COMMUNICATIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 2, Pages 374–379 (December 12, 2022)
https://doi.org/10.1090/cams/15

RIBBON CONCORDANCE OF KNOTS IS A PARTIAL ORDERING

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ABSTRACT. In this note we show that ribbon concordance forms a partial ordering on the set of knots, answering a question of Gordon [Math. Ann. 257 (1981), pp. 157–170, Conjecture 1.1]. The proof makes use of representation varieties of the knot groups to $SO(N)$ and the subquotient relation between them induced by a ribbon concordance.

1. INTRODUCTION

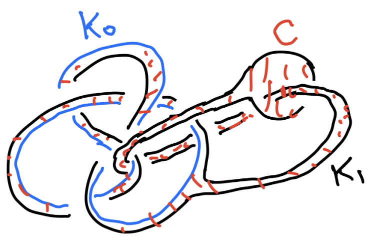

FIGURE 1. A ribbon concordance from a knot K_1 to the figure eight knot K_0

A concordance between knots $K_0, K_1 \subset S^3$ is a smooth embedding of an annulus

$$e : (S^1 \times [0,1], S^1 \times \{0\}, S^1 \times \{1\}) \to (S^3 \times [0,1], K_0 \times \{0\}, K_1 \times \{1\}).$$

We also call the image of the annulus $C = e(S^1 \times [0,1]) \subset S^3 \times [0,1]$ a concordance from K_1 to K_0. If the projection $S^3 \times [0,1] \to [0,1]$ is a Morse function when restricted to C with only critical points of indexes 0 and 1 (so no local maxima), then we say that C is a *ribbon concordance* from K_1 to K_0 (introduced in [7]), and we write $K_1 \geq K_0$ in this case (note: this is the opposite convention of [16] and subsequent papers). Projecting onto S^3, one may see K_1 and K_0 bounding an immersed annulus C with ribbon singularities intersecting only K_1 (see Figure 1 for an example).

The main Conjecture 1.1 of [7] states that this relation is a partial order. The ribbon concordance relation is reflexive and transitive, so the conjecture amounts to asking if it is antisymmetric. That is, if $K_1 \geq K_0$ and $K_0 \geq K_1$, is K_0 isotopic to K_1? Gordon answers this conjecture for knots satisfying various hypotheses, as a special case if K_0 or K_1 is fibered. Much more evidence has been amassed for this conjecture: if $K_0 \geq K_1 \geq K_0$, then K_0 and K_1 have the same S-equivalence class [6, Theorem 1.6], Seifert genus and

Received by the editors January 26, 2022, and, in revised form, April 19, 2022.
2020 *Mathematics Subject Classification.* Primary 57Mxx.
The author was supported by a Simons Investigator grant.

knot Floer homology [16, Theorem 1.4], Khovanov homology [11, Corollary 2], and instanton knot Floer homology [4, Corollary 4.5], [10, Theorem 7.4].

The main result of this note is to answer Gordon's conjecture positively:

Theorem 1.1. *Ribbon concordance is a partial order.*

This will follow pretty immediately from the following (compare [7, Theorem 1.4]):

Theorem 1.2. *Let C be a ribbon concordance from K to K. Then the exterior of C is a relative s-cobordism from the exterior of K to itself.*

In the conclusion we point out that Theorem 1.2 potentially generalizes to homology ribbon cobordism in the sense of [4] and we consider the possibility of answering some other questions from [7, Section 6].

2. Proof of the main theorems

Proof of Theorem 1.2. For $N > 0$, let $R_N(\pi)$ be the representation variety of the group π to $SO(N)$. This is a real algebraic set (the zero-set of polynomials in $\mathbb{R}^k$ for some k) for π finitely generated, with coordinates given by coordinates of the matrices of the generators, and relations given by the rows of the matrices being orthogonal and norm 1, the determinant is 1, and the entries of matrices given by the relators as products of the generator matrices and their inverses/transposes being 0 or 1 (depending on whether it is off- or on-diagonal respectively) to give an identity matrix. Define $R_N(X) = R_N(\pi_1(X))$ for a connected manifold X (we will ignore basepoints as all the spaces are connected and different choices of basepoints will only affect maps between representation varieties up to a change of coordinates).

We have a ribbon concordance $C \subset S^3\times[0,1]$ from $K \subset S^3\times\{1\}$ to $K \subset S^3\times\{0\}$. Let X and X' denote the exterior of K in $S^3\times\{0\}$ and $S^3\times\{1\}$ respectively, and let Y denote the exterior of C in $S^3\times[0,1]$. By [7, Lemma 3.1], $\iota : \pi_1(X') \to \pi_1(Y)$ is surjective (where the map is induced by inclusion), hence the induced map $R_N(Y) \to R_N(X')$ is injective. For our argument, we need to know something slightly stronger, that $R_N(Y) \subseteq R_N(X')$ is an algebraic subset. The point here is that since $\iota : \pi_1(X') \to \pi_1(Y)$ is surjective, we may take a presentation $\pi_1(X') \cong \langle g_1,\dots,g_k | r_1,\dots,r_{k-1}\rangle$, and use the surjection to get a presentation $\pi_1(Y) \cong \langle g_1,\dots,g_k | r_1,\dots,r_{k+n-1}\rangle$, where $r_k,\dots,r_{k+n-1}$ are the extra n relations that hold in $\pi_1(Y)$. Then we see that $R_N(Y)$ is an algebraic subset of $R_N(X')$, with coordinates given by the matrix coordinates of the matrices in $SO(N)$ corresponding to $g_1,\dots,g_k$, together with relations corresponding to the relations defining $SO(N)$ for each matrix and the relators being the identity in $r_1,\dots r_{k-1}$ or $r_1,\dots,r_{k+n-1}$ respectively.

Also by [7, Lemma 3.1] the map $\pi_1(X) \to \pi_1(Y)$ is injective. By [4, Proposition 2.1] the induced map $R_N(Y) \to R_N(X)$ is surjective. Both of these results follow from a result of Gerstenhaber-Rothaus [5, Theorem 1(ii)] which allows one to extend a representation $\rho : \pi_1(X) \to SO(N)$ to a representation $\rho' : \pi_1(Y) \to SO(N)$ which restricts to ρ using the fact that Y has a handle decomposition with n 1-handles and n 2-handles added to a collar neighborhood of X, and so that the 2-handles homologically cancel the 1-handles to obtain a homology cobordism (this is called a ribbon homology cobordism in [4]). Note that the map $R_N(Y) \to R_N(X)$ may be with respect to different coordinates, since the generators of $\pi_1(X)$ may be regarded as a subset of the generators of $\pi_1(Y)$, and hence this polynomial map is a projection onto the subspace

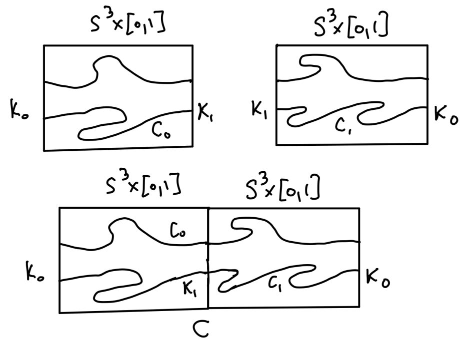

FIGURE 2. Composing ribbon concordances to get a self-concordance

corresponding to the generators of $\pi_1(X)$. There is a polynomial isomorphism from $R_N(X)$ to $R_N(X')$ given by (for example) a sequence of Tietze transformations. Hence we get a surjective polynomial map $R_N(Y) \to R_N(X')$ by composing the projection $R_N(Y) \to R_N(X)$ with the polynomial isomorphism $R_N(X) \to R_N(X')$. We want to show that $\iota : \pi_1(X') \to \pi_1(Y)$ is injective and hence an isomorphism.

Let $c \in \pi_1(X') - \{1\}$, and choose N so that there is a representation $\rho : \pi_1(X') \to SO(N)$ such that $\rho(c) \neq 1$ (using the fact that $\pi_1(X')$ is residually finite, see [8, Theorem 1.1], and that any finite group G embeds into $SO(N)$ for some N). Then $R_N(Y) \subseteq R_N(X')$ is a real algebraic subset with $R_N(Y) \twoheadrightarrow R_N(X')$ a surjective polynomial map by the discussion in the previous two paragraphs. Then by Lemma A.1 $R_N(Y) = R_N(X')$. Thus ρ is the image of a representation $\rho' : \pi_1(Y) \to SO(N)$. Hence $\rho'(\iota(c)) \neq 1$, and therefore $\iota(c)$ is non-trivial in $\pi_1(Y)$. Thus $\iota : \pi_1(X') \to \pi_1(Y)$ is injective, and hence an isomorphism.

The argument finishes the same as the 4th paragraph of the argument of [7, Lemma 3.2] and the proof of [7, Theorem 1.4]. The statement is that the inclusions $X \subset Y$ and $X' \subset Y$ are simple homotopy equivalences, but we will only briefly summarize the part of the argument showing that the inclusions induce isomorphisms of fundamental groups. The map $\pi_1(X) \to \pi_1(Y) \cong \pi_1(X')$ induced by the embedding $X \subset Y$ is injective preserving the peripheral structure (meridian and longitude are sent to meridian and longitude), and thus induces a cover $X \to X'$ by a theorem of Waldhausen [15, Corollary 6.4] (note that there are knots such as the torus knots whose complements self-cover, but the covers do not induce an isomorphism on the peripheral subgroup). Because the peripheral subgroup map is an isomorphism, this cover is index 1 and hence $\pi_1(X) \to \pi_1(Y) \cong \pi_1(X')$ is an isomorphism preserving the peripheral structure. The proof that Y is an s-cobordism is the same as the proof of [7, Theorem 1.4] (this is not necessary for the proof of Theorem 1.1). □

Proof of Theorem 1.1. As observed before, we only need to show that the relation is antisymmetric.

Now, let $K_1 \geq K_0$ by a ribbon concordance C_0 and $K_0 \geq K_1$ by a ribbon concordance C_1. Concatenating the ribbon concordances, we get a ribbon concordance $C = C_0C_1$ from K_0 to K_0 (see Figure 2). Let Y be the exterior of C, Y_i the exterior of C_i, X_i the exterior of K_i. By Theorem 1.2, $\pi_1(X_0) \to \pi_1(Y)$ is an isomorphism (induced by the embedding on the right end of the concordance). Hence the map $\pi_1(X_0) \to \pi_1(Y_1)$ is also an isomorphism. Now again the argument finishes the same as the 4th paragraph of the argument of [7, Lemma 3.2] to show that $\pi_1(X_0) \cong \pi_1(X_1)$ preserving the peripheral subgroups, and hence K_0 and K_1 are isotopic. $\square$

3. Conclusion

We make some remarks on this argument and generalizations and the prospect for addressing some Questions from [7, Section 6].

An earlier version of the proof of Theorem 1.1 used the fact that each knot group embeds into $SO(N)$ for some N [1, 14]. However, we realized that this result is overkill and that we only needed residual finiteness. Theorem 1.1 might generalize to the setting of $\mathbb{Q}$-homology ribbon cobordisms to prove [4, Conjecture 1.1], generalizing [7, Conjecture 1.1]. The proof that a self-homology ribbon cobordism has isomorphic fundamental group (from the right) ought to carry over, but we do not know how to show that it is an s-cobordism (and in general it might not be). The same proof applies to knots which are *strongly homotopy-ribbon concordant* in the sense of [13, Definition 1.1].

One natural question arising from the proof of Theorem 1.1 is whether one may extract an invariant from $R_N(S^3 - K)$ which preserves the partial order. A natural invariant is the ordered list of dimensions of the irreducible components of $R_N(S^3 - K)$, considered with lexicographic ordering. Then this ordering is compatible with the partial ordering of ribbon concordance of knots and thus might give an obstruction to ribbon concordance (but only in one direction for each N since lexicographic order is a total order). For example, if the lexicographic ordering is reversed for two different N, then K_0 and K_1 could not be related by ribbon concordance in either order. Compare to [4, Proposition 1.18].

Another possible invariant is to consider the projection $B_N(K) = im\{R_N(S^3 - K) \to R_N(T^2)\}$, where T^2 is the peripheral torus of $S^3 - K$. In general $B_N(K)$ is only a semi-algebraic set, but the extension lemma [4, Proposition 2.1] implies that $B_N(K_0) = B_N(C) = im\{R_N(Y) \to R_N(T^2)\}$, and $B_N(C) \subset B_N(K_1)$, where C is a ribbon concordance from K_1 to K_0. We have preferred coordinates on $R_N(T^2)$ given by the meridian and longitude, hence we may consider the partial order on knots given by inclusion of $B_N(K)$. One may also consider this partial order for any compact connected Lie group. Hence ribbon concordance is a partial order refining the partial orders of these peripheral algebraic sets. It is likely that these are hard to compute in general, so this may not be a very practical obstruction to ribbon concordance.

One could hope to apply the proof of Theorem 1.1 to answer [7, Question 6.2]. Given a sequence of knots $K_1 \geq K_2 \geq K_3 \geq \cdots$, does there exist n so that $K_m = K_n$ for all $m \geq n$? The proof of Theorem 1.1 shows that the representation varieties $R_N(K_i)$ must stabilize. But to prove injectivity one would need to know that there is a faithful representation independent of N which is not known in general. One special case that might work is if all the K_i are hyperbolic. A conjecture of Chinburg-Reid-Stover [2, Conjecture 1.9] states that there is a curve of characters of $SO(3)$ representations lying on the curve of characters of $PSL_2(\mathbb{C})$ representations containing the discrete faithful representation of a hyperbolic knot group. Assuming this conjecture, one would know that each hyperbolic knot group has a faithful $SO(3)$ representation on this curve since there are only a countable number of non-faithful representations on this curve. Then the proof of Theorem 1.1 considering $SO(3)$ representations would show that Question 6.2 holds for such a sequence. Also assuming [2, Conjecture 1.9], one might be able to show that the main component of the A-polynomial of K_{i+1} divides the A-polynomial of K_i [3], and potentially get some information about [7, Question 6.4].

Appendix A. A lemma in real algebraic geometry (by James Dix)

Here we describe the algebraic geometry needed to obtain the main result. We will be working in the classical setting with real algebraic sets.

Only classical algebraic geometry is required to attain the result of Lemma A.1. This result follows intuitively from the definition of Krull dimension, however an elementary proof using only the Noetherian property of real affine space is presented here. The proof is taken from a discussion on Mathoverflow [9], and a reference for the real algebraic geometry can be found in the appendices of [12].

Lemma A.1. *Let X and Y be real algebraic sets, with $Y \subseteq X$ and a surjective polynomial map $\phi : Y \twoheadrightarrow X$. Then $Y = X$.*

Proof. Assume $Y \subsetneq X$. Define a sequence of algebraic sets Y_i starting with $Y_0 = Y$ and with $Y_{i+1} = \phi^{-1}(Y_i)$. Since $Y \subsetneq X$ and ϕ is surjective, $Y_1 = \phi^{-1}(Y)$ must be a proper subset of Y_0. Then $\phi|_{Y_1}$ gives a surjection $Y_1 \twoheadrightarrow Y_0$, so by the same logic $Y_2 \subsetneq Y_1$.

By induction, the Y_i form a sequence of nested algebraic sets $Y_0 \supsetneq Y_1 \supsetneq \ldots$, which contradicts the Zariski topology on $\mathbb{R}^n$ being a Noetherian topological space [12, Proposition 12.3.3]. □

Acknowledgments

I thank Joshua Greene for introducing me to this problem and Bernd Sturmfels, Johannes Huisman and Rainer Sinn for suggestions regarding real algebraic geometry and Nathan Dunfield, Larsen Linov and Alan Reid for comments and references. I thank James Dix for allowing me to include the argument in the appendix. I thank the referees for their comments.

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