THIRTY-THREE *YES* OR *NO* QUESTIONS ABOUT MAPPINGS, MEASURES, AND METRICS

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1. Introduction

Most problems in the ensuing list are of fairly recent origin. None of them seem easy and some are likely to be very difficult. The formulation of each problem is such that it can be answered by one word only: either *yes* or *no*. (Strictly speaking, it is conceivable that within the same question, the answer sometimes depends on the dimension.) We offer no conjectures or guesses. In many cases, the particular question is just a chosen concise representative from a whole group of related open problems.

Whenever known, we shall point out the original source of a question. Otherwise, the question is either a folk question, a modification of a folk question, or suggested by one or both of the authors. We apologize in advance for all omissions and misquotes.

Practically all the problems require some background definitions; many concepts that are being used have only recently been introduced, and are perhaps not so widely known. Typically, the question is stated first, and the relevant definitions and references are given right afterwords.

To keep this essay brief, we give little or no motivation here. For this purpose, we kindly invite the reader to consult the literature as referred to in the text.

This list of questions was born at the Institut des Hautes Études Scientifiques in August 1996. The choices we made were necessarily partial but still somewhat arbitrary. There certainly are many more problems around these topics that we deem equally worthy.

2. The Questions

Question 1. *Is every strong A_∞ weight in $\mathbf{R}^2$ comparable to the Jacobian of a quasiconformal map $f : \mathbf{R}^2 \to \mathbf{R}^2$?*

A *strong A_∞ weight* in $\mathbf{R}^2$ is a locally integrable nonnegative function w in $\mathbf{R}^2$ so that the induced measure $d\mu = wd\lambda_2$ is doubling and that the distance function

$$D_w(x,y) = (\int_{B_{x,y}} w \, d\lambda_2)^{1/2}, \qquad x, y \in \mathbf{R}^2,$$

is bi-Lipschitz equivalent to a metric in $\mathbf{R}^2$; that is, it is asked there be a metric d_w in $\mathbf{R}^2$ and a constant $C \geq 1$ so that

$$C^{-1}d_w(x,y) \leq D_w(x,y) \leq Cd_w(x,y)$$

for all pairs of points $x, y \in \mathbf{R}^2$. Above, $B_{x,y}$ denotes the (closed) disk in $\mathbf{R}^2$ which contains both x and y and has diameter $|x-y|$, and λ_2 denotes the Lebesgue 2-measure.

A nontrivial Borel measure μ in a metric space X is said to be *doubling* if there is a constant $C \geq 1$ so that

(2.1) $$\mu(2B) \leq C\mu(B)$$

for each ball B in X, where $2B$ denotes a ball of same center but twice the radius of B.

A weight w is said to be *comparable* to the Jacobian $J(x,f)$ of a quasiconformal map f if there is a constant $C \geq 1$ so that

$$C^{-1}J(x,f) \leq w(x) \leq CJ(x,f)$$

for a.e. x. An equivalent way to ask Question 1 is to ask whether the metric space $(\mathbf{R}^2, d_w)$ is bi-Lipschitz equivalent to $\mathbf{R}^2$ with the standard metric, if w is a strong A_∞ weight.

Question 1 is due to David and Semmes [DS1]. For more information about strong A_∞ weights, see [S2], [S4], [S6, Section 19].

The answer to a similar question in dimensions $n \geq 3$ is *no* by [S4].

Question 2. *Is every A_1 weight in $\mathbf{R}^n$, $n \geq 2$, comparable to the Jacobian of a quasiconformal map $f : \mathbf{R}^n \to \mathbf{R}^n$?*

An A_1 *weight* in $\mathbf{R}^n$ is a locally integrable nonnegative function w in $\mathbf{R}^n$ for which there exists a constant $C \geq 1$ so that

$$R^{-n}\int_{B_R} w\,d\lambda_n \ \leq \ C \operatorname{essinf}_{B_R} w$$

for all n-balls B_R of radius $R > 0$. Here λ_n is the Lebesgue n-measure.

Every A_1 weight is a strong A_∞ weight [S2], so in dimension $n = 2$, Question 2 is a special case of Question 1. Question 2 originates in [DS1] as well.

Question 3. *If $(\mathbf{S}^2, d)$ is both linearly locally contractible and Ahlfors 2-regular, is it then quasisymmetrically equivalent to $\mathbf{S}^2$?*

Here $\mathbf{S}^2$ is the standard 2-sphere and $(\mathbf{S}^2, d)$ denotes a metric space that is homeomorphic to $\mathbf{S}^2$.

A metric space is said to be *locally linearly contractible* if there is a constant $C \geq 1$ so that each metric ball of radius $0 < R < C^{-1}$ in the space can be contracted to a point inside a ball of same center but radius CR. See [S5] for more discussion about this condition.

A metric space X is said to be *Ahlfors s-regular* for some real number $s > 0$ if it has Hausdorff dimension s and if there is a constant $C \geq 1$ such that

(2.2) $$C^{-1}R^s \ \leq \ \mathcal{H}_s(B_R) \ \leq \ CR^s$$

for each metric ball B_R of radius $R < \operatorname{diam} X$. Here $\mathcal{H}_s$ denotes the s-dimensional Hausdorff measure in X. It is easy to see that a metric space is Ahlfors s-regular if (2.2) is satisfied by some Borel measure in place of $\mathcal{H}_s$.

A homeomorphism $f : (X, d) \to (X', d')$ between two metric spaces is said to be *quasisymmetric* if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ so that

$$d(x, y) \le t\, d(x, z) \quad \text{implies} \quad d'(f(x), f(y)) \le \eta(t)\, d'(f(x), f(z))$$

for all triples of points x, y, z in X. See [TV1] for the basic theory of quasisymmetric maps.

If d is a smooth Riemannian metric on $\mathbf{S}^2$, then the answer to Question 3 is *yes*, in the sense that the quasisymmetry function η can be made to depend only on the constants associated with the linear local contractibility and Ahlfors regularity conditions. See [S1], [DS2], [HK2].

The answer to a similar question in dimension $n \ge 3$ is *no* by [S3]; also the smooth version fails for $n \ge 3$ by [S3].

Question 4. *If $(\mathbf{S}^2, d)$ is both linearly locally contractible and Ahlfors 2-regular, is it then bi-Lipschitz equivalent to $\mathbf{S}^2$?*

A homeomorphism $f : (X, d) \to (X', d')$ between metric spaces is said to be *bi-Lipschitz* if there is a constant $C \ge 1$ so that

$$(2.3) \qquad\qquad C^{-1} d(x, y) \le d'(f(x), f(y)) \le C d(x, y)$$

for all $x, y \in X$. Bi-Lipschitz maps are always quasisymmetric, so a *yes* answer to Question 4 implies a *yes* answer to Question 3. Moreover, a *yes* answer to Question 4 would imply that the answer to Question 1, and hence to Question 2, is also *yes*. Next, if the answer to both Questions 1 and 3 is *yes*, then the answer to Question 4 is also *yes*.

The answer to a similar question in dimension $n \ge 3$ is false. See [SS] for $n \ge 5$ and [S4] for $n \ge 3$.

Question 5. *If $(\mathbf{S}^n, d)$, $n \ge 2$, is linearly locally contractible, Ahlfors n-regular, and quasisymmetrically three point homogeneous, is it then quasisymmetrically equivalent to $\mathbf{S}^n$?*

A metric space X is said to be *quasisymmetrically three point homogeneous* if for every two triples of distinct points x, y, z and x', y', z' in X there is a quasisymmetric self-homeomorphism f of X so that $f(x) = x'$, $f(y) = y'$, and $f(z) = z'$. Note that both the homogeneity and the linear local contractibility of $(\mathbf{S}^n, d)$ are necessary conditions for the existence of a quasisymmetric map $(\mathbf{S}^n, d) \to \mathbf{S}^n$.

Question 6. *Do smooth n-dimensional chord arc surfaces with small constant in $\mathbf{R}^{n+1}$, $n \ge 3$, admit quasisymmetric parametrizations?*

A *chord arc surface with small constant* is defined in [S1, p. 172]. Such a surface Γ is assumed smooth, and Question 6 asks if there is an η-quasisymmetric map $f : \Gamma \to \mathbf{R}^n$ with η depending only on the parameters associated with Γ. The answer to a similar question in dimension $n = 2$ is *yes* by [S1].

Question 7. *Do smooth n-dimensional chord arc surfaces with small constant in $\mathbf{R}^{n+1}$, $n \ge 2$, admit bi-Lipschitz parametrizations?*

Akin to Question 6, this question asks for the existence of a bi-Lipschitz map $f : \Gamma \to \mathbf{R}^n$ with bi-Lipschitz constant depending only on the parameters associated with the smooth chord arc surface Γ. Clearly, a *yes* answer to Question 7 implies a *yes* answer to Question 6.

Question 8. *If an Ahlfors regular metric space admits a regular map into some Euclidean space, then does it admit a bi-Lipschitz map into another, possibly different, Euclidean space?*

A map from one metric space X into another metric space X' is said to be *regular* if it is Lipschitz and if there is a constant $C \geq 1$ so that the preimage of each ball of radius R in X' can be covered by at most C balls of radius R in X. Recall that a map $f : X \to X'$ is *Lipschitz* if it satisfies the second inequality in (2.3). Regular maps were first considered by David in [Dd]. For more information about regular maps, see [DS4], [S4].

The works [S4] and [HR1] show that there need not be a bi-Lipschitz map from a metric space X into $\mathbf{R}^3$ if there is a regular map from X into $\mathbf{R}^3$, even if X is quasisymmetrically homeomorphic to $\mathbf{R}^3$.

Question 9. *If an Ahlfors n-regular metric space has big pieces of Lipschitz images of $\mathbf{R}^n$, $n \geq 2$, is it then uniformly rectifiable of dimension n?*

A metric space X has *big pieces of Lipschitz images* of $\mathbf{R}^n$ if there exist positive constants θ and M so that for each $x \in X$ and $0 < R < \operatorname{diam} X$ there is a subset E of $\mathbf{R}^n$ and an M-Lipschitz map $f : E \to B(x, R)$ so that the Hausdorff n-measure of $f(E)$ in X is at least θR^n. Here $B(x, R)$ denotes an open ball in X.

An Ahlfors n-regular metric space X is said to be *uniformly rectifiable of dimension n* if there exist positive constants θ and M so that for each $x \in X$ and R as above there is a subset A of the ball $B(x, R)$ so that the Hausdorff n-measure of A is at least θR^n and that A is M-bi-Lipschitz equivalent to a subset of $\mathbf{R}^n$.

The answer to Question 9 is *yes* for Ahlfors regular spaces that are subsets of some Euclidean space by [DS3, Theorem 1.57]. Moreover, positive results for the nonuniform version of Question 9 can be found in [K].

Question 9 is due to David and Semmes [DS4].

Question 10. *Does the space $W \times \mathbf{R}^k$ admit for some $k \geq 1$ a quasisymmetric map onto $\mathbf{R}^{3+k}$, where W is a linearly locally contractible and Ahlfors 3-regular geometric realization of the decomposition space associated with the Whitehead continuum?*

The decomposition space associated with the Whitehead continuum is obtained from $\mathbf{R}^3$ by collapsing the Whitehead continuum to a point; see [Dn]. In [S3], it was shown that this space can be realized geometrically as a linearly locally contractible and Ahlfors 3-regular subset of $\mathbf{R}^4$, which is a smooth submanifold of $\mathbf{R}^4$ save at one point. It is known that $W \times \mathbf{R}$ is homeomorphic to $\mathbf{R}^4$; see [Dn, Section 11].

Question 11. *Does the space $B \times \mathbf{R}^k$ admit for some $k \geq 1$ a quasisymmetric map onto $\mathbf{R}^{3+k}$, where B is a linearly locally contractible and Ahlfors 3-regular geometric realization of the decomposition space associated with the Bing double?*

The decomposition space B is described in [Dn], and its pertinent geometric realization in [S3]. This geometric realization is a hypersurface in $\mathbf{R}^4$ that is smooth off a Cantor set; it is homeomorphic to $\mathbf{R}^3$ but not quasisymmetrically so by [S3].

Question 12. *Is there a quasisymmetric map from a polyhedral Edwards sphere X onto $\mathbf{S}^5$?*

This question was raised by Siebenmann and Sullivan in [SS], who also showed that no homeomorphism from X onto $\mathbf{S}^5$ (with the standard metric) can be Lipschitz. By a *polyhedral Edwards sphere* we mean the double suspension of the

Poincaré homology sphere H^3, where H^3 is being triangulated so that the space $X = \Sigma^2 H^3$ as a finite simplicial complex has a natural barycentric metric. Alternatively, we can embed X piecewise linearly into some Euclidean space and take the induced Euclidean metric. For this problem, it does not matter which underlying triangulation one chooses for H^3.

Due to deep work of Edwards and Cannon, double suspensions of homology spheres are known to be homeomorphic to standard spheres; see [C], [E], and [Dn]. No explicit modulus of continuity of such a homeomorphism is known to the authors. Note that a quasisymmetric map from X onto $\mathbf{S}^5$ would be Hölder continuous.

The next two questions ask further how good a homeomorphism from X to $\mathbf{S}^5$ can be.

Question 13. *Is there a homeomorphism from a polyhedral Edwards sphere X onto $\mathbf{S}^5$ that preserves sets of Hausdorff 5-measure zero?*

We have a natural 5-dimensional Hausdorff measure on X, which restricts to the Lebesgue 5-measure on each 5-simplex. By [HK3, Section 7], a *yes* answer to Question 12 implies a *yes* answer to Question 13.

Question 14. *Is there a homeomorphism from a polyhedral Edwards sphere X onto $\mathbf{S}^5$ that belongs to the Sobolev space $W^{1,p}(X)$ for some $p \geq 1$?*

We say that a homeomorphism $f : X \to \mathbf{S}^5$ belongs to the *Sobolev space $W^{1,p}(X)$* if it belongs to the standard Sobolev space $W^{1,p}(\Delta_5)$ on each 5-simplex Δ_5 on X as a map $\Delta_5 \to \mathbf{S}^5 \subset \mathbf{R}^6$.

Question 15. *If X is a metric space of locally finite Hausdorff n-measure and f a quasisymmetric homeomorphism from X onto $\mathbf{R}^n$, $n \geq 2$, is f then absolutely continuous with respect to the Hausdorff n-measures?*

By *absolute continuity* of f we mean that f carries sets of Hausdorff n-measure zero in X to sets of Hausdorff (Lebesgue) n-measure zero in $\mathbf{R}^n$. This question appears in a paper by Väisälä [V2, Question 5.10] for subsets X of Euclidean space, and the answer is unknown even in that case. The answer is known to be *yes* if X is Ahlfors n-regular [DS1], [H1], [HK3]. It is well known that the answer to a similar question is *no* for $n = 1$.

If f is a quasisymmetric embedding of $\mathbf{R}^n$, $n \geq 2$, into some Euclidean space so that the image $f(\mathbf{R}^n)$ has locally finite Hausdorff n-measure, then f is absolutely continuous in that it carries sets of n-measure zero to sets of n-measure zero; this is due to Väisälä [V2] who extended the earlier result of Gehring [G1], [G2] from the case where f has a quasisymmetric extension to $\mathbf{R}^N$, where $N \geq n$ is such that $f(\mathbf{R}^n) \subset \mathbf{R}^N$.

Question 16. *If X is a metric space and f a quasisymmetric homeomorphism from X onto $\mathbf{R}^n$, $n \geq 2$, is f then absolutely continuous with respect to the Hausdorff n-measures?*

Question 16 asks if f carries sets of Hausdorff n-measure zero in X to sets of Hausdorff (Lebesgue) n-measure zero in $\mathbf{R}^n$. The only difference to Question 15 is that no assumption on the measure properties of X is being made. However, since X is homeomorphic to $\mathbf{R}^n$, its Hausdorff dimension is at least n by the dimension theory [HW, Chapter VII].

Clearly, a *yes* answer to Question 16 implies a *yes* answer to Question 15. Again, the answer is unknown even for subsets X of Euclidean space, and the answer to a similar question is *no* if $n = 1$.

See [H2], [HK1], [S8], [V3] for further discussion and problems along these lines.

Question 17. *Is there an embedding f of $\mathbf{R}^2$ into $\mathbf{R}^3$ such that, for some $C \geq 1$ and $0 < \alpha < 1$,*

$$(2.4) \qquad\qquad C^{-1}|x - y|^\alpha \leq |f(x) - f(y)| \leq C|x - y|^\alpha$$

for all $x, y \in \mathbf{R}^2$?

The answer to a similar question where 2 and 3 are replaced by 1 and 2, respectively, is *yes*, as demonstrated by the standard von Koch snowflake. See [F, p. 121]. The answer is likewise *yes* if one allows the image to lie in a larger dimensional Euclidean space; see [A]. Note that an embedding f as in (2.4) is quasisymmetric, and necessarily $\alpha > 2/3$.

Question 18. *Is there a quasisymmetric embedding f of $\mathbf{R}^2$ into $\mathbf{R}^3$ such that the image $f(\mathbf{R}^2)$ contains no rectifiable curves?*

This question is due to Steffen Rohde. Clearly, if the answer to Question 17 is *yes*, the answer to Question 18 is *yes* as well.

Question 19. *If X is an Ahlfors Q-regular space that admits a weak $(1, 1)$-Poincaré inequality, is Q then an integer?*

An Ahlfors Q-regular space is said to satisfy a *weak $(1, 1)$-Poincaré inequality*, if there are constants $C_1 \geq 1$ and $C_2 \geq 1$ so that

$$\int_{B_R} |u - u_B| \, d\mu \leq C_1 R \int_{B_{C_2 R}} \rho \, d\mu$$

for all balls B_R of radius $0 < R < \operatorname{diam} X$, for all continuous functions u defined in $B_{C_2 R}$, and for all upper gradients ρ of u in $B_{C_2 R}$. A nonnegative Borel function ρ is said to be an *upper gradient* of a function u defined on a metric space if

$$|u(x) - u(y)| \leq \int_\gamma \rho \, ds$$

for all points x and y and for all rectifiable curves γ joining the two points.

See [HK3] and [S5] for more about Poincaré inequalities and upper gradients (called *very weak gradients* in [HK3]).

Carnot groups are examples of Q-regular spaces that are topological manifolds of dimension less than Q but yet admit a weak $(1, 1)$-Poincaré inequality; see [J].

Question 20. *If X is an Ahlfors Q-regular Loewner space for some $Q > 1$, is Q then an integer?*

An Ahlfors Q-regular metric space X is said to be a *Loewner space* if the function

$$\lambda(t) = \inf\{\operatorname{mod}(E, F; X) : \Delta(E, F) \leq t\}$$

is positive for all $t > 0$, where E and F are disjoint nondegenerate continua in X,

$$\Delta(E, F) = \frac{\text{dist}(E, F)}{\min\{\text{diam } E, \text{diam } F\}} \, ,$$

and

$$\text{mod}(E, F; X) = \inf \int_X \rho^Q \, d\mu$$

with the infimum taken over all nonnegative Borel functions ρ such that

$$\int_\gamma \rho \, ds \geq 1$$

for all rectifiable γ joining E and F. Above, $d\mu$ denotes the Hausdorff Q-measure in X.

For a more thorough discussion on Loewner spaces, see [HK3].

It was shown in [HK3] that if a Q-regular space X admits a weak $(1, 1)$-Poincaré inequality, then X is a Loewner space, provided X also satisfies some additional qualitative assumptions (for instance, it suffices to assume that closed balls in X are compact and that X is locally quasiconvex). In particular, Carnot groups are Q-regular Loewner spaces with Q larger than the topological dimension. Furthermore, by [HK3, 6.19], there are Q-regular Loewner spaces that have different topological dimension at different places.

Question 21. *If the first Heisenberg group is deformed by a metric doubling measure, is a (1,1)-Poincaré inequality retained?*

The *Heisenberg group* is assumed equipped with its Carnot metric; thus it is an Ahlfors 4-regular space. For a discussion of the Heisenberg group and the definition of a Carnot metric, see [Pa], [KR1], [KR2].

Recall the definition for a doubling measure from (2.1).

Given a doubling measure μ in an Ahlfors Q-regular space (X, d), we can associate to it a distance function much as in the definition for a strong A_∞ weight:

$$D_\mu(x, y) = \mu(B_{x,y})^{1/Q} \, ,$$

where $B_{x,y} = B(x, d(x, y)) \cup B(y, d(x, y))$ is the union of two (closed) balls centered at the points x and y, and both with radius $d(x, y)$. The distance function D_μ need not satisfy the triangle inequality, although it satisfies the other requirements for a metric. A doubling measure μ is called a *metric doubling measure* if there is a metric d_μ on X which is bi-Lipschitz equivalent to D_μ; that is, if there is a constant $C \geq 1$ such that

$$C^{-1} d_\mu(x, y) \leq D_\mu(x, y) \leq C d_\mu(x, y)$$

for all $x, y \in X$.

It is not hard to see that the space (X, d_μ) is also Ahlfors Q-regular, and it is called a *deformation of X by a metric doubling measure* μ. It is also not hard to see that the identity map $(X, d) \to (X, d_\mu)$ is quasisymmetric.

The answer to Question 21 is *yes* if the Heisenberg group is replaced with $\mathbf{R}^n$. See [DS1], [S5], [S7]. Moreover, in general the answer is *no* for a Q-regular space admitting a weak $(1, 1)$-Poincaré inequality; that is, a deformation with a metric doubling measure may destroy a weak $(1, 1)$-Poincaré inequality on a general Q-regular space by [HK3, 6.19]. However, a somewhat weaker Poincaré inequality is always retained by [KM].

Question 22. *Is the first Heisenberg group minimal in looking down?*

Again, it is understood here that the Heisenberg group is equipped with its Carnot metric.

Question 22 is due to David and Semmes [DS4]. "Looking down" is a partial ordering in the collection of equivalence classes of BPI spaces, where letters BPI stand for "big pieces of itself".

An Ahlfors Q-regular metric space X is said to be a *BPI space* if there are constants $C \geq 1$ and $\theta > 0$ so that for each pair of balls $B(x_1, r_1)$ and $B(x_2, r_2)$ in X with $0 < r_1, r_2 < \operatorname{diam} X$ there is a closed set $A \subset B(x_1, r_1)$ whose Q-Hausdorff measure is at least θr_1^Q and there is a C-conformally bi-Lipschitz embedding $f : A \to B(x_2, r_2)$ with scale factor r_2/r_1.

A bi-Lipschitz map f between metric spaces (X, d) and (X', d') is said to be *C-conformally bi-Lipschitz with scale factor $\lambda > 0$* if f is C-bi-Lipschitz between the metric spaces $(X, \lambda d)$ and (X', d').

Two BPI spaces X and X' of the same dimension Q are said to be BPI *equivalent* if there are positive constants θ and M so that for each $x \in X$, $x' \in X'$ and radii $0 < R < \operatorname{diam} X$, $0 < R' < \operatorname{diam} X'$ there exist a subset A of the ball $B(x, R)$ in X with Q-Hausdorff measure at least θR^Q and an M-conformally bi-Lipschitz embedding $f : A \to B(x', R')$ with scale factor R'/R. It is not hard to see that two BPI spaces of the same dimension Q are BPI equivalent if they have subsets of positive Q-measure that are bi-Lipschitz equivalent.

A BPI metric space X of Hausdorff dimension Q is said to *look down* on another BPI metric space Y, also of Hausdorff dimension Q, if there is a closed set A in X and a Lipschitz map $f : A \to Y$ so that $f(A)$ has positive Hausdorff Q-measure. If also Y looks down on X, then X and Y are *look-down equivalent*.

A BPI space X of dimension Q is said to be *minimal* in looking down if every BPI space Y of same dimension Q that X looks down to is BPI equivalent to X.

Each Euclidean space is minimal in looking down by [DS4].

Question 23. *If two Q-regular BPI metric spaces are look-down equivalent, are they then BPI equivalent?*

Question 23 appears in [DS4], where more discussion and more similar problems can be found.

Question 24. *If f is a Lipschitz map of a subset E in the first Heisenberg group equipped with its Carnot metric into a metric space such that the Hausdorff 4-measure of $f(E)$ is positive, is f then bi-Lipschitz in a subset of E of positive Hausdorff 4-measure?*

If the answer to Question 24 is *yes*, then the answer to Question 22 is *yes* as well. The answer to a similar question where the Heisenberg group is replaced with $\mathbf{R}^n$ is *yes* by [K].

Question 24 is again from [DS4].

Question 25. *Can the T-axis in the first Heisenberg group be mapped onto a locally rectifiable curve by a quasiconformal self-map of the group?*

For a discussion about quasiconformal maps in the Heisenberg group with its Carnot metric, see [KR2]. The definition for quasiconformal maps is taken to be the metric one, where no smoothness is assumed. In fact, quasiconformal self-maps of the Heisenberg group are the same as quasisymmetric maps by [KR2], [HK2].

Question 26. *Is the fixed point set of every quasiconformal reflection $\mathbf{S}^3 \to \mathbf{S}^3$ a topologically tame 2-sphere?*

A quasiconformal reflection on the 3-sphere $\mathbf{S}^3$ is a sense-reversing quasiconformal involution. Question 26 was raised by Heinonen and Yang [HY]. Note that by a result of Tukia and Väisälä [TV2], one can assume that the reflection is in fact bi-Lipschitz.

Question 27. *Can every branched cover $\mathbf{S}^n \to \mathbf{S}^n$, $n \geq 3$, be made BLD by changing the metric in the domain but keeping the space n-regular and linearly locally contractible?*

We define a *branched cover* between manifolds to be a discrete, open, and sense-preserving map. Note in particular that no local tameness assumption is made on the branching locus.

A branched cover $f : X \to Y$ between metric manifolds is said to be *BLD*, or a *map of bounded length distortion*, if there is a constant $L \geq 1$ so that

$$L^{-1} \operatorname{length} \gamma \leq \operatorname{length}(f \circ \gamma) \leq L \operatorname{length} \gamma$$

for all curves γ in X. Question 27 asks if for a given branched cover $f : \mathbf{S}^n \to \mathbf{S}^n$ there is a metric d on $\mathbf{S}^n$ so that $(\mathbf{S}^n, d)$ is an Ahlfors n-regular and locally linearly contractible metric space, and $f : (\mathbf{S}^n, d) \to \mathbf{S}^n$ is a BLD map.

The answer to a similar question for $n = 2$ is *yes*; then, in fact, every branched cover is topologically conjugate to a holomorphic map by a classical theorem of Stoïlow.

Question 27 is due to Heinonen and Rickman [HR2]. See [MV], [HR2] for more information about BLD branched covers.

Question 28. *Is every branched cover $f : \mathbf{S}^n \to \mathbf{S}^n$, $n \geq 3$, topologically conjugate to a quasiregular map?*

Question 28 (in a slightly different local formulation) was stated by Martio, Rickman, and Väisälä in [MRV, 4.11]. For definitions and discussion of quasiregular maps, see the monographs [Re] and [Ri2]. Question 28 asks if for a given branched cover f one can find homeomorphisms h and g of the n-sphere such that $h \circ f \circ g$ is quasiregular. Recall that every nonconstant quasiregular map is a branched cover as defined above.

The answer to a similar question in dimension $n = 2$ is *yes* by Stoïlow's theorem.

Question 29. *Is the topological dimension of the branch set of a quasiregular map $\mathbf{S}^n \to \mathbf{S}^n$, $n \geq 3$, either -1, $n - 2$, or n?*

The *branch set* of a quasiregular map is the closed set of points in the domain where the map does not define a local homeomorphism. Thus, for a quasiregular map $\mathbf{S}^n \to \mathbf{S}^n$, the dimension of the branch set is -1 if and only if the map is homeomorphic and it is n if and only if the map is constant. By a result of Černavskiĭ [C1], [C2] (see also Väisälä [V1]), the topological dimension of the branched set for any branched cover between n-manifolds is at most $n - 2$; in dimension $n = 2$ it is known to be either -1 or $n-2$ by the aforementioned Stoïlow theorem.

The Edwards sphere can be used to show that there are branched covers $\mathbf{S}^n \to \mathbf{S}^n$ for $n \geq 5$ whose branch set is presicely an $(n - 4)$-dimensional sphere. Thus, if the answer to Question 29 is *yes*, then the answer to Questions 11 and 28 is *no*, and if the answer to either Question 11 or Question 28 is *yes*, then the answer to Question 29 is *no*.

Question 29 has been around since the inception of the theory of quasiregular maps. See [Sr] for a recent survey and references.

Question 30. *Does there exist a nonconstant quasiregular map* $\mathbf{R}^4 \to \mathbf{S}^2 \times \mathbf{S}^2 \sharp \mathbf{S}^2 \times \mathbf{S}^2$?

This question is due to Mikhail Gromov and Seppo Rickman. Here $\sharp$ denotes connected sum. It is not difficult to construct a nonconstant quasiregular map $\mathbf{R}^4 \to \mathbf{S}^2 \times \mathbf{S}^2$. See [Ri1].

Question 31. *Does every closed oriented topological n-manifold, $n \geq 4$, admit a branched cover onto* $\mathbf{S}^n$?

By an old theorem of Alexander, the answer is *yes* to every combinatorially triangulable (closed) manifold; in particular the answer is *yes* in dimensions $n = 1, 2, 3$. In fact, Alexander's method only requires that the manifold be homeomorphic to a simplicial complex which is an oriented pseudo-manifold (without boundary) as defined in [Sp, p. 148]. It is known that not every closed four-manifold can so be triangulated (see page xvi in the Introduction of [AM]), but the existence of a triangulation is unknown in dimensions five or more, cf. [GS]. See [Ri3] for a discussion about Alexander's method.

We reiterate that in the above questions branched covers are not required to be piecewise linear, or even locally equivalent to such. (See the definition after Question 27.)

Question 32. *Does every closed oriented PL n-manifold, $n \geq 5$, admit a PL branched cover of degree at most n onto the PL n-sphere?*

We learned Question 32 from Dennis Sullivan. The answer to a similar question is *yes* in dimensions $n = 1, 2, 3, 4$. The case $n = 2$ is classical, the case $n = 3$ was proved independently in the 1970's by Hilden, Hirsch, and Montesinos (see [BE] for an extensive account), while the case $n = 4$ is a recent result of Piergallini [Pi]

Question 33. *Does every closed topological four-manifold admit a metric that makes the manifold Ahlfors 4-regular and locally linearly contractible?*

The answer to a similar question in dimensions other than four is *yes*, because of Sullivan's theorem on the existence of Lipschitz structures [Su]. In contrast to this, recall that in dimension four most manifolds do not admit smooth or even Lipschitz or quasiconformal structures, by the works of Freedman, Donaldson, and Donaldson and Sullivan; see [FQ], [DK], [DS]. Furthermore, as mentioned above after Question 31, there are four manifolds that are not homeomorphic to finite simplicial complexes.

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