B; L_d) \neq 0 \quad \text{for } C_1 d^{q-1} \leq p \leq r_d - C_2 d^{n-q};$$

in this case with $P = 0$, a quick proof is provided in [10] with effective range of p. It is worth noting that Zhou adapted the arguments in [12] to show that the asymptotic nonvanishing theorem also holds for singular projective varieties with a relaxed assumption on B (see [12, Remark 4.2]). The influential paper [12] opens the door to research on the asymptotic behaviors of $K_{p,q}(X, B; L_d)$ for d increasing. We refer to [14] for survey on recent progress on asymptotic syzygies of algebraic varieties.

It is very natural to ask whether the $K_{p,q}(X, B; L_d)$ vanish for the values of p outside the range in the statement of the asymptotic nonvanishing theorem. Ein–Lazarsfeld conjectured that this is indeed the case for $p < O(d^{q-1})$ (see [12, Conjecture 7.1], [14,

Conjecture 1.10]). We confirm this *asymptotic vanishing conjecture* in a more general setting.

Theorem 1.1. *Let X be a projective variety of dimension n, and B be a coherent sheaf on X. Fix an ample divisor A and an arbitrary divisor P on X, and put $L_d := \mathcal{O}_X(dA + P)$ for an integer $d \geq 1$. For each $1 \leq q \leq n + 1$, there is a constant $C > 0$ depending on X, A, B, P such that if d is sufficiently large, then*

$$K_{p,q}(X, B; L_d) = 0 \quad \text{for } 0 \leq p \leq Cd^{q-1}.$$

We give some remarks on vanishing of $K_{p,q}(X, B; L_d)$ for large p. For simplicity, we assume that X is smooth and B is a vector bundle. If $H^i(X, B) = 0$ for $1 \leq i \leq n - 1$, then the duality theorem (cf. [12, Proposition 3.5], [21, Theorem 2.c.6]) says that

$$K_{p,q}(X, B; L_d) = K_{r_d-n-p,n+1-q}(X, B^* \otimes \omega_X; L_d)^*.$$

Then our asymptotic vanishing theorem implies the following: For each $1 \leq q \leq n$, there is a constant $C > 0$ such that if d is sufficiently large, then

$$K_{p,q}(X, B; L_d) = 0 \quad \text{for } p \geq r_d - Cd^{n-q}.$$

However, if $H^{q-1}(X, B) \neq 0$ for some $2 \leq q \leq n$, then $K_{r_d-q+1,q}(X, B; L_d) \neq 0$ for large d (see [12, Remark 5.3]). When X is a smooth projective complex curve and B is a line bundle, vanishing of weight-one syzygies $K_{p,1}(X, B; L_d)$ for large p is determined by the duality theorem and [[13], Theorem B] (see also [29]). This implies Green–Lazarsfeld's gonality conjecture, and a higher dimensional generalization is treated in the work of Ein–Lazarsfeld–Yang [15].

Shortly after the asymptotic vanishing conjecture was proposed, Raicu showed in the appendix of [28] that the general case of the conjecture follows from the case of products of three projective spaces. The case that $q = 1$ in Theorem 1.1 is trivial. To prove Theorem 1.1, it is more than enough to establish the following:

Theorem 1.2. *Let $k \geq 1$ be an integer, $n_1, \ldots, n_k, d_1, \ldots, d_k$ be positive integers, and $b_1, \ldots, b_k$ be integers. Set*

$$X := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k},$$
$$B := \mathcal{O}_{\mathbb{P}^{n_1}}(b_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{n_k}}(b_k),$$
$$L := \mathcal{O}_{\mathbb{P}^{n_1}}(d_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{n_k}}(d_k),$$

and $b := \min\{b_1, \ldots, b_k\}$, $d := \min\{d_1, \ldots, d_k\}$. For each $2 \leq q \leq n_1 + \cdots + n_k + 1$, if $d + b \geq 0$, then

$$K_{p,q}(X, B; L) = 0 \quad \text{for } 0 \leq p \leq (1/n_1! \cdots n_k!)(d^{q-1} + bd^{q-2}).$$

As $K_{p,q+1}(X, B; L) = K_{p,q}(X, B + L; L)$, it is reasonable to assume that $b < d$ in Theorem 1.2. But we do not need this assumption for the proof.

We give a sketch of the proof of Theorem 1.2 for the Veronese case. Let $M_{\mathcal{O}_{\mathbb{P}^n}(d)}$ be the kernel of the evaluation map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(d)$. It is well known that

$$K_{p,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = H^{q-1}(\mathbb{P}^n, \wedge^{p+q-1} M_{\mathcal{O}_{\mathbb{P}^n}(d)}(d)) \quad \text{for } p \geq 0 \text{ and } q \geq 2.$$

The main idea is to work on $\mathbb{P}^{n-1} \times \mathbb{P}^1$ instead of $\mathbb{P}^n$ via the finite map $\sigma : \mathbb{P}^{n-1} \times \mathbb{P}^1 \to \mathbb{P}^n$ given by $(\xi, z) \mapsto \xi + z$, where $\mathbb{P}^n$ is regarded as the Hilbert scheme of n points on

$\mathbb{P}^1$ and σ is the universal family. Note that $\sigma_*(\mathcal{O}_{\mathbb{P}^{n-1}} \boxtimes \mathcal{O}_{\mathbb{P}^1}(n-1)) = \mathcal{O}_{\mathbb{P}^n}^{\oplus n}$. For each $2 \leq q \leq n+1$, the problem is equivalent to showing that

$$H^{q-1}(\mathbb{P}^{n-1} \times \mathbb{P}^1, \wedge^{p+q-1}\sigma^* M_{\mathcal{O}_{\mathbb{P}^n}(d)} \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(d) \boxtimes \mathcal{O}_{\mathbb{P}^1}(d+n-1)) = 0$$
$$\text{for } 0 \leq p \leq O(d^{q-1}).$$

An advantage of working on $\mathbb{P}^{n-1} \times \mathbb{P}^1$ is that we can use the following short exact sequence

$$0 \longrightarrow \bigoplus \mathcal{O}_{\mathbb{P}^{n-1}} \boxtimes \mathcal{O}_{\mathbb{P}^1}(-n) \longrightarrow \sigma^* M_{\mathcal{O}_{\mathbb{P}^n}(d)} \longrightarrow M_{\mathcal{O}_{\mathbb{P}^{n-1}}(d)} \boxtimes \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow 0,$$

which provides a way to proceed by induction on n. By considering the natural filtration of $\wedge^{p+q-1}\sigma^* M_{\mathcal{O}_{\mathbb{P}^n}(d)}$, we reduce the problem to proving that

$$H^{q-1}(\mathbb{P}^{n-1} \times \mathbb{P}^1, \wedge^i M_{\mathcal{O}_{\mathbb{P}^{n-1}}(d)}(d) \boxtimes \mathcal{O}_{\mathbb{P}^1}(a_i)) = 0 \quad \text{for } 0 \leq i \leq O(d^{q-1}),$$

where $a_i = id - (p+q-1-i)n + d + n - 1$. By induction on n, we can assume that $H^j(\mathbb{P}^{n-1}, \wedge^i M_{\mathcal{O}_{\mathbb{P}^{n-1}}(d)}(d)) = 0$ for $0 \leq i \leq O(d^j)$ and $j = q-2, q-1$. By the Künneth formula, it is sufficient to check that

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i)) = 0 \quad \text{when } i \geq O(d^{q-2}).$$

But we have

$$a_i = id + (d + 2n + in - qn) - 1 - pn \geq O(d^{q-1}) - 1 - pn \geq -1$$

as soon as $0 \leq p \leq O(d^{q-1})$. Thus $K_{p,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = 0$ for this range of p. The same argument works for the general Segre–Veronese case.

There has been a great deal of attention to the syzygies of Veronese or Segre–Veronese embeddings, e.g. [4, 6–8, 19, 25, 26, 28, 30]. The syzygies of these varieties have connections to representation theory and combinatorics. It would be exceedingly interesting to know whether the method of the present paper could make progress on the study of the Veronese or Segre–Veronese syzygies.

The paper is organized as follows. After reviewing basic necessary facts in Section 2, we prove Theorem 1.2 in Section 3, where we also show Theorem 1.1 following Raicu's argument in [28]. Section 4 is devoted to presenting some open problems on asymptotic syzygies of algebraic varieties.

2. PRELIMINARIES

We collect basic facts which are used to prove the main theorems of the paper.

2.1. **Koszul cohomology.** Let X be a projective variety, B be a coherent sheaf on X, and L be a very ample line bundle on X, which gives an embedding

$$X \subseteq \mathbb{P}H^0(X, L) = \mathbb{P}^r.$$

Let $S := S(H^0(X, L)) = \bigoplus_{m \geq 0} S^m H^0(X, L)$ be the homogeneous coordinate ring of $\mathbb{P}^r$, and

$$R = R(X, B; L) := \bigoplus_{m \in \mathbb{Z}} H^0(X, B \otimes L^m)$$

be the graded section S-module of B associated to L. Denote by $S_+ \subseteq S$ the irrelevant maximal ideal, and define the *Koszul cohomology group* to be

$$K_{p,q}(X, B; L) := \mathrm{Tor}_p^S(R, S/S_+)_{p+q}.$$

Then R has a minimal free resolution

$$0 \longleftarrow R \longleftarrow E_0 \longleftarrow E_1 \longleftarrow \cdots \longleftarrow E_r \longleftarrow 0$$

where

$$E_p = \bigoplus_q K_{p,q}(X, B; L) \otimes_{\Bbbk} S(-p - q).$$

Notice that $K_{p,q}(X, B; L)$ is the vector space of *p-th syzygies of weight q* and it is the cohomology of the Koszul-type complex

$$\wedge^{p+1} H^0(X, L) \otimes H^0(X, B \otimes L^{q-1}) \longrightarrow \wedge^p H^0(X, L) \otimes H^0(X, B \otimes L^q)$$
$$\longrightarrow \wedge^{p-1} H^0(X, L) \otimes H^0(X, B \otimes L^{q+1}).$$

When $B = \mathcal{O}_X$, we set $K_{p,q}(X, L) := K_{p,q}(X, \mathcal{O}_X; L)$.

Now, let L be a globally generated line bundle on a projective variety X. Consider the evaluation map

$$\mathrm{ev} : H^0(X, L) \otimes \mathcal{O}_X \longrightarrow L,$$

which is surjective since L is globally generated. Denote by M_L the kernel bundle of the evaluation map ev. Then we obtain a short exact sequence of vector bundles on X:

$$(2.1) \qquad 0 \longrightarrow M_L \longrightarrow H^0(X, L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0.$$

We use the following well-known fact to compute the Koszul cohomology group.

Proposition 2.1 (cf. [12, Corollary 3.3]). *Let X be a projective variety, B be a coherent sheaf on X, and L be a very ample line bundle on X. Assume that $H^i(X, B \otimes L^m) = 0$ for $i > 0$ and $m > 0$. Fix $q \geq 2$. Then we have*

$$K_{p,q}(X, B; L) = H^{q-1}(X, \wedge^{p+q-1} M_L \otimes B \otimes L) \text{ for } p \geq 0.$$

In particular, $K_{p,q}(X, B; L) = 0$ for $0 \leq p \leq p_0$ if and only if

$$H^{q-1}(X, \wedge^j M_L \otimes B \otimes L) = 0 \quad \text{for } 0 \leq j \leq p_0 + q - 1.$$

Proof. By taking wedge product of (2.1), we have a short exact sequence

$$(2.2) \qquad 0 \longrightarrow \wedge^{p+1} M_L \longrightarrow \wedge^{p+1} H^0(X, L) \otimes \mathcal{O}_X \longrightarrow \wedge^p M_L \otimes L \longrightarrow 0.$$

By using the Koszul-type complex and chasing through the diagram, we see that

$$K_{p,q}(X, B; L) = H^1(X, \wedge^{p+1} M_L \otimes B \otimes L^{q-1}) \quad \text{for } p \geq 0 \text{ and } q \geq 2.$$

See [3, Section 2.1] or [11, Section 1] for the complete proof. Now, from (2.2), we find that

$$H^1(X, \wedge^{p+1} M_L \otimes B \otimes L^{q-1}) = H^2(X, \wedge^{p+2} M_L \otimes B \otimes L^{q-2})$$
$$= \cdots$$
$$= H^{q-1}(X, \wedge^{p+q-1} M_L \otimes B \otimes L),$$

so the first assertion holds. Thus $K_{p,q}(X, B; L) = 0$ for $0 \leq p \leq p_0$ if and only if

$$H^{q-1}(X, \wedge^j M_L \otimes B \otimes L) = 0 \quad \text{for } q - 1 \leq j \leq p_0 + q - 1.$$

But we get from (2.2) that

$$H^{q-1}(X, \wedge^i M_L \otimes B \otimes L) = H^{q-2}(X, \wedge^{i-1} M_L \otimes B \otimes L^2) = \cdots = H^{q-i-1}(X, B \otimes L^{i+1}) = 0$$

for any $0 \leq i \leq q - 2$. Thus the second assertion holds. $\qquad \square$

We refer to [3, 11, 12, 17, 21] for more details on syzygies and Koszul cohomology.

2.2. **Filtrations for wedge products.** For a short exact sequence $0 \to U \to V \to W \to 0$ of vector bundles on a projective variety X and an integer $k \geq 1$, there is a natural filtration

$$(2.3) \qquad\qquad 0 = F^0 \subseteq F^1 \subseteq \cdots \subseteq F^k \subseteq F^{k+1} = \wedge^k V$$

such that

$$F^{p+1}/F^p = \wedge^{k-p} U \otimes \wedge^p W \quad \text{for all } 0 \leq p \leq k.$$

Lemma 2.2. *Let $0 \to U \to V \to W \to 0$ be a short exact sequence of vector bundles on a projective variety X. If $H^q(X, \wedge^{k-p} U \otimes \wedge^p W) = 0$ for all $0 \leq p \leq k$, then $H^q(X, \wedge^k V) = 0$.*

Proof. By considering the natural filtration (2.3), we immediately obtain the lemma. $\square$

2.3. **Divided and symmetric powers.** Let V be a finite dimensional vector space over $\Bbbk$. For an integer $n \geq 1$, the symmetric group $\mathfrak{S}_n$ naturally acts on the tensor power $T^n V := V^{\otimes n}$ by permuting the factors. The *divided power* of V is the subspace

$$D^n V := \{\omega \in T^n V \mid \sigma(\omega) = \omega \text{ for all } \sigma \in \mathfrak{S}_n\} \subseteq T^n V,$$

while the *symmetric power* $S^n V$ of V is the quotient of $T^n V$ by the subspace spanned by $\sigma(\omega) - \omega$ for all $\omega \in T^n V$ and $\sigma \in \mathfrak{S}_n$. When $n = 0$, we set $T^0 V = D^0 V = S^0 V = \Bbbk$. We have a natural identification

$$D^n V = (S^n V^*)^*.$$

By composing the inclusion of $D^n V$ into $T^n V$ with the projection onto $S^n V$, we have a natural map $D^n V \to S^n V$. This map is an isomorphism in characteristic zero, but it may be neither injective nor surjective in general. We can also define divided and symmetric powers of vector bundles on projective varieties. We refer to [2, Section 3] for more details.

Let $0 \to U \to V \to W \to 0$ be a short exact sequence of vector bundles on a projective variety X with rank $W = 1$. Let $\pi \colon \mathbb{P}(V^*) \to X$ be the natural projection. Note that $\mathcal{O}_{\mathbb{P}(V^*)}(-\mathbb{P}(U^*)) = \mathcal{O}_{\mathbb{P}(V^*)}(-1) \otimes \pi^* W^*$. For $k \geq 0$, we have a short exact sequence on $\mathbb{P}(V^*)$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}(V^*)}(k) \otimes \pi^* W^* \longrightarrow \mathcal{O}_{\mathbb{P}(V^*)}(k+1) \longrightarrow \mathcal{O}_{\mathbb{P}(U^*)}(k+1) \longrightarrow 0.$$

By applying π_*, we get a short exact sequence on X:

$$0 \longrightarrow S^k V^* \otimes W^* \longrightarrow S^{k+1} V^* \longrightarrow S^{k+1} U^* \longrightarrow 0.$$

This construction was suggested by Lawrence Ein, and a purely algebraic construction of this kind of exact sequence can be found in [1, Corollary V.1.15]. By taking the dual, we obtain a short exact sequence on X:

$$(2.4) \qquad\qquad 0 \longrightarrow D^{k+1} U \longrightarrow D^{k+1} V \longrightarrow D^k V \otimes W \longrightarrow 0.$$

We remark that $S^{k+1} V \to S^k V \otimes W$ may not be surjective in positive characteristic.

Now, let C be a smooth projective curve, and L be a line bundle on C. For an integer $k \geq 0$, the symmetric group $\mathfrak{S}_{k+1}$ naturally acts on the $(k+1)$-th ordinary product C^{k+1} of C by permuting the components, and the line bundle

$$L^{\boxtimes k+1} := \underbrace{L \boxtimes \cdots \boxtimes L}_{k+1 \text{ times}}$$

on C^{k+1} descends to a line bundle $T_{k+1}(L)$ on the $(k+1)$-th symmetric product C_{k+1} of C (see [16, Subsection 3.1]). Note that

$$H^0(C_{k+1}, T_{k+1}(L)) = H^0(C^{k+1}, L^{\boxtimes k+1})^{\mathfrak{S}_{k+1}} = D^{k+1}H^0(C, L).$$

If $C = \mathbb{P}^1$ and $L = \mathcal{O}_{\mathbb{P}^1}(d)$ with $d \geq 1$, then $(\mathbb{P}^1)_n = \mathbb{P}^n$ and $T_n(\mathcal{O}_{\mathbb{P}^1}(d)) = \mathcal{O}_{\mathbb{P}^n}(d)$. By an arbitrary characteristic version of Hermite reciprocity (see [2, Remark 3.2]), we have

$$\begin{aligned}
H^0(\mathbb{P}^n, T_n(\mathcal{O}_{\mathbb{P}^1}(d))) &= D^n H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \\
&= D^n(S^d H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))) \\
&= S^d(D^n H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))) \\
&= S^d H^0(\mathbb{P}^n, T_n(\mathcal{O}_{\mathbb{P}^1}(1))).
\end{aligned}$$

2.4. Tautological bundles on projective spaces.

Recall that $\sigma \colon \mathbb{P}^{n-1} \times \mathbb{P}^1 \to \mathbb{P}^n$ is the finite map of degree n given by $(\xi, z) \mapsto \xi + z$ by viewing $\mathbb{P}^n$ as the Hilbert scheme of n points on $\mathbb{P}^1$. For any integer k, the *tautological bundle* on $\mathbb{P}^n$ is defined as

$$E_{n, \mathcal{O}_{\mathbb{P}^1}(k)} := \sigma_*(\mathcal{O}_{\mathbb{P}^{n-1}} \boxtimes \mathcal{O}_{\mathbb{P}^1}(k)),$$

which is a vector bundle of rank n. The tautological bundles on symmetric products of curves play an important role in the study of secant varieties of curves (see [16]).

Lemma 2.3. $E_{n, \mathcal{O}_{\mathbb{P}^1}(k)}$ *is splitting if and only if* $-1 \leq k \leq n - 1$. *In this case,*

$$E_{n, \mathcal{O}_{\mathbb{P}^1}(k)} = \mathcal{O}_{\mathbb{P}^n}^{\oplus k+1} \oplus \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n-1-k}.$$

Proof. Since $\sigma \colon \mathbb{P}^{n-1} \times \mathbb{P}^1 \to \mathbb{P}^n$ is a finite map, we have

$$H^i(\mathbb{P}^n, E_{n, \mathcal{O}_{\mathbb{P}^1}(k)}(m)) = H^i(\mathbb{P}^{n-1} \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^{n-1}}(m) \boxtimes \mathcal{O}_{\mathbb{P}^1}(m+k)).$$

for any $i \geq 0$ and $m \in \mathbb{Z}$. The Künneth formula shows that

$$\begin{aligned}
H^i(\mathbb{P}^{n-1} &\times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^{n-1}}(m) \boxtimes \mathcal{O}_{\mathbb{P}^1}(m+k)) \\
&= \left(H^{i-1}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(m)) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m+k)) \right) \\
&\qquad \oplus \left(H^i(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(m)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m+k)) \right).
\end{aligned}$$

Then we see that

$$\begin{aligned}
H^i(\mathbb{P}^n, E_{n, \mathcal{O}_{\mathbb{P}^1}(k)}(m)) &= 0 \text{ for } 2 \leq i \leq n-2 \text{ and } m \in \mathbb{Z}, \\
H^1(\mathbb{P}^n, E_{n, \mathcal{O}_{\mathbb{P}^1}(k)}(m)) &= 0 \text{ for } m \in \mathbb{Z} \Longleftrightarrow k \geq -1, \\
H^{n-1}(\mathbb{P}^n, E_{n, \mathcal{O}_{\mathbb{P}^1}(k)}(m)) &= 0 \text{ for } m \in \mathbb{Z} \Longleftrightarrow k \leq n-1.
\end{aligned}$$

By the Horrocks criterion, the first assertion of the lemma follows. Now, we observe that

$$h^0(\mathbb{P}^n, E_{n, \mathcal{O}_{\mathbb{P}^1}(k)}) = k+1,$$

and

$$h^0(\mathbb{P}^n, E_{n,\mathcal{O}_{\mathbb{P}^1}(k)}(1)) = n(k+2)$$
$$= (n+1)(k+1) + n - 1 - k.$$

This implies the second assertion of the lemma. □

Remark 2.4. The following alternative approach to Lemma 2.3 was suggested by Lawrence Ein. Let D_n be the image of the injective map $\mathbb{P}^{n-1} \times \mathbb{P}^1 \to \mathbb{P}^n \times \mathbb{P}^1$ given by $(\xi, z) \mapsto (\xi + z, z)$. Note that $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^1}(-D_n) = \mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-n)$. For any integer k, we have a short exact sequence on $\mathbb{P}^n \times \mathbb{P}^1$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(k-n) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{O}_{\mathbb{P}^1}(k) \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}} \boxtimes \mathcal{O}_{\mathbb{P}^1}(k) \longrightarrow 0.$$

Let $p \colon \mathbb{P}^n \times \mathbb{P}^1 \to \mathbb{P}^n$ be the projection to the first component. When $-1 \leq k \leq n-1$, by applying p_*, we obtain a short exact sequence on $\mathbb{P}^n$:

$$0 \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow E_{n,\mathcal{O}_{\mathbb{P}^1}(k)} \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-n)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow 0.$$

Thus

$$E_{n,\mathcal{O}_{\mathbb{P}^1}(k)} = (H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) \otimes \mathcal{O}_{\mathbb{P}^n}) \oplus (H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-n)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1)).$$

When $k \leq -2$ or $k \geq n$, it is easy to check that $E_{n,\mathcal{O}_{\mathbb{P}^1}(k)}$ is not splitting.

Lemma 2.5. *Let Y be a projective variety, and $\sigma \colon Y \times \mathbb{P}^{n-1} \times \mathbb{P}^1 \to Y \times \mathbb{P}^n$ be the finite map given by $(y, \xi, z) \mapsto (y, \xi + z)$. If M is a vector bundle on $Y \times \mathbb{P}^n$, then*

$$H^q(Y \times \mathbb{P}^n, M) = 0 \iff H^q(Y \times \mathbb{P}^{n-1} \times \mathbb{P}^1, \sigma^* M \otimes (\mathcal{O}_{Y \times \mathbb{P}^{n-1}} \boxtimes \mathcal{O}_{\mathbb{P}^1}(n-1))) = 0$$

for any $q \geq 0$.

Proof. By Lemma 2.3 and the projection formula,

$$\sigma_*(\sigma^* M \otimes (\mathcal{O}_{Y \times \mathbb{P}^{n-1}} \boxtimes \mathcal{O}_{\mathbb{P}^1}(n-1))) = M^{\oplus n}.$$

Since σ is a finite map, the assertion immediately follows. □

3. Asymptotic vanishing theorem

The aim of this section is to prove Theorem 1.1. First, we construct a short exact sequence of vector bundles, which allows us to give a quick proof of Theorem 1.2 by induction on dimension. We then explain how one can deduce Theorem 1.1 from Theorem 1.2.

3.1. **Short exact sequence.** Let C be a smooth projective curve, and L_C be a line bundle on C. Let Y be a smooth projective variety, and L_Y be a line bundle on Y. Fix an integer $k \geq 0$. Assume that the line bundle $L_Y \boxtimes T_{k+1}(L_C)$ on $Y \times C_{k+1}$ is globally generated. Notice that

$$H^0(Y \times C_{k+1}, L_Y \boxtimes T_{k+1}(L_C)) = H^0(Y, L_Y) \otimes D^{k+1} H^0(C, L_C).$$

We have a short exact sequence on $Y \times C_{k+1}$:

$$0 \longrightarrow M_{L_Y \boxtimes T_{k+1}(L_C)} \longrightarrow H^0(Y, L_Y) \otimes D^{k+1} H^0(C, L_C) \otimes \mathcal{O}_{Y \times C_{k+1}} \longrightarrow L_Y \boxtimes T_{k+1}(L_C)$$
$$\longrightarrow 0.$$

We can view $C_{k+1} = \{\text{effective divisors of degree } k+1 \text{ on } C\}$ as the Hilbert scheme of $k+1$ points on C. Let

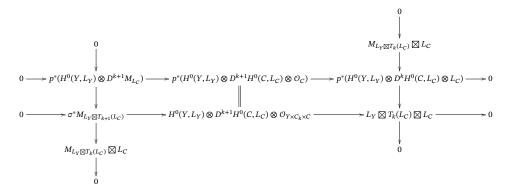$$\sigma : Y \times C_k \times C \longrightarrow Y \times C_{k+1}$$

be the finite morphism given by $(y, \xi, z) \mapsto (y, \xi + z)$, and $p : Y \times C_k \times C \to C$ be the projection to the last component. By taking σ^* of the above exact sequence on $Y \times C_{k+1}$, we get a short exact sequence on $Y \times C_k \times C$:

$$0 \to \sigma^* M_{L_Y \boxtimes T_{k+1}(L_C)} \to H^0(Y, L_Y) \otimes D^{k+1} H^0(C, L_C) \otimes \mathcal{O}_{Y \times C_k \times C}$$
$$\to L_Y \boxtimes T_k(L_C) \boxtimes L_C \to 0,$$

By taking p_* and considering (2.4), we get a short exact sequence on C:

$$0 \to H^0(Y, L_Y) \otimes D^{k+1} M_{L_C} \to H^0(Y, L_Y) \otimes D^{k+1} H^0(C, L_C) \otimes \mathcal{O}_C$$
$$\to H^0(Y, L_Y) \otimes D^k H^0(C, L_C) \otimes L_C \to 0.$$

Then we obtain the following commutative diagram with exact sequences on $Y \times C_k \times C$:

Now, assume that $C = \mathbb{P}^1$ and $L_C = \mathcal{O}_{\mathbb{P}^1}(d)$ with $d \geq 1$. For an integer $n \geq 1$, we have

$$(\mathbb{P}^1)_n = \mathbb{P}^n, \ T_n(\mathcal{O}_{\mathbb{P}^1}(d)) = \mathcal{O}_{\mathbb{P}^n}(d), \ M_{L_Y \boxtimes T_n(\mathcal{O}_{\mathbb{P}^1}(d))} = M_{L_Y \boxtimes \mathcal{O}_{\mathbb{P}^n}(d)}.$$

Since $M_{\mathcal{O}_{\mathbb{P}^1}(d)} = \bigoplus \mathcal{O}_{\mathbb{P}^1}(-1)$, it follows that

$$D^n M_{\mathcal{O}_{\mathbb{P}^1}(d)} = \left(S^n M^*_{\mathcal{O}_{\mathbb{P}^1}(d)} \right)^* = \bigoplus \mathcal{O}_{\mathbb{P}^1}(-n).$$

Then the left vertical short exact sequence in the above commutative diagram gives a short exact sequence on $Y \times \mathbb{P}^{n-1} \times \mathbb{P}^1$:
(3.1)
$$0 \longrightarrow \bigoplus \mathcal{O}_{Y \times \mathbb{P}^{n-1}} \boxtimes \mathcal{O}_{\mathbb{P}^1}(-n) \longrightarrow \sigma^* M_{L_Y \boxtimes \mathcal{O}_{\mathbb{P}^n}(d)} \longrightarrow M_{L_Y \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(d)} \boxtimes \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow 0.$$

When Y is a point, the exact sequence (3.1) is

$$0 \longrightarrow \bigoplus \mathcal{O}_{\mathbb{P}^{n-1}} \boxtimes \mathcal{O}_{\mathbb{P}^1}(-n) \longrightarrow \sigma^* M_{\mathcal{O}_{\mathbb{P}^n}(d)} \longrightarrow M_{\mathcal{O}_{\mathbb{P}^{n-1}}(d)} \boxtimes \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow 0.$$

When $n = 1$, the finite map σ is an isomorphism and the exact sequence (3.1) is

$$0 \longrightarrow \bigoplus \mathcal{O}_Y \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow M_{L_Y \boxtimes \mathcal{O}_{\mathbb{P}^1}(d)} \longrightarrow M_{L_Y} \boxtimes \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow 0.$$

3.2. **Case of product of projective spaces.** In this subsection, we prove Theorem 1.2. Recall that

$$X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k},$$
$$B = \mathcal{O}_{\mathbb{P}^{n_1}}(b_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{n_k}}(b_k),$$
$$L = \mathcal{O}_{\mathbb{P}^{n_1}}(d_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{n_k}}(d_k),$$

and $b = \min\{b_1, \ldots, b_k\}$, $d = \min\{d_1, \ldots, d_k\}$. Fix $2 \leq q \leq n_1 + \cdots + n_k + 1$. Our aim is to show that if $d + b \geq 0$, then

$$K_{p,q}(X, B; L) = 0 \quad \text{for } 0 \leq p \leq (1/n_1! \cdots n_k!)(d^{q-1} + bd^{q-2}).$$

We put

$$Y := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_{k-1}},$$
$$B_Y := \mathcal{O}_{\mathbb{P}^{n_1}}(b_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{n_{k-1}}}(b_{k-1}),$$
$$L_Y := \mathcal{O}_{\mathbb{P}^{n_1}}(d_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{n_{k-1}}}(d_{k-1}),$$

and $n := n_k$. Then

$$X = Y \times \mathbb{P}^n, \ B = B_Y \boxtimes \mathcal{O}_{\mathbb{P}^n}(b_k), \ L = L_Y \boxtimes \mathcal{O}_{\mathbb{P}^n}(d_k).$$

As $md + b \geq 0$ for any $m > 0$, we have

$$H^i(X, B \otimes L^m) = 0 \quad \text{for } i > 0 \text{ and } m > 0.$$

By Proposition 2.1, for $p \geq 0$, we have

$$K_{p,q}(X, B; L) = H^{q-1}\big(Y \times \mathbb{P}^n, \wedge^{p+q-1} M_{L_Y \boxtimes \mathcal{O}_{\mathbb{P}^n}(d_k)} \otimes ((L_Y + B_Y) \boxtimes \mathcal{O}_{\mathbb{P}^n}(d_k + b_k))\big).$$

We proceed by induction on $n_1 + \cdots + n_k$. If $n_1 + \cdots + n_k = 1$, then $q = 2$ and the problem is to check the cohomology vanishing

$$H^1\big(\mathbb{P}^1, \wedge^{p+1} M_{\mathcal{O}_{\mathbb{P}^1}(d)} \otimes \mathcal{O}_{\mathbb{P}^1}(d + b)\big) = 0 \quad \text{for } 0 \leq p \leq d + b.$$

As $\wedge^{p+1} M_{\mathcal{O}_{\mathbb{P}^1}(d)} \otimes \mathcal{O}_{\mathbb{P}^1}(d + b) = \bigoplus \mathcal{O}_{\mathbb{P}^1}(d + b - p - 1)$ and $d + b - p - 1 \geq -1$, the desired cohomology vanishing immediately follows.

Assume that $n_1 + \cdots + n_k \geq 2$. Fix $0 \leq p \leq (1/n_1! \cdots n_k!)(d^{q-1} + bd^{q-2})$. By Lemma 2.5, it is sufficient to show the cohomology vanishing on $Y \times \mathbb{P}^{n-1} \times \mathbb{P}^1$:

$$H^{q-1}\big(\wedge^{p+q-1} \sigma^* M_{L_Y \boxtimes \mathcal{O}_{\mathbb{P}^n}(d_k)} \otimes ((L_Y + B_Y) \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(d_k + b_k) \boxtimes \mathcal{O}_{\mathbb{P}^1}(d_k + b_k + n - 1))\big) = 0,$$

where $\sigma: Y \times \mathbb{P}^{n-1} \times \mathbb{P}^1 \to Y \times \mathbb{P}^n$ is the finite map given by $(y, \xi, z) \mapsto (y, \xi + z)$. By considering the short exact sequence (3.1) and applying Lemma 2.2 to

$$\wedge^{p+q-1} \sigma^* M_{L_Y \boxtimes \mathcal{O}_{\mathbb{P}^n}(d_k)},$$

we can reduce the problem to proving the following:

$$H^{q-1}\big(Y \times \mathbb{P}^{n-1} \times \mathbb{P}^1, (\wedge^i M_{L_Y \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(d_k)} \otimes ((L_Y + B_Y) \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(d_k + b_k))) \boxtimes \mathcal{O}_{\mathbb{P}^1}(a_i)\big) = 0$$

for $0 \leq i \leq p + q - 1 \ (\leq (1/n_1! \cdots n_k!)(d^{q-1} + bd^{q-2}) + q - 1)$, where

$$a_i := id_k - (p + q - 1 - i)n + d_k + b_k + n - 1 = i(d_k + n) + d_k + b_k + 2n - qn - 1 - pn.$$

By the Künneth formula, it is equivalent to showing that
(3.2)
$$H^{q-1}\big(Y \times \mathbb{P}^{n-1}, \wedge^i M_{L_Y \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(d_k)} \otimes ((L_Y + B_Y) \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(d_k + b_k))\big) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i)) = 0$$

and
(3.3)
$$H^{q-2}(Y \times \mathbb{P}^{n-1}, \wedge^i M_{L_Y \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(d_k)} \otimes ((L_Y + B_Y)) \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(d_k + b_k)) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i)) = 0$$

for $0 \leq i \leq p + q - 1$.

By induction and Proposition 2.1, we can assume that

$$(3.4) \qquad H^{q-1}(Y \times \mathbb{P}^{n-1}, \wedge^i M_{L_Y \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(d_k)} \otimes ((L_Y + B_Y) \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(d_k + b_k))) = 0$$

for $0 \leq i \leq (1/n_1! \cdots n_{k-1}!(n_k - 1)!)(d^{q-1} + bd^{q-2}) + q - 1$ when $q \geq 2$, and

$$(3.5) \qquad H^{q-2}(Y \times \mathbb{P}^{n-1}, \wedge^i M_{L_Y \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(d_k)} \otimes ((L_Y + B_Y) \boxtimes \mathcal{O}_{\mathbb{P}^{n-1}}(d_k + b_k))) = 0$$

for $0 \leq i \leq (1/n_1! \cdots n_{k-1}!(n_k - 1)!)(d^{q-2} + bd^{q-3}) + q - 2$ when $q \geq 3$. Observe that
(3.2) immediately follows from (3.4). It only remains to check (3.3). If $q = 2$, then

$$a_i = i(d_k + n) + d_k + b_k - 1 - np \geq d_k + b_k - 1 - np \geq d + b - 1 - np \geq -1$$

since $p \leq (1/n_1! \cdots n_k!)(d + b) \leq (1/n)(d + b)$. Thus

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i)) = 0,$$

so the cohomology vanishing (3.3) holds for $q = 2$. Next, we consider the case that
$q \geq 3$. If $0 \leq i \leq (1/n_1! \cdots n_{k-1}!(n_k - 1)!)(d^{q-2} + bd^{q-3}) + q - 2$, then (3.3) immediately
follows from (3.5). If $i \geq (1/n_1! \cdots n_{k-1}!(n_k - 1)!)(d^{q-2} + bd^{q-3}) + q - 1$, then

$$a_i = id_k + \underbrace{(d_k + b_k + 2n + in - qn)}_{\geq 0} - 1 - np$$

$$\geq id - 1 - np$$

$$\geq (1/n_1! \cdots n_{k-1}!(n - 1)!)(d^{q-1} + bd^{q-2}) - 1 - np$$

$$\geq -1$$

since $p \leq (1/n_1! \cdots n_{k-1}! n!)(d^{q-1} + bd^{q-2})$. Thus

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i)) = 0,$$

so the cohomology vanishing (3.3) holds in this case as well. We have shown (3.2) and
(3.3), and they imply $K_{p,q}(X, B; L) = 0$ as desired. We complete the proof of Theo-
rem 1.2.

3.3. General case.

In this subsection, we prove Theorem 1.1. As we mentioned in the
introduction, Raicu proved that Theorem 1.1 can be deduced from Theorem 1.2 for the
case $k = 3$ (see [28, Corollary A.5]). Here we reproduce his proof for the completeness.
Recall that X is an n-dimensional projective variety, B is coherent sheaf, A is an ample
divisor, P is an arbitrary divisor on X, and $L_d := \mathcal{O}_X(dA + P)$ for an integer $d \geq 1$. Our
aim is to show that for each $2 \leq q \leq n + 1$ (the case that $q = 1$ is trivial), there is a
constant $C > 0$ depending on X, A, B, P such that if d is sufficiently large, then

$$K_{p,q}(X, B; L_d) = 0 \quad \text{for } 0 \leq p \leq Cd^{q-1}.$$

We can choose integers $a_1, a_2, a_3 \geq 1$ with $\gcd(a_1, a_2 + a_3) = 1$ such that

$$A_1 := a_1 A, \ A_2 := a_2 A + P, \ A_3 := a_3 A - P$$

are very ample and the natural maps

$$(3.6) \ S^{m_1} H^0(X, A_1) \otimes S^{m_2} H^0(X, A_2) \otimes S^{m_3} H^0(X, A_3) \longrightarrow H^0(X, m_1 A_1 + m_2 A_2 + m_3 A_3)$$

are surjective for all $m_1, m_2, m_3 > 0$. We may assume that $a_1 \gg a_2 + a_3$. Note that a_1, a_2, a_3 are depending only on X, A, P. As d is sufficiently large, we can find integers $d_1, d_2 \geq 1$ such that $d_1 \approx d/2a_1, d_2 \approx d/2(a_2 + a_3)$ and $d + a_3 = a_1 d_1 + (a_2 + a_3)d_2$. Let $d_3 := d_2 - 1$. Then $d = a_1 d_1 + a_2 d_2 + a_3 d_3$, and $L_d = \mathcal{O}_X(d_1 A_1 + d_2 A_2 + d_3 A_3)$. Note that $d_1 < d_3 < d_2$.

Next, consider the commutative diagrams

$$X \hookrightarrow \mathbb{P}H^0(X, A_1) \times \mathbb{P}H^0(X, A_2) \times \mathbb{P}H^0(X, A_3) =: \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3} = Y$$

$$\mathbb{P}^r := \mathbb{P}H^0(X, L_d) \hookrightarrow \mathbb{P}H^0(Y, \mathcal{O}_{\mathbb{P}^{n_1}}(d_1) \boxtimes \mathcal{O}_{\mathbb{P}^{n_2}}(d_2) \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}}(d_3)) =: \mathbb{P}^N.$$

Clearly, n_1, n_2, n_3 are depending only on X, A, P. Notice that $\mathbb{P}^r$ is a linear subspace of $\mathbb{P}^N$ by the surjectivity of (3.6). We can regard B as a coherent sheaf on Y, $\mathbb{P}^r$, and $\mathbb{P}^N$. The syzygies of B on $\mathbb{P}^N$ are the syzygies of B on $\mathbb{P}^r$ tensoring with a Koszul complex of linear forms. By letting

$$L_Y := \mathcal{O}_{\mathbb{P}^{n_1}}(d_1) \boxtimes \mathcal{O}_{\mathbb{P}^{n_2}}(d_2) \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}}(d_3),$$

we see that

$$(3.7) \qquad \min\{p \mid K_{p,q}(X, B; L_d) \neq 0\} = \min\{p \mid K_{p,q}(Y, B; L_Y) \neq 0\}.$$

Let $S := S(H^0(X, A_1) \oplus H^0(X, A_2) \oplus H^0(X, A_3))$ be the total coordinate ring of $Y = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$ with the usual $\mathbb{Z}^3$-grading. Then

$$M := \bigoplus_{(a_1, a_2, a_3) \in \mathbb{Z}_{\geq 0}^3} H^0(Y, B \otimes \mathcal{O}_{\mathbb{P}^{n_1}}(a_1) \boxtimes \mathcal{O}_{\mathbb{P}^{n_2}}(a_2) \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}}(a_3))$$

is a finitely generated graded S-module. Consider the minimal free resolution of M:

$$0 \longleftarrow M \longleftarrow E_0 \longleftarrow E_1 \longleftarrow \cdots \longleftarrow E_m \longleftarrow 0$$

where

$$E_i = \bigoplus_{b^j = (b_1^j, b_2^j, b_3^j) \in S_i} F_{i,b^j} \otimes S(b^j)$$

for some finite dimensional vector space F_{i,b^j} over $\Bbbk$ and finite subsets $S_i \subseteq \mathbb{Z}^3$. Let

$$B_Y^j := \mathcal{O}_{\mathbb{P}^{n_1}}(b_1^j) \boxtimes \mathcal{O}_{\mathbb{P}^{n_2}}(b_2^j) \boxtimes \mathcal{O}_{\mathbb{P}^{n_3}}(b_3^j)$$

and

$$b := \min\{b_1^j, b_2^j, b_3^j \mid b^j \in S_i, 0 \leq i \leq m\}.$$

Note that b is depending only on X, A, B, P. Now, fix $2 \leq q \leq n+1 \ (\leq n_1 + n_2 + n_3 + 1)$. By Theorem 1.2, for $0 \leq i \leq m$ and $b^j \in S_i$, we have

$$(3.8) \quad K_{p-i,q+i}(Y, B_Y^j; L_Y) = 0 \quad \text{for } 0 \leq p \leq (1/n_1! \, n_2! \, n_3!)(d_1^{q+i-1} + b d_1^{q+i-2}) + i.$$

Then (3.7) and [28, Theorem A.1] (cf. [21, Proposition 1.d.3]) show that

$$(3.9) \ K_{p,q}(X, B; L_d) = K_{p,q}(Y, B; L_Y) = 0 \quad \text{for } 0 \leq p \leq (1/n_1! \, n_2! \, n_3!)(d_1^{q-1} + b d_1^{q-2}).$$

Recall that the numbers $a_1, a_2, a_3, n_1, n_2, n_3, b$ are depending only on X, A, B, P but not on d. Since $d_1 \approx d/2a_1$ grows linearly in d and d is sufficiently large, Theorem 1.1 follows.

Remark 3.1. Instead of applying [28, Theorem A.1], one can alternatively prove (3.9) as follows. From the minimal free resolution of M, we have an exact sequence on Y:

$$0 \longleftarrow B \longleftarrow \mathcal{E}_0 \longleftarrow \mathcal{E}_1 \longleftarrow \cdots \longleftarrow \mathcal{E}_m \longleftarrow 0$$

where

$$\mathcal{E}_i = \bigoplus_{b^j \in S_i} F_{i,b^j} \otimes B_Y^j.$$

By using Proposition 2.1 and chasing through the above exact sequence, we see that

$$K_{p,q}(Y, B; L_Y) = H^{q-1}(Y, \wedge^{p+q-1} M_{L_Y} \otimes B \otimes L_Y) = 0$$

is implied by

$$K_{p-i,q+i}(Y, B_Y^j; L_Y) = H^{q+i-1}(Y, \wedge^{(p-i)+(q+i)-1} M_{L_Y} \otimes B_Y^j \otimes L_Y) = 0$$

for all $0 \le i \le m$ and $b^j \in S_i$. Thus (3.7) and (3.8) imply (3.9).

4. Open problems

In this section, we discuss some open problems and conjectures. Let X be a smooth projective variety of dimension n, and B be a coherent sheaf on X. Fix an ample divisor A and an arbitrary divisor P on X, and put $L_d := \mathcal{O}_X(dA + P)$ for an integer $d \ge 1$.

For each $2 \le q \le n+1$, it would be extremely interesting to find an explicit constant $c > 0$ in terms of X, A, B, P, and q, d such that if d is sufficiently large, then

$$K_{p,q}(X, B; L_d) = 0 \text{ for } 0 \le p \le c \text{ and } K_{c+1,q}(X, B; L_d) \ne 0.$$

However, this problem is already very difficult for $q = 2$. A generalization of Mukai's conjecture (cf. [11, Conjecture 4.2]) asks whether the property N_d holds for $K_X + (n+2+d)A$ when X is a smooth projective complex variety. But it is widely open even when $n = 2$ and $d = 0$. Moreover, Fujita's conjecture, which predicts that $K_X + (n+2+d)A$ is very ample for $d \ge 0$, is unknown when $n \ge 3$. However, when A is very ample, Ein–Lazarsfeld established in [11, Theorem 1] that

$$K_{p,q}(X, K_X + (n+1+d)A) = 0 \quad \text{for } 0 \le p \le d \text{ and } q \ge 2.$$

It is reasonable to expect extending this result for $q \ge 3$.

Problem 4.1. Let X be a smooth projective complex variety of dimension n, and A be a very ample divisor on X. For each $2 \le q \le n+1$ and $d \ge 0$, find an explicit polynomial $P(x)$ of degree $q - 1$ such that

$$K_{p,q}(X, K_X + (n+1+d)A) = 0 \quad \text{for } 0 \le p \le P(d)$$

One can also consider the effective asymptotic vanishing problem for the syzygies of products of projective spaces.

Problem 4.2. Let $k \ge 1$ be an integer, $n_1, \ldots, n_k, d_1, \ldots, d_k$ be positive integers, and $b_1, \ldots, b_k$ be integers. Set

$$X := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k},$$
$$B := \mathcal{O}_{\mathbb{P}^{n_1}}(b_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{n_k}}(b_k),$$
$$L := \mathcal{O}_{\mathbb{P}^{n_1}}(d_1) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{n_k}}(d_k),$$

For each $2 \leq q \leq n_1 + \cdots + n_k + 1$, find a constant $c > 0$ in terms of $b_1, \ldots, b_k, d_1, \ldots, d_k$, and q such that if $d_1, \ldots, d_k$ are sufficiently large, then

(4.1) $K_{p,q}(X, B; L) = 0$ for $0 \leq p \leq c$ and $K_{c+1,q}(X, B; L) \neq 0$.

In Theorem 1.2, we prove that if $d + b \geq 0$, then

$$K_{p,q}(X, B; L) = 0 \quad \text{for } 0 \leq p \leq (1/n_1! \cdots n_k!)(d^{q-1} + bd^{q-2}),$$

where $b := \min\{b_1, \ldots, b_k\}$, $d := \min\{d_1, \ldots, d_k\}$. By a small modification of the proof, we can improve the bound on p, but our method only gives a bound on p depending on dimensions $n_1, \ldots, n_k$. It is expected that the constant c in Problem 4.2 is independent of $n_1, \ldots, n_k$. It would be also interesting to find a constant $d > 0$ in terms of $b_1, \ldots, b_k, n_1, \ldots, n_k$, and q such that if $d_1, \ldots, d_k \geq d$, then (4.1) holds. When $k = 1$, there is a precise prediction on c and d (see [12, Conjecture 7.5], [14, Conjecture 2.3]).

Conjecture 4.3 (Ein–Lazarsfeld). *Fix $n \geq 1$, $b \geq 0$, and $0 \leq q \leq n$. If $d \geq b + q + 1$, then*

$$K_{p,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(b); \mathcal{O}_{\mathbb{P}^n}(d)) = 0 \quad \text{for } 0 \leq p \leq \binom{d+q}{q} - \binom{d-b-1}{q} - q - 1.$$

Notice that the conjecture gives the precise vanishing range because Ein–Erman–Lazarsfeld [10, Theorem 2.1] (see also [14, Theorem 2.1]) proved that

$$K_{p,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(b); \mathcal{O}_{\mathbb{P}^n}(d)) \neq 0$$

for all

$$\binom{d+q}{q} - \binom{d-b-1}{q} - q \leq p \leq \binom{d+n}{n} + \binom{d+n-q}{n-q} - \binom{n+b}{q+b} - q - 1,$$

In [26], Ottaviani–Paoletti conjectured that if $n \geq 3$, $d \geq 3$, then $\mathcal{O}_{\mathbb{P}^n}(d)$ satisfies the property N_{3d-3}. They also consider the cases that $n \leq 2$ or $d \leq 2$, but these cases are already settled. By [26, Theorem 1.6], the property N_{3d-3} for $\mathcal{O}_{\mathbb{P}^n}(d)$ is implied by that $K_{p,2}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = 0$ for $0 \leq p \leq 3d - 3$. Thus Conjecture 4.3 for $b = 0$ and $q = 2$ is equivalent to Ottaviani–Paoletti's conjecture. At this moment, we only know that $\mathcal{O}_{\mathbb{P}^n}(d)$ satisfies the property N_{d+1} by Bruns–Conca–Römer [7], and a small change of the proof of Theorem 1.2 yields that $\mathcal{O}_{\mathbb{P}^3}(d)$ satisfies the property N_{d+2}. A new idea might be needed to solve Conjecture 4.3.

It is also a fascinating problem to study the asymptotic behavior of the *Betti numbers*

$$k_{p,q}(X, B; L_d) := \dim K_{p,q}(X, B; L_d)$$

when d is sufficiently large (see [12, Problem 7.3]). In this direction, Ein–Erman–Lazarsfeld conjectured that for each $1 \leq q \leq n$, the Betti numbers $k_{p,q}(X, L_d)$ converge to a normal distribution (see [9, Conjecture B], [14, Conjecture 3.2]). This normal distribution conjecture has not been verified even for $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, and it seems that the conjecture is already very challenging for Veronese embeddings (cf. [5, 6]).

Conjecture 4.4 (Ein–Erman–Lazarsfeld). *Fix $n \geq 1$ and $1 \leq q \leq n$. Then there is a normalizing function $F_q(d)$ such that*

$$F_q(d) \cdot k_{p_d,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow e^{-a^2/2}$$

as $d \to \infty$ and $p_d \to r_d/2 + a\sqrt{r_d}/2$.

Notice that

$$k_{p,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \frac{1}{n} \cdot h^q\big(\mathbb{P}^{n-1} \times \mathbb{P}^1, \wedge^{p+q}\sigma^* M_{\mathcal{O}_{\mathbb{P}^n}(d)} \otimes (\mathcal{O}_{\mathbb{P}^{n-1}} \boxtimes \mathcal{O}_{\mathbb{P}^1}(n-1)))\big)$$

for $p \geq 0$ and $1 \leq q \leq n$. It is tempting to wonder if there is a clever way to compute this Betti number.

We refer to [6] and [12, 14] for more problems and conjectures on syzygies of Veronese embeddings and asymptotic syzygies of algebraic varieties, respectively.

Acknowledgments

References

[1] Kaan Akin, David A. Buchsbaum, and Jerzy Weyman, *Schur functors and Schur complexes*, Adv. in Math. **44** (1982), no. 3, 207–278, DOI 10.1016/0001-8708(82)90039-1. MR658729

[2] Marian Aprodu, Gavril Farkas, Ştefan Papadima, Claudiu Raicu, and Jerzy Weyman, *Koszul modules and Green's conjecture*, Invent. Math. **218** (2019), no. 3, 657–720, DOI 10.1007/s00222-019-00894-1. MR4022070

[3] Marian Aprodu and Jan Nagel, *Koszul cohomology and algebraic geometry*, University Lecture Series, vol. 52, American Mathematical Society, Providence, RI, 2010, DOI 10.1090/ulect/052. MR2573635

[4] Juliette Bruce, *Asymptotic syzygies in the setting of semi-ample growth*, Preprint, arXiv:1904.04944, 2019.

[5] Juliette Bruce, Daniel Corey, Daniel Erman, Steve Goldstein, Robert P. Laudone, and Jay Yang, *Syzygies of $\mathbb{P}^1 \times \mathbb{P}^1$: data and conjectures*, J. Algebra **593** (2022), 589–621, DOI 10.1016/j.jalgebra.2021.10.023. MR4352589

[6] Juliette Bruce, Daniel Erman, Steve Goldstein, and Jay Yang, *Conjectures and computations about Veronese syzygies*, Exp. Math. **29** (2020), no. 4, 398–413, DOI 10.1080/10586458.2018.1474506. MR4165980

[7] Winfried Bruns, Aldo Conca, and Tim Römer, *Koszul homology and syzygies of Veronese subalgebras*, Math. Ann. **351** (2011), no. 4, 761–779, DOI 10.1007/s00208-010-0616-1. MR2854112

[8] Winfried Bruns, Aldo Conca, and Tim Römer, *Koszul cycles*, Combinatorial aspects of commutative algebra and algebraic geometry, Abel Symp., vol. 6, Springer, Berlin, 2011, pp. 17–33, DOI 10.1007/978-3-642-19492-4_2. MR2810423

[9] Lawrence Ein, Daniel Erman, and Robert Lazarsfeld, *Asymptotics of random Betti tables*, J. Reine Angew. Math. **702** (2015), 55–75, DOI 10.1515/crelle-2013-0032. MR3341466

[10] Lawrence Ein, Daniel Erman, and Robert Lazarsfeld, *A quick proof of nonvanishing for asymptotic syzygies*, Algebr. Geom. **3** (2016), no. 2, 211–222, DOI 10.14231/AG-2016-010. MR3477954

[11] Lawrence Ein and Robert Lazarsfeld, *Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension*, Invent. Math. **111** (1993), no. 1, 51–67, DOI 10.1007/BF01231279. MR1193597

[12] Lawrence Ein and Robert Lazarsfeld, *Asymptotic syzygies of algebraic varieties*, Invent. Math. **190** (2012), no. 3, 603–646, DOI 10.1007/s00222-012-0384-5. MR2995182

[13] Lawrence Ein and Robert Lazarsfeld, *The gonality conjecture on syzygies of algebraic curves of large degree*, Publ. Math. Inst. Hautes Études Sci. **122** (2015), 301–313, DOI 10.1007/s10240-015-0072-2. MR3415069

[14] Lawrence Ein and Robert Lazarsfeld, *Syzygies of projective varieties of large degree: recent progress and open problems*, Algebraic geometry: Salt Lake City 2015, Proc. Sympos. Pure Math., vol. 97, Amer. Math. Soc., Providence, RI, 2018, pp. 223–242, DOI 10.4310/pamq.2005.v1.n2.a8. MR3821151

[15] Lawrence Ein, Robert Lazarsfeld, and David Yang, *A vanishing theorem for weight-one syzygies*, Algebra Number Theory **10** (2016), no. 9, 1965–1981, DOI 10.2140/ant.2016.10.1965. MR3576117

[16] Lawrence Ein, Wenbo Niu, and Jinhyung Park, *Singularities and syzygies of secant varieties of nonsingular projective curves*, Invent. Math. **222** (2020), no. 2, 615–665, DOI 10.1007/s00222-020-00976-5. MR4160876

[17] David Eisenbud, *The geometry of syzygies*, Graduate Texts in Mathematics, vol. 229, Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry. MR2103875

[18] David Eisenbud, Mark Green, Klaus Hulek, and Sorin Popescu, *Restricting linear syzygies: algebra and geometry*, Compos. Math. **141** (2005), no. 6, 1460–1478, DOI 10.1112/S0010437X05001776. MR2188445

[19] Mihai Fulger and Xin Zhou, *Schur asymptotics of Veronese syzygies*, Math. Ann. **362** (2015), no. 1-2, 529–540, DOI 10.1007/s00208-014-1125-4. MR3343887

[20] F. J. Gallego and B. P. Purnaprajna, *Projective normality and syzygies of algebraic surfaces*, J. Reine Angew. Math. **506** (1999), 145–180, DOI 10.1515/crll.1999.506.145. MR1665689

[21] Mark L. Green, *Koszul cohomology and the geometry of projective varieties*, J. Differential Geom. **19** (1984), no. 1, 125–171. MR739785

[22] Mark L. Green, *Koszul cohomology and the geometry of projective varieties. II*, J. Differential Geom. **20** (1984), no. 1, 279–289. MR772134

[23] Mark Green and Robert Lazarsfeld, *On the projective normality of complete linear series on an algebraic curve*, Invent. Math. **83** (1986), no. 1, 73–90, DOI 10.1007/BF01388754. MR813583

[24] M. Green and R. Lazarsfeld, *Some results on the syzygies of finite sets and algebraic curves*, Compositio Math. **67** (1988), no. 3, 301–314. MR959214

[25] Luke Oeding, Claudiu Raicu, and Steven V. Sam, *On the (non-)vanishing of syzygies of Segre embeddings*, Algebr. Geom. **6** (2019), no. 5, 571–591, DOI 10.14231/ag-2019-026. MR4009174

[26] Giorgio Ottaviani and Raffaella Paoletti, *Syzygies of Veronese embeddings*, Compositio Math. **125** (2001), no. 1, 31–37, DOI 10.1023/A:1002662809474. MR1818055

[27] Giuseppe Pareschi, *Syzygies of abelian varieties*, J. Amer. Math. Soc. **13** (2000), no. 3, 651–664, DOI 10.1090/S0894-0347-00-00335-0. MR1758758

[28] Claudiu Raicu, *Representation stability for syzygies of line bundles on Segre-Veronese varieties*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 6, 1201–1231, DOI 10.4171/JEMS/611. MR3500834

[29] Jürgen Rathmann, *An effective bound for the gonality conjecture*, Preprint, arXiv:1604.06072, 2016.

[30] Elena Rubei, *A result on resolutions of Veronese embeddings* (English, with English and Italian summaries), Ann. Univ. Ferrara Sez. VII (N.S.) **50** (2004), 151–165. MR2159811

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