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PROPERTIES OF CONVERGENCE GROUPS AND SPACES

ERIC M. FREDEN

Abstract. This paper discusses algebraic and topological conditions that are consequences of a convergence group action.

0. Introduction

Two generalizations of Möbius groups were introduced in 1986. Gromov defined the geometric notion of negatively curved (word hyperbolic) groups and spaces [Gr]. At the same time, Gehring and Martin gave a simple topological condition (the convergence property) that all Möbius groups obey [GM1]. While the former subject has seen great activity in recent years, the so-called convergence groups have only recently assumed a prominent role. That negatively curved groups display convergence properties is found in [B1], [F], [MS], [T1], and [W]. More recently, Bowditch has shown a partial converse [B2].

The aim of this paper is to illustrate the considerable restrictions that accompany the convergence action of a (nonelementary) group G on a compact Hausdorff space X. Under mild hypotheses, the limit set of G must be metrizable with cardinality $2^{\aleph_0}$. There is some control on the type and number of normal subgroups. Endomorphisms of G with finite kernel are automorphisms. There is a class of finitely generated convergence groups with solvable word problem.

Section 1 deals with necessary background material along with an example showing that every finitely generated group can be realized as an elementary convergence group. In section 2, an esoteric example of a convergence action on a nonmetrizable space is given. The triple space is defined and used to prove various topological properties of the limit set. Section 3 is concerned with group theoretic consequences of a convergence action.

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1. Preliminaries and examples

The principal properties of convergence groups can be found in [GM1] and [T1]. The former paper shows that many of the properties of Möbius groups do not depend on an analytic structure but rather on a dynamic topological condition. Let G be a group of homeomorphisms of S^n. Then G is a (discrete) convergence group if given any infinite family of distinct $g_m \in G$, there are (not necessarily

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distinct) points $x, y \in S^n$ and a (sub)sequence $\{g_j\}_{j=1}^{\infty}$ such that

$$g_j(z) \to x \quad \text{locally uniformly on} \quad S^n \setminus \{y\}, \quad \text{and}$$

$$g_j^{-1}(z) \to y \quad \text{locally uniformly on} \quad S^n \setminus \{x\}.$$

Tukia's paper [T1] extends most of the theory to a group of homeomorphisms of a first countable, compact Hausdorff space X. Recall that the *ordinary set* $\Omega(G) \subset X$ consists of the points where G acts properly discontinuously while its complement $L(G)$ is the *limit set*. A convergence group G is *elementary* if $L(G)$ contains less than three points, *nonelementary* otherwise. If G is virtually abelian or torsion or a finite extension of Z, then G is elementary (see section 5 of [GM1]). The limit set of a nonelementary group is closed, perfect, and contained in the closure of the orbit Gx of any $x \in X$ (see 2S of [T1]). If $L(G)$ is locally connected, then it is connected (6.21 of [GM1]). Every $g \in G$ is one of three mutually exclusive types: g is *elliptic* if g is torsion, *parabolic* if g is nontorsion and fixes exactly one $x \in L(G)$, *loxodromic* if g is nontorsion and fixes exactly two limit points. Evidently any subgroup of a convergence group is a convergence group.

It is unnecessary to assume that the action of a convergence group G on X is effective. It is immediate that the set of $g \in G$ that fixes every point of X must be a finite normal subgroup of G. In fact, every finite normal subgroup of G acts trivially on $L(G)$ by 3.9 of [GM2].

Proposition 1.1. *Suppose G is a convergence group on X and N is a finite normal subgroup. Then $H = G/N$ inherits a natural convergence action on $L(G)$ and $L(G) \subset L(H)$.* ■

Thus there is no loss of generality in assuming that G acts faithfully. I will make this assumption throughout.

Example 1.2. Let G be any finitely generated group with fixed presentation and finite generating set. Let Γ be the Cayley graph associated with the given presentation. Define $X = \Gamma \cup \infty$ as the one-point compactification of Γ. Then the left action of G on Γ extends continuously to X by defining $g(\infty) = \infty$ for all $g \in G$. Given any distinct sequence g_n of group elements, there is a subsequence g_m such that the word length of g_m is a strictly increasing function of m. The fact that the left G-action is properly discontinuous on Γ implies that for any compact $C \subset \Gamma$, and for all big m, $g_m(C)$ is contained in $\Gamma \setminus C$. Thus G is a convergence group with $L(G) = \{\infty\}$. In particular if G is an infinite Burnside group, this gives an example of a purely torsion convergence group with nonempty limit set. ■

Example 1.3. If $L(G)$ contains exactly two points, then an easy argument (G has two ends) shows that G is a finite extension of $\mathbf{Z}$. ■

Thus the class of (infinite) elementary groups is either too large or too small depending on the cardinality of the limit set. Nonelementary groups will be the principal topic of discussion for the balance of the text.

2. Topological considerations

There is a condition implied by the convergence property for groups of homeomorphisms involving the triple space. The original construction in the classical case seems to be due to Cheeger and, independently, Tukia. Assign to each distinct triple (x, y, z) of ideal boundary points of hyperbolic n-space $\mathbf{H}^n$ an approximate barycenter, b, of the corresponding ideal triangle. More specifically, b is the (hyperbolic) orthogonal projection of z onto the geodesic with endpoints x, y. The following is a well-known generalization.

Definition 2.1. The *triple space* associated to a topological space X is the set

$$\mathcal{T} = \{(x_1, x_2, x_3) \in X^3 : x_1 \neq x_2 \neq x_3 \neq x_1\}$$

endowed with the subspace topology of X^3.

The following characterization of convergence groups has appeared in several places. The proof is almost immediate.

Proposition 2.2. *Let X be compact Hausdorff. If G is a (discrete) convergence group on X, then G acts properly discontinuously on $\mathcal{T}$.* ■

There is a converse to 2.2 when X is first countable. For general X, the convergence property can be broadened by replacing sequences and subsequences with nets and subnets. In this context, proper discontinuity of G on $\mathcal{T}$ is equivalent to G being a (generalized) discrete convergence group on X. The details can be found in [B1].

There is a natural compactification $\overline{\mathcal{T}}$ of the triple space. Let $\Delta \subset X^3$ be the diagonal set

$$\Delta = \{(x_1, x_2, x_3) : \text{ two coordinates are equal}\}.$$

Identifying (x_1, x_2, x_3) with $x \in X$ whenever at least two coordinates equal x defines a quotient map $f : \Delta \to X$ which can be continuously extended by the identity to all of X^3. $\overline{\mathcal{T}}$ is the image of X^3 by f, and any convergence action of G on X can be extended to $\overline{\mathcal{T}}$ with $L(G) \subset X \cong \partial\mathcal{T}$. (Details can be found in [B1].)

Given a nonelementary convergence group G acting on a compact Hausdorff space X, it is often convenient to ignore the ordinary set $\Omega(G)$. Indeed Example 2.8 and Theorem 2.17 below show that the limit set is often well-behaved while the ordinary set can be pathological. Unless there is a definite geometric setting (e.g. $\Omega(G)$ is standard hyperbolic space or a Cayley graph), I have found it useful to dispense with the ordinary set and replace it with the triple space if a domain of discontinuity is required.

Question 2.3. Are there uncountable (discrete) nonelementary convergence groups? By a result of Kulkarni, any convergence group G acting on a metric space is countable [K], consequently the base space for an uncountable group must be very large. Furthermore, such G cannot act cocompactly on its corresponding triple space.

The following is true regardless of the cardinality of G. Let $g \in G$ be loxodromic with attracting and repelling fixed points $x^+, x^- \in L(G)$, respectively. Let U be any neighborhood of x^+ with $x^- \notin \overline{U}$. Then the convergence property immediately shows that $\{g^n(U) : n \in \mathbf{Z}^+\}$ is a countable neighborhood base for x^+.

Remark 2.4. This rules out the possibility of any loxodromic action on $X = I^I$, where I is the closed unit interval, as no point in I^I has a countable neighborhood base (example 125 of [SS]).

For the remainder of this section, assume that G is a countable (discrete) nonelementary convergence group.

Definition 2.5. A limit point z is *parabolic* (resp. *loxodromic*) if z is a fixed point of some parabolic (resp. loxodromic) $h \in G$. If G has an infinite torsion subgroup H, then necessarily there is a point $z \in L(G)$ fixed by every $h \in H$ (see [GM2]). Also call such z parabolic. (It is conjectured that no such H exists when G is nonelementary.) A parabolic point p is *bounded* if the stabilizer of p acts cocompactly on $X \setminus \{p\}$. A point $x \in X$ is a *conical limit point* (synonyms are *point of approximation* and *radial limit point*) if there exist a convergence sequence $\{g_n\}_{n=1}^{\infty}$ and points $a, b \in X$ such that

$$g_n(x) \to a \quad \text{and} \quad g_n(y) \to b \neq a \quad \text{locally uniformly on} \quad X \setminus \{x\}.$$

Alternately, x is conical if there exist $y \neq x$, a (one-to-one) sequence x_i with limit x, and a convergence sequence $\{g_n\}_{n=1}^{\infty}$ such that $(g_i(x), g_i(y), g_i(x_i))$ converges to $(a, b, c) \in \mathcal{T}$. The equivalence of these definitions is a simple exercise.

It is easy to see that every loxodromic fixed point is conical. On the other hand, no parabolic point can be conical [T2]. Assume x is a conical limit point. As in the definition, if U is any open set containing a with $x \notin \overline{U}$, then $\{g^{-n}(U) : n \in \mathbf{Z}^+\}$ is a countable neighborhood base for x.

Suppose that p is a bounded parabolic point with stabilizer G_p. Any neighborhood of p in X looks like $X \setminus K$ where $K \subset X \setminus \{p\}$ is compact. Let $C \subset X \setminus \{p\}$ be a compact neighborhood that contains a fundamental set for the action of G_p on $X \setminus \{p\}$. By proper discontinuity (the fact that G_p acts discontinuously on $X \setminus \{p\}$ is fairly obvious; a reference is [T2]) the set $S = \{g \in G_p : gC \cap K \neq \emptyset\}$ is finite. Then

$$V_S = \{p\} \cup \bigcup_{g \in G_p \setminus S} gC$$

is a neighborhood satisfying $p \in V_S \subset X \setminus K$. The collection $\mathcal{V} = \{V_S : S \subset G_p \text{ is finite}\}$ forms a countable neighborhood base for p.

In the case of discrete Möbius groups of divergence type, almost all limit points are conical [N]. This is also the case for geometrically finite convergence groups (see Proposition 2.14 below), so it is perhaps not unreasonable to assume that X is, in general, first countable. Also, all results of [T1] are valid in this context.

Since the orbit of any $z \in X$ is dense in $L(G)$, the following is immediate.

Proposition 2.6. *Let G be a countable convergence group on a first countable compact Hausdorff space X. Then card$(L(G)) \leq 2^{\aleph_0}$.* ∎

It is possible to have a convergence group acting on a nonmetrizable space.

Example 2.7. Let $X_1 = (-\infty, +\infty]$ and $X_2 = [-\infty, +\infty)$. Define X as the topological sum (disjoint union) $X_1 \sqcup X_2$. Give X the *weak parallel line* topology. A basic open set is of two types: $U = (a, b] \sqcup (a, b)$ or $W = (c, d) \sqcup [c, d)$, where a, b, c, d are any extended real numbers (see Figure 2A). Define a $\mathbf{Z}$-action on X by $g(x \sqcup y) = x + 1 \sqcup y + 1$. Then $G = \langle g \rangle$ is a loxodromic group on X, with limit points $\emptyset \sqcup +\infty$ and $-\infty \sqcup \emptyset$. X is compact Hausdorff, first countable, separable, but totally disconnected and not second countable (example 95 of [SS]). ∎

To my knowledge the following is the first example of a nonelementary convergence action on a nonmetrizable compact Hausdorff space.

Example 2.8. Let $F = \langle a, b \rangle$ be free on two generators. The standard Cayley graph Γ is an infinite simplicial tree with vertex valence 4 (Figure 2B).

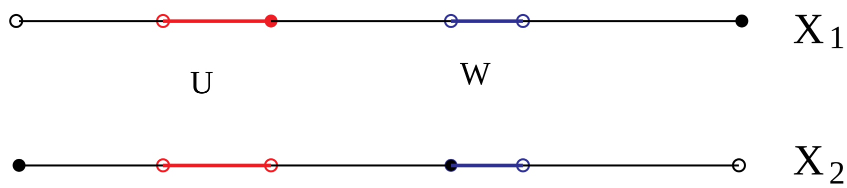

FIGURE 2A

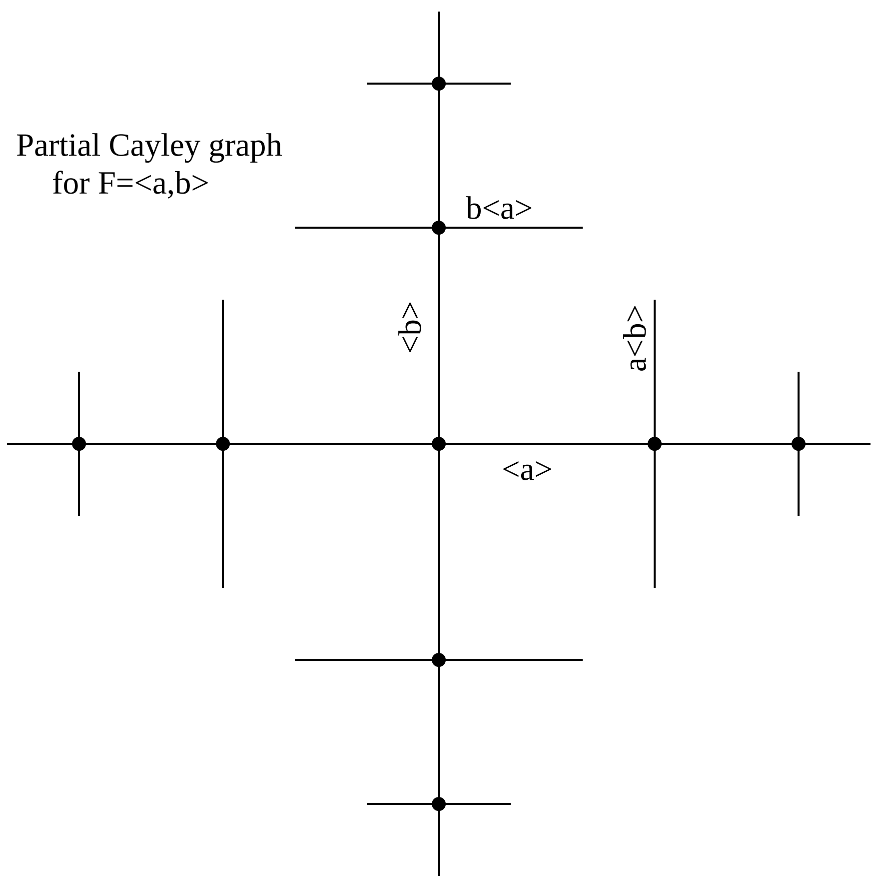

FIGURE 2B

Consider the subgraph corresponding to the subgroup $\langle a\rangle$. This can be visualized as a (horizontal) copy of the real line $\mathbf{R}$ embedded in Γ. Each $\langle a\rangle$-coset corresponds to an embedding of $\mathbf{R}$ "parallel" to that of $\langle a\rangle$. Similarly $\langle b\rangle$ and its cosets are vertical lines in Γ. Pull each $\langle b\rangle$-coset away from the vertices where it crosses other cosets and introduce new points to fill the holes. Replace each coset with a copy of X from Example 2.7. Some care must be given to ensure that all orientations are consistent (see Figure 2C). The resulting set $Y = Y_1 \sqcup Y_2$ can be visualized as two conglomerates of lines. To ensure that Y is compact Hausdorff and that the induced F-action (by the usual left multiplication) makes F a convergence group, I must describe the topology.

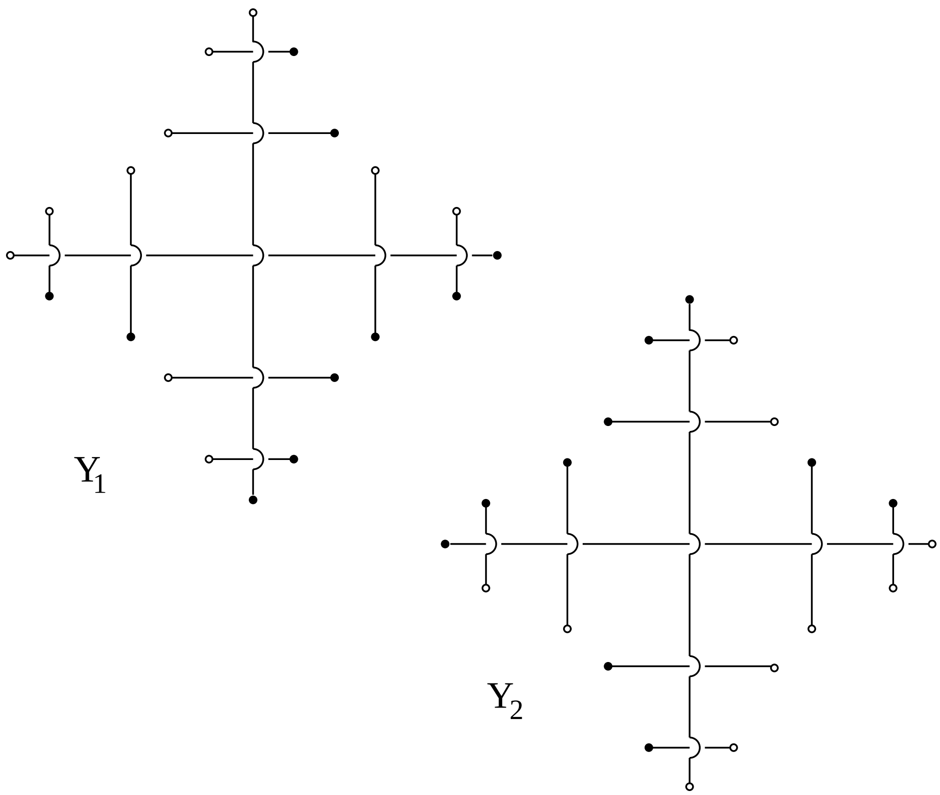

FIGURE 2C

Pick a (former) vertex $O_1 \in Y_1$ as an origin and let O_2 be the corresponding point in Y_2. Suppose p_1 is any other point. If p_1 is not a former vertex, there is a natural division of Y_1 into two pieces at p_1. There is a similar division of Y_2 at the corresponding p_2 (see Figure 2D). Define the *cone* based at p_1 as the (disjoint) union of the pair of pieces that do not contain O_1 and O_2, respectively. The points p_1, p_2 will or will not be included in the cone in analogy to the weak parallel line topology. In the case that q_1 (and hence q_2) is a former vertex, modify the cone as indicated in Figure 2D.

The group F will translate cones in the obvious way. Let $\mathcal{S}$ be the collection of all cones and their F translates. Define the "weak parallel tree" topology on Y by declaring $\mathcal{S}$ to be a subbase. It is readily checked that a finite intersection of cones can always be written as a finite union of other cones and "bounded" open sets of the type in Example 2.7.

It is clear that Y is Hausdorff and $F \subset Homeo(Y)$. Since the only neighborhoods of endpoints of Y are those from $\mathcal{S}$, F is a convergence group on Y, with $L(F)$ being the endpoints. Note that this limit set is (homeomorphic to) a standard Cantor set which **is** metrizable. Let $\mathcal{V}$ be any open cover of Y. Then $L(F)$ being compact is necessarily covered by finitely many cones $S_1, S_2, \ldots, S_n \in \mathcal{V}$. The remainder

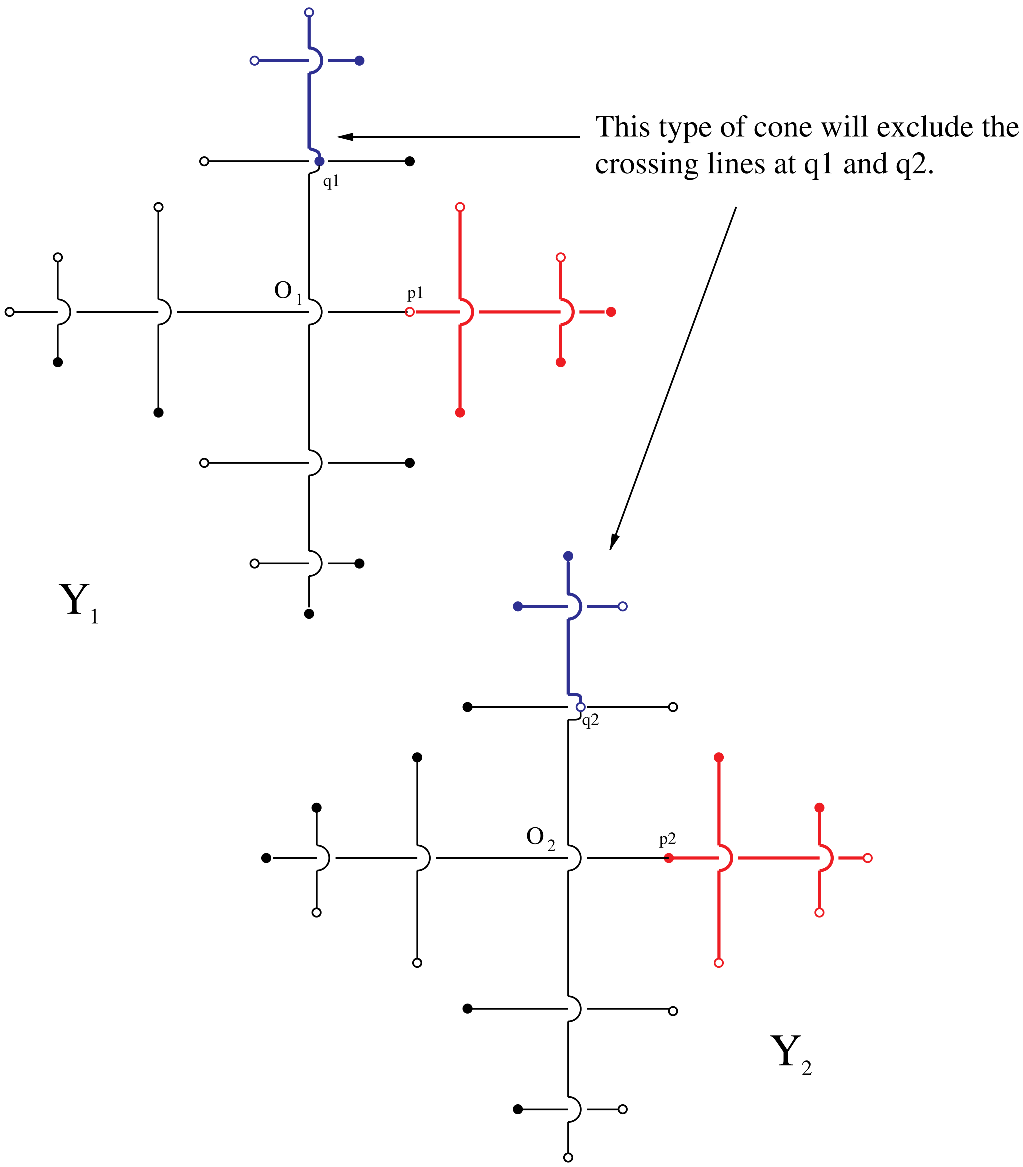

FIGURE 2D

$Y \setminus \bigcup S_i$ consists of finitely many segments each homeomorphic to the weak parallel line. Therefore finitely many $V \in \mathcal{V}$ cover this remainder. The conclusion is that Y is a nonmetrizable compact Hausdorff space admitting a discrete nonelementary convergence group action. ■

The proofs of the following two lemmas are easy exercises and are left to the reader.

Lemma 2.9. *Suppose X is a locally compact Hausdorff space. Then X is σ-compact if and only if X is Lindelöf.* ■

Lemma 2.10. *Let X be compact Hausdorff and first countable. Then $\mathcal{T}$ is locally compact Hausdorff. Also $\overline{\mathcal{T}}$ is first countable.* ■

Convention 2.11. For the balance of this section, I will discard any existing ordinary set and assume that G is a finitely generated group of the first kind, i.e. $X = L(G)$, and that X is locally connected (and hence connected by 6.21 of [GM1]).

With the preceding hypotheses, it is possible [K] to produce a closed connected "fundamental domain" $D \subset \mathcal{T}$ for the action of G. (In 2.2 of [K], the author assumes second countability only to ensure that the projection $\pi : \mathcal{T} \to D$ is proper.)

Suppose that $\mathcal{T}/G$ is noncompact and for a given topological end $E \subset \mathcal{T}/G$ there is a connected (necessarily noncompact) neighborhood $W \subset \mathcal{T}/G$ of E, which lifts to a connected $\widetilde{W} \subset \mathcal{T}$. Assume that the stabilizer stab$(\widetilde{W})$ is also the stabilizer G_p of some parabolic fixed point p and that $g\widetilde{W} \cap \widetilde{W} = \emptyset$ for all $g \notin G_p$. Under these circumstances, define E as a *parabolic end* and W as a *parabolic cusp* associated with p. Note that each conjugate gG_pg^{-1} is in one-to-one correspondence with a translate $g\widetilde{W}$, all translates are pairwise disjoint, and each projects to the same cusp W (so actually a parabolic end is associated with an orbit of parabolic points). If $\mathcal{T}/G$ has only finitely many ends, each parabolic, then it is possible to choose finitely many orbits of precisely invariant $\widetilde{W}$ as above so that the entire collection is pairwise disjoint. In this case $\mathcal{T}/G$ is the union of a compact set and finitely many noncompact parabolic cusps.

Definition 2.12. A convergence group G is *GF1* provided that $\mathcal{T}/G$ has finitely many ends, each being a parabolic end. If $L(G)$ consists solely of conical and bounded parabolic points, then G is defined to be *GF2*. For the purposes of this paper, G is *geometrically finite* if it is GF1 and GF2.

Remark 2.13. Bowditch shows that GF1 and GF2 are equivalent conditions when G is a Möbius group (or more generally the fundamental group of any Riemannian manifold with pinched negative curvature) ([B3], [B4]). It is possible that GF1 and GF2 are equivalent in the present context, but that is the subject of another paper [T2]. Furthermore, with the assumption of a connected limit set for G, each stabilizer G_p of a bounded parabolic point p will (by a standard argument) be finitely generated.

The discussion after Definition 2.5 implies

Proposition 2.14. *If G is geometrically finite, then $X = L(G)$ is first countable.* ■

Lemma 2.15. *If, in addition, there is a fundamental domain $D \subset \mathcal{T}$, with $Z = \overline{D} \cap \partial\mathcal{T}$ discrete, then D is σ-compact.*

Proof. First, observe that a closed set $C \subset \mathcal{T}$ is compact if and only if as a subset of X^3, C is disjoint from some neighborhood of the big diagonal Δ. Let $\mathcal{V}_z = \{V_{z,n} : n = 1, 2, 3, \dots\}$ be a countable neighborhood base for each $z \in Z$. Without loss of generality, $(\bigcup \mathcal{V}_z) \cap (\bigcup \mathcal{V}_w) = \emptyset$ whenever $z \neq w$ (Z is discrete). Define for each $n = 1, 2, 3, \dots$ the compact sets

$$K_n = \overline{D} \setminus \bigsqcup_{z \in Z} V_{z,n}.$$

Then $\bigcup_{n=1}^{\infty} K_n = D$. ■

Remark 2.16. The hypothesis "Z is discrete" can be relaxed to "the set of cluster points of Z is discrete" (or even "the set of cluster points of the cluster points of Z is discrete", etc.). The proof is easily adapted and the conclusion remains valid.

The following argument is generalized from 7.6 of [B2]

Theorem 2.17. *If G is geometrically finite, then $X = L(G)$ is metrizable.*

Proof. Since X is compact Hausdorff, it suffices to show X is second countable. Let D be a fundamental domain. By Lemmas 2.9 and 2.15, D is Lindelöf. Let $\mathcal{U}$ be a countable open cover of D by product neighborhoods of the form $U_1 \times U_2 \times U_3$ where the closures of each factor are pairwise disjoint in X. Then $G\mathcal{U}$ covers $\mathcal{T}$. By hypothesis there are countably many parabolic fixed points p_i in X, each with its own countable neighborhood base $\mathcal{V}_i$. Adjoin to the collection of all the first factor sets of $\mathcal{U}$ a collection of neighborhood bases for each parabolic point and call this (countable) collection $\mathcal{B}$. More succinctly,

$$\mathcal{B} = \Big\{ B \subset X : B = g(U_1) \text{ for some } U_1 \times U_2 \times U_3 \in \mathcal{U}, g \in G \text{ or } B \in \mathcal{V}_i \text{ for some } i \Big\}.$$

The claim is that $\mathcal{B}$ is a base for the topology of $X = L(G)$.

Let $x \in X$. Every neighborhood of x looks like $X \setminus K$ where $K \subset X \setminus \{x\}$ is compact. Pick such a compact K. If x is a parabolic fixed point, use the construction of the previous paragraph to find $B \in \mathcal{B}$ such that $x \in B \subset X \setminus K$. Suppose x is conical. Using the alternate definition from 2.5, there is $y \in X$ and a sequence x_n converging to x such that $(x, y, x_n) \in \mathcal{T}$ for all n. There exist $g_n \in G$ and distinct $a, b, c \in X$ such that $g_n(x, y, x_n)$ converges to $(a, b, c) \in \mathcal{T}$. If necessary, pass to a subsequence so that $g_n(X \setminus \{x\})$ converges to b locally uniformly. The limit triple $(a, b, c) \in U \times V \times W$ for some $U \times V \times W \in \mathcal{B}$, and so for big n, $g_n(x) \in U$ and $g_n(K) \subset V$ (and $U \cap V = \emptyset$). Therefore $x \in g_n^{-1}(U) \subset X \setminus K$, establishing the claim that $\mathcal{B}$ is a base. ∎

Remark 2.18. The fact that the nonconical limit points are parabolic is irrelevant. It is sufficient that $L(G)$ is first countable, all but countably many limit points are conical, and that D is σ-compact. If all limit points are conical, then G is cocompact on $\mathcal{T}$ and in fact G is a negatively curved group [B2].

Conjecture 2.19. The limit set of a countable nonelementary discrete G is always metrizable and finite dimensional.

3. Algebraic considerations

Various algebraic restrictions on convergence groups have been proven over the years. Virtually abelian groups are elementary and nonelementary groups contain a free subgroup of rank two ([GM1], [T1]). Purely torsion convergence groups are elementary with at most one limit point ([GM1], [B1]). Cocompact groups contain no $\mathbf{Z} \oplus \mathbf{Z}$ subgroups and have no infinite torsion subgroups ([B2], [Gr]).

It will be necessary to assume that G is countable and $L(G)$ is first countable for several results of this section. The following is a standard fact (for example, 3.8 of [GM2]).

Proposition 3.1. *If N is an infinite normal subgroup of a nonelementary convergence group G, then $L(N) = L(G)$.* ∎

Consequently, any virtually elementary group is elementary, and the only elementary normal subgroups of nonelementary G are finite. Such subgroups will act trivially on $L(G)$ by 3.9 of [GM2]. Furthermore, any amenable, solvable, nilpotent, or virtually nilpotent convergence group is elementary (this can be deduced from the fact that a nonelementary group contains a free subgroup of rank two).

Proposition 3.2. *If $L(G)$ is not a singleton, then G satisfies the ascending chain condition on finite normal subgroups.*

Proof. The conclusion is trivial if $L(G) = \emptyset$ and almost trivial (Example 1.3) if $|L(G)| = 2$. Suppose that G is nonelementary and that $N_1 \triangleleft N_2 \triangleleft N_3 \triangleleft \cdots$ is an infinite chain of distinct finite normal subgroups. Then

$$N = \bigcup_i N_i$$

is an infinite normal subgroup of G, all elements of which are elliptic. Thus N, and hence G, are elementary, a contradiction. ∎

Recently, Sela has shown that cocompact convergence groups are hopfian, i.e. any endomorphism is an isomorphism [S]. His arguments are geometric and do not seem to be reproducible in the present context. Nonetheless, some partial results can be derived.

The following proof is based on ideas from [MKS].

Theorem 3.3. *Suppose $\varphi : G \to G$ is an epimorphism. If the kernel of φ is finite, then φ is an isomorphism.*

Proof. Assume $K_1 = ker(\varphi)$ is finite, and $G \cong G/K_1$. If K_1 is not trivial, then (by the standard isomorphism theorems) there exists $K_2 \triangleleft G$ such that $K_1 \triangleleft K_2$ and $K_1 \cong K_2/K_1$. Note that K_1 has finite index in K_2, so K_2 is finite. Thus

$$G \cong G/K_1 \cong \frac{G}{(K_2/K_1)} \cong \frac{G/K_1}{(K_2/K_1)} \cong G/K_2.$$

Continuing by induction gives an infinite chain of distinct finite normal subgroups $K_1 \triangleleft K_2 \triangleleft K_3 \triangleleft \cdots$, a contradiction. Therefore $K_1 = ker(\varphi)$ must be trivial. ∎

Let X be fixed. Consider the lattice $\mathcal{G}$ of all convergence groups (of the first kind) acting on a given X, partially ordered by inclusion. Every $G \in \mathcal{G}$ acts properly discontinuously on the associated triple space $\mathcal{T}$. It follows that (up to finite extensions) the cocompact G are maximal in $\mathcal{G}$.

Theorem 3.4. *If H is a convergence group of the first kind (i.e. the ordinary set is empty) acting on X, $N \triangleleft H$, and the action of N on $\mathcal{T}$ is cocompact, then H is a finite extension of N and H acts cocompactly on $\mathcal{T}$.*

Proof. Since N is a normal subgroup, N is a convergence group with limit set all of X. Let C be a compact fundamental region for the action of N on $\mathcal{T}$ and D a closed fundamental region for H. Without loss of generality, assume $D \subset C$. Then D is also compact. H acts properly discontinuously on $\mathcal{T}$, so there are only finitely many $h \in H$ for which $h(D) \cap C \neq \emptyset$. Therefore H is a finite extension. ∎

Remark 3.5. By [B2], both N and H will be negatively curved groups.

Recall that the *word problem* for a group G with presentation $G = \langle C|R\rangle$ asks about the existence of an algorithm that determines if an arbitrary word in the given generators represents the identity in G. If such an algorithm exists, then the presentation is said to have a *solvable word problem* [MKS]. In the case that G is a negatively curved group (i.e. a convergence group acting cocompactly on its triple space) it is possible to adapt Dehn's algorithm to solve the word problem [C]. In a nonelementary convergence group, the stabilizer of three limit points is always finite. This allows

Proposition 3.6. *Suppose that $C = \{c_1, c_2, \ldots, c_k\}$ is a finite generating set for a torsion-free nonelementary convergence group G acting on X, a compact Hausdorff space. Let $z \in X$ have orbit Gz. If for each generator c_i and $x \in Gz$ there is an algorithm that computes the value of $c_i(x)$, then the word problem for G is solvable.*

Proof. Let w be a word in the generators, $z \in X$, and $x_1, x_2, x_3 \in Gz$ be any distinct triple. Then w is the identity if and only if $w(x_i) = x_i$ for $i = 1, 2, 3$. ∎

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Uintah Basin Education Center, 987 E. Lagoon St. (124-9), Roosevelt, Utah 84066

E-mail address: `freden@math.byu.edu`