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UNBOUNDED COMPONENTS IN PARAMETER SPACE OF RATIONAL MAPS

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Abstract. Using pinching deformations of Riemann surfaces, we give several sufficient criteria for the space of quasiconformal deformations of rational map R of degree d to have non-compact closure in the space Rat_d of rational maps of degree d modulo conjugation by Möbius transformations.

0. Introduction

Let $d > 1$ be an integer. Then the space of rational maps of degree d may be identified with an open subset of $\mathbb{C}P^{2d+1}$ via

$$R(z) = \frac{a_d z^d + \cdots + a_0}{b_d z^d + \cdots + b_0} \to [a_d : \cdots : a_0 : b_d : \cdots : b_0] \in \mathbb{C}P^{2d+1} - V(\Delta),$$

where $V(\Delta)$ is the locus where the resultant of numerator and denominator vanishes. The group $PSL_2(\mathbb{C})$ of Möbius transformations acts on this space by conjugation, yielding a Hausdorff quotient space Rat_d whose elements represent Möbius conjugacy classes of rational maps degree d. In general, if $R_n \to R$ in Rat_d, the dynamics of the limiting map R may be quite different from that of the R_n—consider e.g. the complicated behavior in the quadratic family $z^2 + c$. However, by work of [MSS] and [L], one knows that there is an open dense subset $W \subset Rat_d$ consisting of structurally stable maps such that if $R_n, R \in W$ and $R_n \to R$, then the dynamics of R_n converge to that R in the sense that there exist quasiconformal conjugacies h_n conjugating R to R_n such that $h_n \to id$. Thus, changes in the dynamics can occur only on ∂W.

In this case when a sequence R_n is "unbounded", the limiting dynamical system R does not necessarily exist. As an example, consider $z^2 + c_n$ with $c_n \to \infty$. For n large enough the maps lie in a single component of W (the complement of the Mandelbrot set) but no limiting quadratic map exists, regardless of how the maps are normalized.

In this paper, we study the question of unboundedness of certain kinds of deformation spaces associated with rational maps.

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Definition 0.1. Let $R \in \mathbb{C}P^{2d+1}$ be a rational map. The component of **J**-stability of R is the following space.

$$qc_J(R) = \{F \in \mathbb{C}P^{2d+1} : \text{ there are neighborhoods } U_R \text{ and } U_F \text{ of } \mathbf{J}(R) \text{ and } \mathbf{J}(F), \text{ respectively and a quasiconformal homeomorphism } h_F : U_R \to U_F \text{ such that } F = h_F \circ R \circ h_F^{-1}\}/PSL_2(\mathbb{C}).$$

If $qc_J(R)$ has non-compact closure in Rat_d, we say that $qc_J(R)$ is unbounded. Equivalently, $qc_J(R)$ is unbounded if there is a sequence of Möbius conjugacy classes in $qc_J(R)$ such that any corresponding sequence of representatives in $\mathbb{C}P^{2d+1} - V(\Delta)$ leaves every compact subset.

If R is hyperbolic (i.e. every critical point converges to an attracting cycle), then $qc_J(R)$ is known as the *hyperbolic component* containing R.

Definition 0.2. Let $R \in \mathbb{C}P^{2d+1}$ be a rational map. The space of quasiconformal deformations is defined as:

$$qc(R) = \{F \in \mathbb{C}P^{2d+1} : \text{ there is a quasiconformal automorphism } h_F \text{ of the Riemann sphere } \overline{\mathbb{C}} \text{ such that } F = h_F \circ R \circ h_F^{-1}\}/PSL_2(\mathbb{C}).$$

The subspace $qc(R)$ is an everywhere dense subset of $qc_J(R)$; the complement consists of maps whose dynamics near the Julia set is essentially the same, but whose critical points in the Fatou set behave in an essentially different way. In this paper, we give conditions on the dynamics of a rational map R which imply that the J-stable component of R is unbounded (see Theorem A below). Theorem A is related to the following two problems:

1. (i) (C. McMullen [MM1]) *If R is a rational map of degree d with connected Julia set and no common periodic point on the boundary of two components of the Fatou set, is the closure of the J-stable component of R compact in Rat_d?*

Recently C. McMullen reformulated this problem (see [MM2]) as follows.

(ii) *Let R be a hyperbolic rational map. If the Julia set of R is a Sierpinski carpet, is the closure of the J-stable component of R compact in Rat_d?*

The Julia set $J(R)$ is a Sierpinski carpet means that $J(R)$ is obtained from the sphere by removing a countable dense set of open disks, bounded by disjoint Jordan curves whose diameters tend to zero.

2. (J. Milnor [M]) *Let $d = 2$. How can one decide whether a given hyperbolic component has compact closure in Rat_2 or whether it is unbounded?*

To state our main theorems, we briefly recall a description of these quotient Riemann surfaces, and the notion of an accessible point in the Julia set.

Quotient Riemann surfaces. Due to Sullivan [S] there are only 5 types of periodic domains which can occur in the Fatou set for a rational map R. Furthermore there exists a Riemann surface associated with each type of these domains.

The cases with a fundamental domain:

(1) Let D be an attractive periodic domain. Then the associated Riemann surface $\mathbf{T}_D$ is a torus with marked points $a_1, \dots, a_l$, where a_i correspond to those orbits of critical points of the full orbit of D which do not land at the attractive

cycle. Denote by S_D the surface $\mathbf{T}_D \backslash \{a_1, \dots, a_l\}$ and by D_R the full orbit of D minus the closure of the full orbit of the set of critical points. Then there is an unbranched covering

$$P_D : D_R \to \mathbf{S}_D.$$

(2) Let D be a parabolic periodic domain. Then the associated Riemann surface $\mathbf{Sp}_D$ is a twice punctured sphere with marked points $a_1, \dots, a_l$, where a_i again correspond to the orbits of critical points of the full orbit of D. Denote by S_D the surface $\mathbf{Sp}_D \backslash \{a_1, \dots, a_l\}$ and by D_R the full orbit of D minus the set of full orbits of critical points. Then there is an unbranched covering

$$P_D : D_R \to \mathbf{S}_D.$$

Foliated cases:

(1) Let D be a periodic Siegel disk of period k. Then associated Riemann surface $\mathbf{D}_D$ is the unit disk conformally isomorphic to D with marked points $a_1, \dots, a_{l+1}$ and is equipped with the cyclic group $\mathbf{G}_D$ generated by a rotation by angle $2\pi i\alpha$, where α is irrational and is determined by the dynamics of R. Here the point $a_1 \in \partial \mathbf{D}_D$. The points $a_2, \dots, a_{l+1} \in \mathbf{D}_D \backslash \partial \mathbf{D}_D$ correspond to the first hits of forward orbits of the critical points into D. The group $\mathbf{G}_D$ corresponds to the action $R^k : D \to D$. Then there exists a branched covering P_D from the full orbit of D onto $\mathbf{D}_D$ semi-conjugating R^k with the generator of $\mathbf{G}_D$. The covering P_D is branched over the points $a_2, \dots, a_{l+1}$. Denote by S_D the surface $\mathbf{D}_D \backslash \big\{a_2, \dots, a_{l+1}\big\}$.

(2) Let D be a periodic Herman ring of period k. Then the associated Riemann surface $\mathbf{H}_D$ is a circular ring conformally equivalent to D with marked points $a_1, \dots, a_{l+2}$ and is equipped with the cyclic group $\mathbf{G}_D$ generated by rotation by angle $2\pi i\alpha$, where α is irrational and is determined by the dynamics of R. Here the points a_1 and a_2 belong to different components of $\partial \mathbf{H}_D$. The points $a_3, \dots, a_{l+2}$ correspond to the first hits of forward orbits of critical points. The group $\mathbf{G}_D$ corresponds to the action $R^k : D \to D$. Then there exists a branched covering P_D from the full orbit of D onto $\mathbf{H}_D$ semi-conjugating R^k with the generator of $\mathbf{G}_D$. The covering P_D is branched over the points $a_3, \dots, a_{l+2}$. Denote by S_D the surface $\mathbf{H}_D \backslash \big\{a_3, \dots, a_{l+2}\big\}$.

(3) Let D be a superattractive periodic domain of period k. Then the associated Riemann surface S_D is a circular ring with marked points $a_1, \dots, a_{l+2}$ and equipped with the group $\mathbf{G}_D$ generated by rotations by angles $2\pi i\alpha, \alpha = \frac{1}{\delta^n}, n = 1, 2, \dots,$ where δ is the local degree of R^k at the superattractive point. Here points a_1 and a_2 belong to different components of ∂S_D and all points $a_1, \dots, a_{l+2}$ correspond to the forward orbits of critical points. In this case there is no map from the full orbit of D onto S_D.

In the last three cases, the Riemann surface is foliated by concentric circles.

Denote by S_R the disjoint union $\sqcup S_D$. Let F_R be the Fatou set $\mathbf{F}(R)$ minus the full orbits of all superattractive periodic domains together with the closure of the full orbit of all critical points into S_R. Then there exists an unbranched covering $P_R : F_R \to S_R$.

Accesses.

Definition 0.3. Let $D \subset \overline{\mathbb{C}}$ be an open subset. Then a point $x \in \partial D$ is called accessible if there exists an arc (path) $\gamma \in D$ landing at x.

Let $x \in \partial D$ be a periodic point accessible from D. Let ω_1 and $\omega_2 \subset D$ be two arcs (paths) landing at the point x. Then ω_1 is equivalent to ω_2 if these two paths are homotopic by a homotopy leaving the boundary of D fixed. The equivalence class of an arc $\omega \subset D$ landing at the point $x \in \partial D$ is called an access to x from D and is denoted by $(x, [\omega])$.

Let $D \subset \mathbf{F}(R)$ be an attractive (parabolic) periodic domain of a given rational map R. Let $\gamma \subset S_D$ be a closed simple geodesic. Then a connected component $\beta \in P_R^{-1}(\gamma) \cap D$ is called a main component of the lift of γ if β lands at the attractive (parabolic) periodic point.

An access $(x, [\omega])$ is called geodesic if there is a geodesic $\gamma \subset \mathbf{S}_D$ and a main component β of the lift of γ such that $(x, [\beta])$ is equivalent to $(x, [\omega])$.

A periodic point $x \in \partial D$ is called geodesically accessible if x admits a geodesic access from D.

We call two geodesic accesses $(x, [\beta])$ and $(y, [\beta_1])$ independent if two geodesics γ and $\gamma_1 \subset S_R$ corresponding to these accesses are either mutually disjoint or coincide.

Main results. Consider the following sets of rational maps.

$\mathbf{W}_1 = \{$rational maps R with disconnected Julia set$\}$

$\mathbf{W}_2 = \{$rational maps R with connected Julia set such that there is a periodic component D of the Fatou set and a periodic point $x \in \partial D$ having more than one access from $D\}$

$\mathbf{W}_3 = \{$rational maps R with connected Julia set having two periodic components D_1 and D_2 of the Fatou set such that the intersection $\partial D_1 \cap \partial D_2$ contains two periodic points x and y accessible from both D_1 and $D_2\}$

Theorem A. *Let R be a rational map of degree d.*

(1) *Let $R \in \mathbf{W}_1$. Then the component of $\mathbf{J}$-stability $qc_J(R)$ is unbounded.*
(2) *Let $R \in \mathbf{W}_2 \cup \mathbf{W}_3$. Assume there is a map $R_1 \in qc_J(R)$ such that the accesses in the definitions of $\mathbf{W}_2$ and $\mathbf{W}_3$ are geodesic and independent for R_1. Then $qc_J(R)$ is unbounded.*

Corollary A. *Let $R \in \mathbf{W}_2 \cup \mathbf{W}_3$ be of degree d. Then there is an integer $N(R)$ such that $qc_J(R^n)$ is unbounded for all n divisible by $N(R)$.*

The following example shows that $N(R)$ can be greater than 1.

Example A. Let R be a rational map of degree d with a completely invariant parabolic domain D. Assume that $\mathbf{J}(R)$ is connected and $\mathbf{Sp}_D$ is a twice punctured sphere with only one marked point. Then the closure of the space $qc(R)$ is bounded in the space $\mathbf{Rat}_d$. Moreover,

(1) if $\deg(R) = 2$, then $qc_J(R)$ is bounded;
(2) if $\deg(R) = 2$ and R has an attractive periodic point of a period $p > 1$, then $qc_J(R^p)$ is unbounded.

The condition in Theorem A is based on the existence of a geodesic access for the corresponding periodic points. However it is easy to see that periodic points may be non-geodesically accessible even when the Julia set is a Jordan curve. We study the existence of independent geodesic accesses for the maps in $\mathbf{W}_2$ and $\mathbf{W}_3$ (see Proposition 3.1 and Theorem 4.1 below). This leads us to the following results.

Let $L(A)$ be the length (period) of a periodic set A determined by a rational map R.

Theorem B. *Let $R \in \mathbf{W}_3$ be a rational map of degree d and let D_1 and D_2 be attracting or superattracting periodic domains of R. Let d_i^j be the degrees of the restrictions $R|_{D_j^i}$, where $D_j^i = R^i(D_j)$, $j = 1, 2$. Assume that $L(x), L(y) \leq \min(L(D_1), L(D_2))$ and*

$$\frac{d_1^j d_2^j \dots d_{L(D_j)-1}^j - 1}{2} \leq \sum_{i=1}^{L(D_j)} (d_i^j - 1), \text{ for } j = 1, 2.$$

Then $qc_J(R)$ is unbounded.

Corollary B. *Let $R \in \mathbf{W}_3$ be a rational map of degree d and let the domains D_1 and D_2 be attracting (superattracting) and invariant. Assume that x and y are fixed points. Then $qc_J(R)$ is unbounded.*

Theorem C. *Let $R \in \mathbf{W}_2$. Let D be the domain and x be the point from the definition of $\mathbf{W}_2$. Assume that*

(1) *the domain D is attracting (superattracting) and $L(x) \leq L(D)$,*
(2) *Let d_i be the degrees of the restrictions $R|_{D_i}$, where $D_i = R^i(D)$. Suppose $d_i = 1$ for all i except i_0.*

Then $qc_J(R)$ is unbounded.

Corollary C. *Let R be a rational map of degree d with a completely invariant domain D. Assume that the fixed points of R are non-indifferent. Then $qc_J(R)$ is unbounded.*

Outline of proof. The main tool to prove the above theorems is application of the pinching construction to the Riemann surfaces associated with rational maps.

Let S_R be the Riemann surface associated with a rational map R. Let S be a component of S_R equipped with the canonical hyperbolic metric and let $\gamma \subset S$ be a closed, simple (without self-intersection) geodesic (or a leaf of foliation in the case of foliated Riemann surface, see Remark 1 below). Then there is (see [Str]) a small annular neighborhood $A_\gamma \subset S$ of γ and a conformal map $h : C \to A$ from the standard ring $C = \{z : 1/r < |z| < r\}$ onto A such that $h^{-1}(\gamma) = \{z : |z| = 1\}$. Then the modulus $m(A)$ of A is given by $\ln(r)/\pi$.

Consider the quasiconformal map $F(z) = z|z|$. Let $C_n = F^{-n}(C)$. Then $C_{n+1} \subset C_n$. Denote by μ_n the Beltrami differential of F^n on C_n and consider on C the new Beltrami differentials ν_n by

$$\nu_n = \begin{cases} 0 & \text{on } C \backslash C_1; \\ \mu_i & \text{on } C_i \backslash C_{i+1}, \text{ for } 1 \leq i \leq n-1; \\ \mu_n & \text{on } C_n. \end{cases}$$

Now transfer ν_n to the full orbit of D by means of h and the covering P_R. Extend ν_n by zero to the Riemann sphere to obtain a sequence ω_n of Beltrami differentials on $\overline{\mathbb{C}}$ invariant with respect to the map R (that is $\omega_n(R(z))\overline{R'(z)} = \omega_n(z)R'(z)$). The Measurable Riemann Mapping Theorem ([Ab]) then provides a sequence H_n of quasiconformal automorphisms of the Riemann sphere such that the maps $R_n = H_n \circ R \circ H_n^{-1}$ are rational. If for some normalization (for $\{H_n\}$) there exists a subsequence $\{R_{n_j}\}$ converging to a map R_∞ such that $\deg(R_\infty) = \deg(R)$, then we call R_∞ *a pinching deformation of R along the geodesic γ* and denote this map by R_γ. The sequences R_n and H_n of rational maps and quasiconformal homeomorphisms we call *the pinching sequences of maps and homeomorphisms*, respectively.

Roughly speaking, we use Remark 3 below to show that pinching deformations shrink independent accesses to points and that we loose in the limit a portion of the Julia set, contradicting the assumption that R_n converges. Thus we add one more line to Sullivan's dictionary between Kleinian groups and Holomorphic dynamics. Later on, Kevin Pilgrim ([Pil]) pointed out the following analogy:

Independent accesses in Theorem A correspond to cylinders for hyperbolic three manifolds.

From this point of view a reformulation of the question (1) (version (ii)) translates a well-known Thurston compactness result for convex cocompact geometrically finite hyperbolic acylindrical three-manifolds with non-empty incompressible boundary (see [Thu]).

Remarks.

Remark 1. In the foliated cases a geodesic always means a leaf of the foliation. Hence, the pinching deformation is well defined since $F(z)$ commutes with the rotation of the annulus C.

Remark 2. Assume that there exists a collection of simple closed geodesics $\gamma_1, \dots, \gamma_n$ on S such that $\gamma_i \cap \gamma_j = \varnothing$ for $i \neq j$. Then the pinching deformations may be defined along these geodesics simultaneously. If the pinching deformation of R along $\gamma_1, \dots, \gamma_n$ exists, we denote it by $R_{\gamma_1,\dots,\gamma_n}$. (Note: The deformations depend on $A_1, A_2, \dots$.)

Remark 3. Let A_n be $h^{-1}(C_n)$ and suppose $\Psi_n : S \to S_{R_n}$ is a quasiconformal homeomorphism with dilatation equal to ν_n. Then it is easy to show that moduli of the annuli $B_n \cup B_n{}' = \Psi_n(A_n \backslash A_{n+1})$ coincide and are equal to $\ln(r)/2\pi$ for all n. Furthermore the maps $\Psi_i \circ \Psi_n^{-1}$ are conformal on both components K_n and $K_n{}'$ of $\Psi_n(A \backslash A_{n+1})$ for all $i \geq n+1$, and

$$m(K_n), m(K_n{}') \geq \sum_{i=0}^{n} m(B_i) = \frac{n \ln(r)}{2\pi}.$$

In particular, the maps $\Psi_i \circ \Psi_n^{-1}$ are conformal on $\{\Psi_n(S \backslash A)\} \cup \{K_n \cup K_n{}'\}$ for all $i \geq n+1$.

We conclude the introduction with the following useful lemma whose proof is an application of the quasiconformal surgery of A. Douady and J. Hubbard ([DH]) and M. Shishikura ([Sh]).

Lemma 1. *Let R be a rational map and let $D \subset \mathbf{F}(R)$ be an attractive (superattractive) or parabolic periodic domain. Denote by d_i the degree of the restriction $R|_{D_i}$, where $D_i = R^i(D)$. Then there is a rational map $R_1 \subset qc_J(R)$ and a homeomorphism $h : \mathbf{J}(R) \to \mathbf{J}(R_1)$ conjugating R to R_1 such that*

(1) *the domain $D_1 \subset \mathbf{F}(R_1)$ bounded by $h(\partial D)$ is attracting or parabolic, respectively;*

(2) *in the case of the connected Julia set, for any* i *the domains* $D_1^i = R(D_1)$ *contain* $d_i - 1$ *simple critical points of* R_1 *and their forward orbits are infinite;*
(3) *there are no critical points in* $\bigcup_{i=1}^k \{R_1^i(D_1)\}$ *having intersecting forward orbits.*

Contents. In §1, we prove Theorem A and Corollary A. In §2, we prove the claims in Example A. §3 and §4 prove Theorems B and C as well as Corollaries B and C.

Remark 4. This paper is based on the preprint of 1993, [Mak1].

1. Proofs of Theorem A and Corollary A

We start with the next simple lemma.

Lemma 1.1. *Let* R *be a rational map and* $\gamma_1, \dots, \gamma_n \in S_R$ *be a collection of simple closed geodesics (or leafs of invariant foliations) such that* $\gamma_i \cap \gamma_j = \varnothing$ *for* $i \neq j$*. Assume that there is no pinching deformation along* $\gamma_1, \dots, \gamma_n$*. Then* $qc_J(R)$ *is unbounded.*

Proof. Choose some normalization for the pinching sequence $\{H_n\}$ of quasiconformal homeomorphisms and assume that $qc_J(R)$ has compact closure in Rat_d. Then there exists a sequence $\{g_i\}$ of Möbius maps and a subsequence $\{H_{n_i}\}$ such that the maps $R_i = g_i \circ H_{n_i} \circ R \circ H_{n_i}^{-1} \circ g_i^{-1}$ converge to a rational map R_∞ of degree $d = \deg(R)$. The critical and fixed points of R_{n_j} converge respectively to critical and fixed points of R_∞. Now, let c_1, c_2, c_3 be different critical points of R_∞ (or critical points and critical values if R_∞ does not have 3 different critical points, or critical points and fixed point if R_∞ is conjugated with $z \to z^d$ or $z \to \frac{1}{z^d}$) and let x_1, x_2, x_3 be corresponding points for R such that $g_i \circ H_{n_i}(x_j) \to c_j$, respectively. Now choose a new normalization for H_n, namely: $H_n = \alpha_n \circ \Psi_n$, where $\Psi_n(x_j) = x_j$. Then $g_i \circ \alpha_{n_i}(x_j) \to c_j$ and hence $\lim(g_i \circ \alpha_{n_i}) = g_0$ does exist and g_0 is a non-degenerate Möbius map. We conclude that $\Psi_{n_i} \circ R \circ \Psi_{n_i}^{-1} \to g_0^{-1} \circ R_\infty \circ g_0$, contradicting the assumption that there is no pinching deformation. ■

Next, we divide the proof of Theorem A into 3 cases, according to whether R belongs to $\mathbf{W}_1$, $\mathbf{W}_2$, or $\mathbf{W}_3$.

1) $R \in \mathbf{W}_1$**.** To start with we assume that $\mathbf{F}(R)$ contains a cycle of Herman rings of period k. Let D be a component of this cycle and $l \subset D$ be a leaf of the invariant foliation which does not intersects orbits of the critical points of R. Let $A \subset D$ be a small invariant ring neighborhood of l and R_n and h_n be the pinching sequences of rational maps and quasiconformal homeomorphisms, respectively. Assume that for some normalization there is a limit map R_∞ of the same degree as R. Then the maps h_n on $C_1 \cup C_2 = D \backslash A$ form normal families of conformal maps. Indeed, as in Lemma 1.1 above, consider triples of points c_1, c_2, c_3 and x_1, x_2, x_3 for maps R and R_∞, respectively with $h_n(c_i) \to x_i$ (arguments are up to a subsequence). Then the functions from the following family

$$f_n = \frac{h_n(z) - h_n(c_1)}{h_n(z) - h_n(c_2)} \frac{h_n(c_3) - h_n(c_2)}{h_n(c_3) - h_n(c_1)}$$

omit the values $0, 1$ and ∞ on $\overline{\mathbb{C}} - \{\bigcup_i c_i\}$.

We show that in one of the families $\mathfrak{F}_i = \{h_n|_{C_i}\}$, $i = 1, 2$, all limits are constants. Indeed, assume, say $\mathfrak{F}_1$, has a non-constant limit map H_∞ and $\{H_{n_j}\} \to H_\infty$ is a convergent sequence. Then

$$R^k_\infty(H_\infty(C_1)) = H_\infty(C_1)$$

and $H_\infty(C_1)$ is an annulus with definite modulus. But the moduli $m(H_{n_j}(A)) \to \infty$ and thus the spherical diameter of $H_{n_j}(C_2)$ tends to zero. Hence all limit maps for $\mathfrak{F}_2$ are constants. We need the following lemma.

Lemma 1.2. *Let R be a rational map of degree d. Let $\mathbf{Per}_{k,n}(R)$ be the set of periodic points of the period l with $k \le l \le n$. Then there is an integer N_R such that*

(1) *the set $\mathbf{Per}_{k,n}(R)$ consists of repelling periodic points for any $n \ge k \ge N_R$;*
(2) *for any $N_1 \ge N \ge N_R$, there is a neighborhood $U_{N,N_1} \subset \mathbb{C}P^{2d+1}$ of R such that*

$$\operatorname{card}(\mathbf{Per}_{k,n}(F)) = \operatorname{card}(\mathbf{Per}_{k,n}(R))$$

for all $N \le k \le n \le N_1$ and $F \in U_{N,N_1}$;
(3) *for any given $N_1 \ge N \ge N_R$ as well $F_i \in U_{N,N_1}$ and $\{F_i\} \to R$, then*

$$\mathbf{Per}_{k,n}(F_i) \to \mathbf{Per}_{k,n}(R)$$

uniformly. Moreover for all sufficiently large i there are homeomorphisms f_i mapping the sets $\mathbf{Per}_{k,n}(F_i)$ onto $\mathbf{Per}_{k,n}(R)$ and conjugating the action R with the action F_i. The homeomorphisms f_i depend continuously on F_i and $f_i \to \mathrm{id}$ as $i \to \infty$.

Proof. The number of non-repelling points of R is bounded and this gives (1). The cases (2) and (3) follow by analyzing the solutions of the equation

$$F^m_t(z) - z = 0, \text{ for } N \le k \le m \le n \le N_1,$$

where $F_t(z) \in U_{N,N_1}$ and U_{N,N_1} is a neighborhood such that all solutions of the equation above with the initial conditions

$$F^m_0(z_0) = R^m(z_0) = z_0, \text{ for } z_0 \in \mathbf{Per}_{k,n}(R)$$

are well defined on U_{N,N_1}. ■

Now we return back to Theorem A. We have that all limit functions of one of the families $\mathfrak{F}_i$ are constants. Then the spherical diameter of one of the components B^j_1 and B^j_2 of $h_{n_j}(\overline{\mathbb{C}}\backslash A)$ tends to 0 as $n_j \to \infty$ for any convergent subsequence $\{h_{n_j}\} \subset \{h_n\}$. Let $diam(B^j_1) \to 0$ and let x and y be different points of $\mathbf{Per}_{k,n}(R) \cap B_1$ with $k \ge N_{R_\infty}$. Then by the above construction we have $x_j = h_{n_j}(x)$ and $y_j = h_{n_j}(y) \in \mathbf{P}er_{k,n}(R_{n_j}) \cap B^j_1$. Therefore $\lim_{i\to\infty} x_{j_i} = \lim_{i\to\infty} y_{j_i}$ for some $\{j_i\} \subset \{j\}$. This contradicts Lemma 1.2.

Now we examine the general case. Let D be a non-simply connected Fatou component and O be the periodic component of the forward orbit of D. Then O is either attractive (superattractive), parabolic, or Siegel. Let $O = R^k(D)$.

Assume first that there is a Jordan curve $\gamma \subset O$ (in the Siegel case γ is a leaf of foliation) satisfying:

(1) there exists a small ring neighborhood $\mathcal{A} \subset D$ of γ such that $R^n|_{\mathcal{A}}$ is univalent for all $n \ge 1$;
(2) there is a component $\Omega \subset \bigcup_n R^{-n}(\mathcal{A})$ separating components of ∂D;
(3) $R^n(\mathcal{A}) \cap R^m(\mathcal{A}) = \varnothing$ for any $n \ne m$ in the parabolic and attractive cases.

Claim. The space $qc(R)$ is unbounded.

Proof of the claim. Conditions (1), (2), and (3) imply that the projection $P_R(\gamma) \subset S_R$ is a closed Jordan curve. We may assume γ is a geodesic. Under assumption (2) there exists a component $O_1 \subset D$ of $P_R^{-1}(S_R \setminus P_R(\mathcal{A}))$ separating components of ∂D. Let h_n and R_n be the pinching sequences of quasiconformal homeomorphisms and rational maps, determined by γ. Suppose there is a limit map R_∞ of degree $d = \deg(R)$. Then there is a subsequence $\{n_j\} \subset \{n\}$ such that the family $\mathfrak{F} = \{h_{n_j}|_{O_1}\}$ is normal. Lemma 1.2 implies that all limit maps of $\mathfrak{F}$ cannot be constants. Let H_∞ be a limit map and Q a component of $\mathbf{F}(R_\infty)$ containing the set $Q_1 = H_\infty(O_1)$. Then Q_1 separates components of ∂Q.

Remark 3 implies that for any N there exists N_1 such that for any fixed $s \geq N_1$ the maps

$$\psi_j = h_{i_j} \circ h_s^{-1}$$

are conformal on the component $O_s \subset \mathbf{F}(R_s)$ containing the set $h_s(O_1)$ and the set $O_s \setminus h_s(O_1)$ consists of the rings $B_l, l = 1, \ldots, k, k \geq 2$ with moduli $m(B_s) \geq N$. Therefore, if Ψ_∞ is a limit map for ψ_j, then

$$\Psi_\infty|_{h_s(O_1)} = H_\infty \circ h_s^{-1}|_{h_s(O_1)}.$$

Moreover any component of $\overline{\mathbb{C}} \setminus H_\infty(O_1)$ contains a ring $C_l = \Psi_\infty(B_l)$ for some l and $\partial C_l \cap H_\infty(\overline{O_1}) \subset \partial H_\infty(O_1)$. We conclude that the set $\overline{\mathbb{C}} \setminus Q$ consists of a finite number of points since the moduli of C_l may be arbitrarily large. This contradicts Lemma 1.2. ∎

Now we show an existence of a rational map $R_1 \subset qc_J(R)$ and a curve $\gamma \subset \mathbf{F}(R_1)$ satisfying conditions (1), (2), and (3).

Let O be a Siegel disk. Define γ to be an invariant leaf whose interior (up to a Möbius change of coordinates) contains all first hits of forward orbits of the critical points of the full orbit of O. Note that in our case the full orbit of O contains at least two critical points. Then passing to the branched covering $R^k : D \to O$ it is easy to see that γ satisfies conditions (1), (2), and (3).

Let O be an attractive periodic domain of the period k. Then we define γ to be the boundary of a small disk neighborhood U of an attractive periodic point such that $R(U) \subset U$ and $R|_U$ is univalent. Then (1) and (3) are satisfied. We now show (2). If O is not simply connected, then there is an iterated preimage of γ separating components of ∂O and (2) is satisfied.

If O is simply connected, then all components of $R^{-n}(U) \cap D$ are simply connected for any n. Let U_n be a component of $R^{-n}(U) \cap D$ containing all first hits of the forward orbits of critical points from the full orbit of O. Consider the branched covering $R^m : D \to O$. Then ∂U_n is homotopic to ∂O, thus $R^{-l}(\partial U_n)$ is homotopic to ∂D and (2) is satisfied.

The superattractive case is reduced to the attractive one by Lemma 1.

Let O be a parabolic periodic domain. Then by Lemma 1 we have a map $R_1 \in qc_J(R)$ and a homeomorphism $h : \mathbf{J}(R) \to \mathbf{J}(R_1)$ such that

(1) the component O_1 bounded by $h(\partial O)$ is parabolic, and
(2) the sets of forward orbits of critical values belonging to the forward orbit of O_1 are mutually disjoint.

Then by [Mak, Theorem 1] there is a horoneighborhood $U \subset O_1$ of the parabolic point such that

(1) $R|_U$ is univalent, $R(U) \subset U$ and
(2) $U \backslash R(U)$ contains all first hits of the orbits of critical values from the full orbit of O_1.

Let $\gamma \subset U \backslash R(U)$ be a closed Jordan curve whose interior contains all first hits from (2). Again consider the branched covering $R^l : D_1 \to O_1$, where D_1 is bounded by $h(\partial D)$. Then we conclude that γ satisfies (1), (2), and (3) since the interior of γ contains all critical values of this covering and $\pi_1(D_1) \neq 1$.

The case $R \in \mathbf{W}_1$ is proved.

2) $R \in \mathbf{W}_2$**.** We start with Maskit inequalities.

Lemma 1.3 (The Maskit inequalities)**.** *Let* $\mathbf{S}$ *be a hyperbolic surface of finite topological type (that is* $\pi_1(\mathbf{S})$ *is finitely generated) and* γ *be a simple closed geodesic on* $\mathbf{S}$*. If* $\mathcal{A}$ *is a ring neighborhood of* γ *in* $\mathbf{S}$*, let* $m(\mathcal{A})$ *be the extremal length of the family of all rectifiable curves in* $\mathcal{A}$ *freely homotopic to* γ*. Then*

1) $L_\gamma \leq 2\pi m(\mathcal{A}) \leq \pi L_\gamma \exp(L_\gamma/2)$*, where* L_γ *is the length of* γ*.*

2) Let $\mathbf{T}$ *be a torus and* $p : \mathbb{C}\backslash\{0\} \to \mathbf{T}$ *the unbranched covering with the covering group* $G = \langle g = \lambda z \rangle$*, for some* $|\lambda| > 1$*. Let* $[c]$ *be a generator of* $\pi_1(\mathbb{C}\backslash\{0\})$ *represented by a curve* c *with counterclockwise orientation. Let* Σ, Γ *be the generators of the fundamental group of the torus* $\mathbf{T}$ *such that*

$$p^*(g) = \Gamma, \quad p^*([c]) = \Sigma.$$

Let γ *be a simple closed Jordan curve homologous to a loop* $\Sigma^{-p}\Gamma^q$*. Then the lifting* $p^{-1}(\gamma)$ *consists of* q *curves* $\gamma_1, \dots, \gamma_q$*. If* $\mathcal{A} \subset \mathbf{T}$ *is a small ring neighborhood of* γ*, then*

$$\frac{\ln^2|\lambda| + (\arg(\lambda) - 2\pi p/q)^2}{2\pi \ln|\lambda|} \leq m(\mathcal{A}).$$

The number p/q *is called the* combinatorial rotation number of the curves $\gamma_1, \dots, \gamma_q$ with respect to the point $z = 0$.

If $|\lambda| < 1$*, then in the above terms we have that*

$$\frac{\ln^2|\lambda| + (\arg(\lambda) + 2\pi p/q)^2}{2\pi \ln|\lambda|} \leq m(\mathcal{A}),$$

and the combinatorial rotation number of curves $\gamma_1, \dots, \gamma_q$ *with respect to the point* $z = 0$ *is equal to* $-p/q$*. In other words, the combinatorial rotation numbers* α_0, α_∞ *of the curves* $\gamma_1, \dots, \gamma_q$ *with respect to* 0 *and* ∞*, respectively, satisfy the equation*

$$\alpha_0 + \alpha_\infty = 0.$$

3) Let B *be a topological ring in* $\mathbb{C}\backslash\{0\}$ *separating* 0 *and* ∞ *and suppose* $g^n(B) \cap B = \varnothing$ *for some* $g(z) = \lambda z, |\lambda| > 1$*, and any* n*. If* B *is conformally equivalent to* $\mathcal{A}$*, then*

$$\frac{2\pi}{\ln|\lambda|} \leq m(\mathcal{A}).$$

Moreover all above inequalities are quasiconformally quasinvariant.

Proof. See [Mas1], [Mas2]. The case (1) is the corollary of the well known collar lemma. The cases (2) and (3) are results of comparison of $m(B)$ with the Euclidean metric on $\mathcal{A}$. ■

Proposition 1.1. *Let R be a rational map and D be an attractive (parabolic) periodic domain. Let $S_D \subset S_R$ be the Riemann surface associated with D and $\gamma_1, \dots, \gamma_n$ closed simple geodesics on S_D with non-empty sets of main components and $\gamma_i \cap \gamma_j = \varnothing$ for $i \neq j$. Assume the pinching deformation $R_{\gamma_1,\dots,\gamma_n}$ does exist. Then there is a continuous map $h : D \to \overline{\mathbb{C}}$ such that h is a diffeomorphism on $D \backslash P^{-1}(\gamma_1, \dots, \gamma_n)$ and*

$$h \circ R = R_{\gamma_1,\dots,\gamma_n} \circ h.$$

Proof. Without loss of generality we can analyze the case with only one geodesic $\gamma \subset S_D$.

Choose some ring neighborhood $\mathcal{A} \subset \mathbf{S}_D$ of γ. Let D_j, $j = 1, \dots, m$ be the components of $D \backslash P_R^{-1}(\mathcal{A})$ touching the attractive (parabolic) point $z_0 \in D (z_0 \in \partial D)$. Then there is an integer k such that for any $l \geq 1$, $R^{kl}(D_j) = D_j$ and each D_j contains at least one critical point of R^{kl}. Consider pinching sequences of quasiconformal maps h_i. Then there is a subsequence $\{i_l\} \subset \{i\}$ such that families $\mathfrak{F}_j = \{h_{i_l}|_{D_j}\}$ are normal. Note that we can assume that the sequence $\{h_{i_l}|_{D_j}\}$ is convergent for any j. If $R_l = h_{i_l} \circ R \circ h_{i_l}^{-1}$ and $z_l = h_{i_l}(z_0)$, then the limit $x = \lim z_l$ is a *parabolic periodic point.* To see this, if z_l are parabolic periodic points, then x is parabolic obviously. Now assume that z_l are attractive points. Consider the geodesic γ. Since γ has main components, γ is homotopic to a loop $\Sigma^{-p}\Gamma^q$ (see the Maskit inequalities (2)) with non-zero q. Then by Remark 3 and the Maskit inequalities (2) we see the multiplier of the point x is equal to $\exp(-2\pi i \frac{p}{q})$.

Denote by H_j the limit maps for $\{h_{i_l}|_{D_j}\}$, respectively. Then for all j these maps satisfy:

(i) $H_j \neq const$,
(ii) if $R^k(D_i) = D_j$, then

$$H_j \circ R^k = R_\gamma^k \circ H_i.$$

To prove (i), suppose $H_j = a$ for some j and number $a \in \mathbb{C}$. The region D_j contains a critical point of R_γ^n, thus a has to be a critical point for R_γ^n. The point x must coincide with a. Then a is a periodic point for R_γ^n, contradicting the parabolicity of x.

(ii) is obvious.

Denote by O_j the component of $\mathbf{F}(R_\gamma)$ containing $H_j(D_j)$. Then all these components touch the point $x = H_1(z_0) = \dots = H_m(z_0)$ and O_j belongs to the immediate basin of attraction of x for each $j = 1, \dots, m$.

Claim. The set $O_i \backslash H_i(D_i)$ consists of the full orbit of two, invariant under $R_\gamma^{n=lk}$, topological disks w_1^i and w_2^i both touching the point x. These disks are called petals of the point x in O_i.

Proof of Claim. To start with we show that $O_i \backslash H_i(D_i)$ does not contain critical points. Otherwise H_i induces a conformal map H_i^* from the component $B \subset S_D \backslash \mathcal{A}$ into S_{O_i} (or from components B_1 and $B_2 \subset \mathbf{S}_D \backslash \mathcal{A}$ in the parabolic case) such that there is a component of $S_{O_i} \backslash H_i^*(B)$ containing a marked point α. By definition of the pinching deformation we know that the maps $\varphi_j = h_j \circ h_s^{-1}$ are induced by the maps $\Phi_j = \Psi_j \circ \Psi_s^{-1}$ (see Remark 3) which are conformal on the rings

$K_s \cup K_s' \subset \Psi_s(\mathcal{A})$. The moduli of these rings satisfy

$$m(K_s), m(K_s') \geq s\frac{\ln r}{2\pi}.$$

Therefore the map $H_i^* \circ h_s^{-1}$ may be extended to a conformal embedding $\hat{H}_i$ from $K_s \cup K_s'$ into S_{O_i}. Thus for all large s the set $\hat{H}_i(K_s \cup K_s')$ contains the point α. This contradicts the number of marked points of the component $S_D \backslash \mathcal{A}$.

Hence the set $S_{O_i} \backslash H_i(B)$ consists of two punctured disks W_1^i and W_2^i without marked points. Therefore we conclude that the set $W_i = P_{R_\gamma}^{-1}(W_1^i \cup W_2^i)$ is the full orbit of two components w_1^i and $w_2^i \subset W_i \cap O_i$ touching the point x. The map R_γ^n restricted to any component of W_i is one-to-one. ■

This Claim implies that the map H_i defines a continuous map on any component β of $\partial D_i \cap P_R^{-1}(\partial \mathcal{A})$ and $H_i(x_\beta) = H_i(y_\beta)$, where $x_\beta \cup y_\beta = \partial\beta$. On $\beta \backslash \partial\beta$ the map H_i is a homeomorphism.

Denote by B^* the set $S_D \backslash \gamma$. Then $B^* \supset B$ ($\supset \{B_1 \cup B_2\}$ in the parabolic case). Let Ψ_i^* be a diffeomorphism from B^* on S_{O_i} coinciding with H^* on B. This diffeomorphism is unique up to isotopy on B^*.

Note that actually Ψ_i^* is a limit of $\{h_{i_l}\}$ since, by construction, the homeomorphisms h_i have locally uniformly bounded distortion on any compact subset in B^*.

Then Ψ_i^* may be lifted from the component D_i^* of $D \backslash P_R^{-1}(\gamma)$ containing the set D_i to a diffeomorphism Ψ_i from D_i^* onto O_i conjugating the action R^n with the action R_γ^n. Let C be a component of $D_i^* \backslash D_i$ and $\beta = \partial C \cap P_R^{-1}(\gamma)$, then $\Psi_i(C \cup \beta) = \overline{w}$ for some $w \in W_i$. Define Ψ_i on β as $\Psi_i(\partial\beta)$. Then it is easy to see that by means of the actions of R^n and R_γ^n we obtain a continuous map from $D_i^* \cup \{\overline{D_i^*} \cap P_R^{-1}(\gamma)\}$ onto O_i. If $\beta \in \overline{D_i^*} \cup \overline{D_j^*} \cap P_R^{-1}(\gamma)$, then $\Psi_i(\beta) = \Psi_j(\beta)$ by construction. Therefore the maps Ψ_i define a continuous map Ψ from $\bigcup_i \{D_i^* \cup \{\overline{D_i^*} \cap P_R^{-1}(\gamma)\}\}$ onto $\bigcup_i O_i$, which semi-conjugates the action R^k with the action R_γ^k. By induction we complete this proposition. ■

Theorem 1.1. *Let R be a rational map with connected Julia set and no superattractive periodic domains. Let $\gamma_1, \dots, \gamma_n \subset S_R$ be a family of closed simple geodesics with $\gamma_i \cap \gamma_j = \varnothing$ if $i \neq j$. Assume that the pinching deformation $R_{\gamma_1,\dots,\gamma_n}$ does exist. Then there is a continuous map h from $\mathbf{F}(R)$ into $\overline{\mathbb{C}}$ such that h is a diffeomorphism on $\mathbf{F}(R) \backslash \{\bigcup_i P_R^{-1}(\gamma_i)\}$ and*

$$h \circ R = R_{\gamma_1,\dots,\gamma_n} \circ h.$$

Moreover h maps each connected component of $\bigcup_i P_R^{-1}(\gamma_i)$ to a single point.

Proof. Let $\{R_{n_i}\}$ be a convergent pinching subsequence of rational maps. Then,

(1) critical points of R_{n_i} converge uniformly to critical points of $R_{\gamma_1,\dots,\gamma_n}$,
(2) non-repelling periodic points of R_{n_i} converge uniformly to non-repelling periodic points of $R_{\gamma_1,\dots,\gamma_n}$, and
(3) the repelling periodic points of large enough period converge to the repelling periodic points of $R_{\gamma_1,\dots,\gamma_n}$ by Lemma 1.2.

Thus we conclude that there exists a subsequence $\{h_i\} \subset \{h_{n_i}\}$ converging on $\mathbf{F}(R) \backslash \bigcup_i P_R^{-1}(\mathcal{A}_i)$ to a non-constant conformal map, where $\mathcal{A}_i \subset S_R$ are some annular neighborhoods of γ_i, respectively. The argument given in Proposition 1.1 completes the proof. ■

Theorem 1.1 implies that if a pinching deformation R_γ does exist along a geodesic γ, then the set $P_R^{-1}(\gamma)$ does not contain finitely many components whose closure separates the Riemann sphere. This is an important observation for the remainder of the proofs.

Let $R \in \mathbf{W}_2$ and $R_1 \in qc_J(R)$. Let D be a component of $\mathbf{F}(R_1)$ and $x \in \partial D$ be an accessible periodic point with at least two independent accesses.

Under the assumptions we have, there is either one geodesic γ and two main components β_1 and β_2 of the lifting of γ defining the different accesses to x, or there are two geodesics γ_1 and γ_2 with main components β_1 and β_2 defining two different accesses to x, respectively. Again, consider pinching along either γ or $\gamma_1 \cup \gamma_2$. If either R_γ or R_{γ_1,γ_2} does exist, then Theorem 1.1 gives a continuous map h from D into $\overline{\mathbb{C}}$ semi-conjugating R with R_γ. The restriction of h on the closure of any component of $P_R^{-1}(\gamma)$ is a constant map. In both cases the set $\overline{\beta_1} \cup \overline{\beta_2}$ defines a closed Jordan curve separating $\overline{\mathbb{C}}$ such that both components B_1 and B_2 of $\overline{\mathbb{C}} \backslash \{\overline{\beta_1} \cup \overline{\beta_2}\}$ contain points of $\mathbf{J}(R_1)$. Hence we have

$$h|_{\{\overline{\beta_1} \cap \overline{\beta_2}\}} = const$$

and we loose a portion of the Julia set by taking a limit. This contradicts Lemma 1.2, and proves the case when $R \in \mathbf{W}_2$. ■

3) $R \in \mathbf{W}_3$**.** Consider R_1 and the geodesics generating accesses from D_1 and D_2 to periodic points x and y, respectively. Again pinching along these geodesics gives the desired contradiction. The case $R \in \mathbf{W}_3$ is done. ■

Proof of Corollary A. To start with we consider the Blaschke maps with connected Julia sets.

Proposition 1.2. *Let B be a Blaschke map of degree $d > 2$ with connected Julia set, and $\mathbf{Fix}(B)$ be the set of fixed repelling points of B. Let the component $S_\Delta \subset S_B$ associated with unit disk Δ be either a torus with $d-1$ punctured points or a sphere with $d+1$ punctured points. Then for any point $x \in \mathbf{Fix}(B)$ there exists a geodesic $\gamma_x \subset S_\Delta$ such that the main component β_x of the lift of γ_x lands at the point x. Moreover $\beta_x \cap \beta_y = \varnothing$ for $x \neq y \in \mathbf{Fix}(B)$.*

Proof. Let S and T be natural generators of $\pi_1(\mathbf{T}_\Delta)$ (in the parabolic case T is the natural generator of $\pi_1(\mathbf{Sp}_\Delta)$) such that $P_B^*(B) = T$.

Let $\gamma_1, \ldots, \gamma_{d-1}$ be different geodesics on S_Δ freely homotopic to T on $\mathbf{T}_\Delta$ such that $S_\Delta \backslash \{\bigcup_i \gamma_i\}$ consists of the union of non-degenerate rings with only one puncture. In the parabolic case let $\gamma_1, \ldots, \gamma_{d-2}$ be different geodesics on S_Δ freely homotopic to T on $\mathbf{Sp}_\Delta$ such that $S_\Delta \backslash \{\bigcup_i \gamma_i\}$ consists of two twice punctured disks (each disk containing exactly one puncture of $\mathbf{Sp}_\Delta$) and the union of non-degenerate rings with only one puncture.

Now, let β_i be the main components of the lifting of γ_i, respectively. Then,

(1) β_i is the unique main component of $P_B^{-1}(\gamma_i)$;
(2) β_i lands at some point $x_i \in \mathbf{Fix}(B)$;
(3) $B(\beta_i) = \beta_i$ and $B|_{\overline{\beta_i}}$ is one-to-one.

Assume that two $\beta_i \neq \beta_j$ land at a common point y. Then the interior $L \subset \Delta$ of the loop $\overline{\beta_i} \cup \overline{\beta_j} \subset \overline{\Delta}$ contains at least one critical point of B. But L cannot contain components of $B^{-n}(\overline{\beta_i} \cup \overline{\beta_j})$ for any $n \geq 0$, because $L \cap \partial\Delta = y$. The conditions (1)–(3) imply $B(L) = L$ and $B|_L$ is one-to-one, a contradiction. ■

Now let $R \in \mathbf{W}_2 \cup \mathbf{W}_3$ and $N(R)$ be the minimal integer such that for the map $R_1 = R^{N(R)}$ in the definition of $\mathbf{W}_2 \cup \mathbf{W}_3$ the domains D, D_1, D_2 are invariant, the periodic points x and y are fixed, and all accesses to the points x and y from D, D_1, D_2 are invariant as well.

By Lemma 1 we find a rational map $R_2 \in qc_J(R_1)$ such that the Fatou set of R_2 does not contain a superattractive periodic domain. If O is a periodic domain of $\mathbf{F}(R_2)$ and $x \in O$ is a critical point, then $R_2''(x) \neq 0$ and the forward orbit of x does not intersect the forward orbit of any other critical point (or a periodic point, whenever O is an attractive periodic domain). Then in the case $R_2 \in \mathbf{W}_2$ the component $S_D \subset S_{R_2}$ is either a torus with $d-1$ punctures or a sphere with $d-3$ punctures, where $d = \deg\{R_2|_D\}$. In the case $\mathbf{R}_2 \in \mathbf{W}_3$ we have a similar conclusion for S_{D_1} and S_{D_2}.

Therefore, we complete the proof of Corollary A by considering the Blaschke models for either the actions $R_2 : D \to D$ or the actions $R_2 : D_i \to D_i, i = 1, 2,$ and applying Proposition 1.2 and Theorem A. ■

2. Example A

Let R be a rational map with a completely invariant domain $D \subset \mathbf{F}(R)$ and let $\mathbf{T}(S_R) = \mathbf{T}(S_D) \times \mathbf{T}(S_1) \times \cdots \times \mathbf{T}(S_n)$ be the Teichmüller space of the Riemann surface S_R. Then (see [S]) there is a branched covering π from $\mathbf{T}(S_R)$ into Rat_d. In the absence of invariant line fields on $\mathbf{J}(R)$ the image $\pi(\mathbf{T}(S_R))$ coincides with $qc(R)$.

Let $S_D \subset S_R$ be the component associated with D and $\mu \in \mathbf{T}(S_D)$ be a conformal structure. Denote by B_μ the set $\{(\omega, \nu_1, \ldots, \nu_n) \in \mathbf{T}(S_D) \times \mathbf{T}(S_1) \times \cdots \times \mathbf{T}(S_n);$ $\omega \equiv \mu\}$. Then we have the following simple conclusion.

Proposition 2.1. *Let R be a rational map with a completely invariant domain D and $\mu \in \mathbf{T}(S_D)$ be a conformal structure. Then the closure of $\pi(B_\mu)$ in Rat_d is a compact subset.*

Proof. Let R_i be any sequence from $\pi(B_\mu)$ and h_i be a sequence of quasiconformal automorphisms of the Riemann sphere such that $R_i = h_i \circ R_1 \circ h_i^{-1}$, where $h_1 = id$. Then we can assume that

(1) the maps h_i are conformal on the completely invariant domain D_1 of R_1;
(2) the point ∞ belongs to D_1 and $h_i(z) = z + O(1/z)$ for $z \to \infty$.

In other words the maps $\{h_i|_{D_1}\}$ define a normal family. Let R_∞ be a limit map for a subsequence $\{R_{i_j}\}$. Then R_∞ is a rational map. If H_∞ is a limit map for $\{h_{i_j}|_{D_1}\}$, then by (2) $H_\infty \neq$ *const* and $R_\infty = H_\infty \circ R \circ H_\infty^{-1}$ on $H_\infty(D_1)$. We conclude that $\deg(R) = \deg(R_\infty)$. The proposition is proved. ■

Case 1. Let R_1 and $R_2 \in qc_J(R_1)$ be parabolic degree two maps with connected Julia sets. Let h_1 and h_2 be conformal maps from completely invariant domains D_1 and D_2 respectively onto the upper half plane mapping the parabolic fixed points onto ∞, respectively, and such that $h_1 \circ R_1 \circ h_1^{-1} = h_2 \circ R_2 \circ h_2^{-1} = z - \frac{1}{z}$. If h is a quasiconformal map from the definition of $qc_J(R)$, then $H = h_2 \circ h \circ h_1^{-1}$ is the identity on the real line (since H preserves the orientation, commutes with $z - \frac{1}{z}$ and fixes the point ∞). Therefore by setting

$$\widehat{h} = \begin{cases} h_2^{-1} \circ h_1 & \text{on } D_1; \\ h & \text{on } U_h \backslash D_1 \end{cases}$$

we obtain an extension of h to D_1. The arguments of Proposition 2.1 complete the proof of the case (1).

Case 2. Let S be an invariant union of periodic Fatou components for a given rational map R. Suppose that ∂S contains a periodic point x of period n. Consider the accesses from interior of S to x. An access $(x, [\gamma])$ is periodic if there is an integer $k > 1$, such that the access $(R^k(x), [R^k(\gamma)])$ defines the same access $(x, [\gamma])$. Then following [GM] we say that the periodic point x has *a combinatorial rotation number* $\frac{p}{q}$ if there is a periodic access $(x, [\gamma])$, such that γ performs a $\frac{p}{q}$ rotation around x under iterations of R^n.

Let R be a parabolic degree two map with connected Julia set. Assume that R has an attractive periodic point z of the period $k \geq 2$. Then R has a repelling fixed point x with non-trivial combinatorial rotation number $\alpha = \frac{p}{k}$. Choose $R_1 = R^k$. Then Proposition 1.2 and Theorem A imply that the space $qc_J(R_1)$ is unbounded in Rat_{2^k}. This gives (2). This shows that Theorem B fails in general for maps from $\mathbf{W}_2$. ∎

3. Proofs of Theorem B and Corollary B

Definition 3.1. Let D be an invariant attractive domain for a given rational map R. An annulus $\mathcal{C} \subset D$ is called a fundamental domain if $P_R(\mathcal{C}) = \mathbf{S}_D$ and the restriction of P_R to the interior of $\mathcal{C}$ is one-to-one and to $\partial\mathcal{C}$ is two-to-one.

A fundamental domain $\mathcal{C}$ is called tame if the interior of $\mathcal{C}$ contains all critical values of R in D and all first hits of the forward orbits of critical points belonging to the back orbit of D minus the back orbit of the attractive point $z \in D$.

Remark 3.1. Let D be an attractive periodic domain of period m for a given rational map R and suppose there are no critical points in the forward orbit of D whose forward orbits intersect. Then there is a tame fundamental domain $\mathcal{C} \subset D$ for R^m (see [Mak, Theorem 1]).

Proposition 3.1. *Let $B : \Delta \to \Delta$ be a hyperbolic Blaschke map of degree $d \geq 2$ with connected Julia set. Assume the number critical values of B in the unit disk Δ is greater than $\frac{d-1}{2}$. Then any two fixed points x and $y \in \partial\Delta$ are independent accessible from Δ.*

Proof. Denote by $\mathbf{Fix}_B$ the set of all repelling fixed points of B and fix a tame fundamental domain $\mathcal{C}$. Consider the torus $\mathbf{T}_\Delta$. Let $\pi_1(\mathbf{T}_\Delta)$ be generated by S $= [P_B(\partial\mathcal{C})]$ and $T = P^*_B(B)$. Further, as in Proposition 1.2, let $\gamma_1, \ldots, \gamma_m$ be different geodesics on S_Δ freely homotopic to T and such that $S_\Delta \backslash \{\bigcup_i \gamma_i\}$ is a union of non-degenerate annuli with only one puncture. Then the main components β_i of the geodesics γ_i land at *different* repelling fixed points of B. Denote by A the set of all these points. Then $card(A) > \frac{d-1}{2}$.

Now, let $\alpha \subset S_\Delta \subset \mathbf{T}_\Delta$ be a geodesic freely homotopic to S and let $\mathcal{A} \subset S_\Delta$ be a small annular neighborhood of S. Consider the twist map $\varphi : \mathcal{A} \to \mathcal{A}$, given by

$$\varphi(t, \theta) = (t, \theta + 2\pi(t - 1))$$

in the annular coordinates $\langle t, \theta \rangle, 1 \leq t \leq 2, 0 \leq \theta \leq 2\pi$ on $\mathcal{A}$. Then tameness of the fundamental domain $\mathcal{C}$ implies that

1. φ may be extended to a homeomorphism $\Phi : \overline{\Delta} \to \overline{\Delta}$ commuting with B and
2. $\Phi|_{\partial\Delta}$ is a map of order $d-1$. In other words Φ acts transitively on the set of fixed points of B.

Now our goal is to show the existence of the integer $k < d-1$ such that $\Phi^k(A)$ contains both points x and y.

Without loss of generality we may assume that $x \in A$. Let $n_1 = 0 < n_2 < \cdots < n_m < d-1$ be a collection of non-negative integers such that $\Phi^{n_i}(x) \neq x$ for $i \neq 1$ and $\Phi^{n_i}(x) \in A$. Assume that $y \notin \bigcup_{i=1}^m \Phi^{n_i}(A)$. Then the points $y_i = \Phi^{-n_i}(y) \notin A$ for $i = 1, \ldots, m$. But $m > \frac{d-1}{2}$ and hence $card(\mathbf{Fix}_B) \geq card(A \cup \{\bigcup_{i=1}^m y_i\}) > m + \frac{d-1}{2} > d-1$, a contradiction. ∎

Now, return back to Theorem B. Let N be $\min(L(D_1), L(D_2))$. We use Lemma 1 to find a rational map $R_1 \in qc_J(R) \subset \mathbf{W}_3$ such that

(1) the respective periodic domains of $\mathbf{F}(R)$ (again denoted by D_1 and D_2) are attracting;
(2) there are no critical points in forward orbits of D_1 and D_2 having common full orbit;
(3) if ℓ_j is a number of punctures of S_{D_j}, then

$$2\ell_j > d_1^j d_2^j \ldots d_{L(D_j)-1}^j - 1.$$

Suppose $S_{D_1} = S_{D_2}$. Then it is easy to see that all accesses to x and y from D_1 and D_2 are R_1^N-invariant. If h is a Riemann map from D_1 onto the unit disk Δ and $B = h \circ R_1^N \circ h^{-1} : \Delta \to \Delta$ is the Blaschke model for $R_1^N : D_1 \to D_1$, then there are two different fixed points x' and y' on $\partial\Delta$ such that $\lim_{r\to 1} h(rx') = x$ and $\lim_{r\to 1} h(ry') = y$. We observe that B satisfies the assumptions of Proposition 3.1. Thus Proposition 3.1 and Theorem A complete the proof of this case.

Suppose S_{D_1} and S_{D_2} are different. If we show that all accesses from D_1 and D_2 to the points x and y are invariant, then by the argument above we complete the proof of Corollary B.

One can show (using a linearization near the point x) that if one access from either D_1 or D_2 is invariant, then all accesses from D_1 and D_2 are invariant.

Let B be the component of $\overline{\mathbb{C}} \setminus D_1$ containing $\overline{D_2}$. Then $R_1^{kN}(\overline{D_2}) \subset B$ for all k. Therefore the accesses from D_1 to x and y are invariant. Theorem B and Corollary B are proved. ∎

4. Proof of Theorem C and Corollary C

Let $R : \partial\Delta \to \partial\Delta$ be a continuous endomorphism. Choose a natural order on $\partial\Delta$, that is $x = \exp(2i\pi t) \leq y = \exp(2i\pi\theta)$ if $0 \leq t \leq \theta < 1$. Let $X = (x_1 < x_2 < \cdots < x_n)$ be a periodic cycle of R of period n and assume there is no $k < n$ such that $R^k(x_i) = x_i$ for any i. Denote by $\langle x, y\rangle$ the arc going from x to y in the counterclockwise direction. A map f_X is called a *linearization* of R with respect to X if $f_X(x_i) = R(x_i)$ and the map f_X is linear on the arcs $\langle x_i, x_{i+1}\rangle$ in the coordinates $0 \leq \theta \leq 2\pi$, where $x = \exp(2i\pi\theta) \subset \partial\Delta$. It is clear that if $R(z) = z^d$, for some $d > 1$ and $R|_{\langle x_i, x_{i+1}\rangle}$ is one-to-one for some i, then $f_X|_{\langle x_i, x_{i+1}\rangle} = R|_{\langle x_i, x_{i+1}\rangle}$.

Further we say that a periodic cycle $X = (x_1, \ldots, x_n)$ of a continuous endomorphism R of unit circle admits a *combinatorial number* (or *combinatorial rotation number*, however this term is already occupied) $\alpha = \frac{p}{n}$, where $\frac{p}{n}$ is in lowest terms, if for any $i = 1, \ldots, n$ the interior of the arc $\langle x_i, R(x_i)\rangle$ contains $(p-1)$ points of X. It is clear that the linearization of a periodic cycle X is a homeomorphism iff X admits a combinatorial number. In this case f_X preserves the natural orientation of the unit circle $\partial\Delta$ and is piecewise linear. Hence the map $z \to |z|^d f_X(arg(z))$ is quasiconformal.

Theorem C is an immediate corollary of Theorem A and the following theorem.

Theorem 4.1. *Let $R(z) = z^d, d \geq 2$ and $X = (x_1 < \cdots < x_n), n > 1$ a periodic cycle. Then X has a combinatorial number if and only if there exists a Blaschke map B and a homeomorphism $h : \partial\Delta \to \partial\Delta$ such that $B = h \circ R(z) \circ h^{-1}$ and the periodic cycle $X' = h(X)$ is geodesically accessible from the unit disk Δ.*

Proof. The direction "$\Leftarrow$" is obvious.

Suppose that X has a combinatorial number $\frac{p}{n}$. Then $R(x_i) = x_{i+p}$ for all i. Let f be the linearization of R with respect to X. Denote by g the map $z \to |z|^d f(\arg(z))$. If r_i is a radius going from the point x_i to zero, then $g(r_i) = R(r_i)$.

Choose $0 < t < 1$ and rings $C_t^0 = \{z, t^d \leq |z| \leq t\}$ and $C_t^1 = \{z, t^{d^2} \leq |z| \leq t^d\}$. Then $R(C_t^0) = g(C_t^0) = C_t^1$. Denote by α_i^j the segments $r_i \cap C_t^j$. Then $R|_{\alpha_i^0} = g|_{\alpha_i^0}$. Denote by D_t^j the disk components in $\Delta \backslash C_t^j$, respectively.

Now, our goal is to construct a branched covering $\mathbf{F}$ from Δ to Δ such that

(1) $\mathbf{F} = R$ on the ring component of $\Delta \backslash C_t^0$ and
(2) $\mathbf{F}|_{(\partial D_t^0 \cup (\bigcup_i \alpha_i^0))} = g|_{(\partial D_t^0 \cup (\bigcup_i \alpha_i^0))}$.

Construction. Denote by Δ_i^j the piece of C_T^j between α_i^j and α_{i+1}^j in counterclockwise direction.

(1) Set $\mathbf{F} := R$ on $\Delta \backslash C_t^0$.
(2) If $R(\Delta_i^0) = \Delta_{i+p}^1$, then by definition of g we have $R|_{\Delta_i^0} = g|_{\Delta_i^0}$. In this case we set

$$\mathbf{F} := g \text{ on the set } \Delta_i^0.$$

(3) Let $R(\Delta_i^0) \supset \Delta_{i+p}^1$ and $R(\Delta_i^0) \neq \Delta_{i+p}^1$. Then $R(\Delta_i^0) = C_t^1$. Let n_i be the degree of the covering $R(\Delta_i^0) \to \Delta_{i+p}^1$. Choose any point $z_i^1 \in \Delta_{i+p}^1$ and some point $z_i^0 \in R^{-1}(z_i^1) \in \Delta_i^0$. Let $\ell_i \subset \Delta_i^1$ be an arc going from z_i^1 to the point y_i of the intersection of α_{i+p}^1 and the circle $\{|z| = t^d\}$ (see Figure 2). Let $\omega_i^1 < \cdots < \omega_i^{n_i}$ be the points of $R^{-1}(y_i) \cap \Delta_i^0$ and let $\ell_i^k \subset \Delta_i^0$ be arcs going from the point z_i^0 to ω_i^k, respectively (see Figure 1). Let $\gamma_i^j \subset \{|z| = t\}$ be the arcs $\langle \omega_i^k, \omega_i^{k+1} \rangle$ and $\gamma_{n_i} \subset \{|z| = t\}$ be the arc $\{\Delta_i^0 \cap \{|z| = t\}\} \backslash \bigcup_j^{n_i - 1} \gamma_i^j$.

Consider a partition of Δ_i^0 and Δ_i^1 into sets as shown in Figures 1 and 2. Define a map $\mathbf{F}$ on these sets as follows.

(a) Set $\mathbf{F} := g$ on $\overline{V}$.
(b) Consider W. Let Ψ be a quasiregular branched covering from W onto W_1 mapping ℓ_i^j onto ℓ_i such that on any component of $\overline{W} \backslash \bigcup_j \overline{\ell}_i^j$ the map Ψ is one-to-one and z_i^0 is the unique point of branching for Ψ of local degree n_i (see Figure 3). Then we set

$$\mathbf{F} := \Psi \text{ on } \overline{W}.$$

(c) Consider U_k. Then $\partial U_k = \beta_k \cup \gamma_i^k$, where $\beta_k \in \partial W$. The map $R|_{\gamma_i^k} \cup \Psi|_{\beta_k}$ maps ∂U_k onto $\partial W_1 \cup \partial D_t^0$. Let ϕ_k be a map from $\overline{U}_k$ onto $D_t^0 \backslash W_1$ quasiconformal

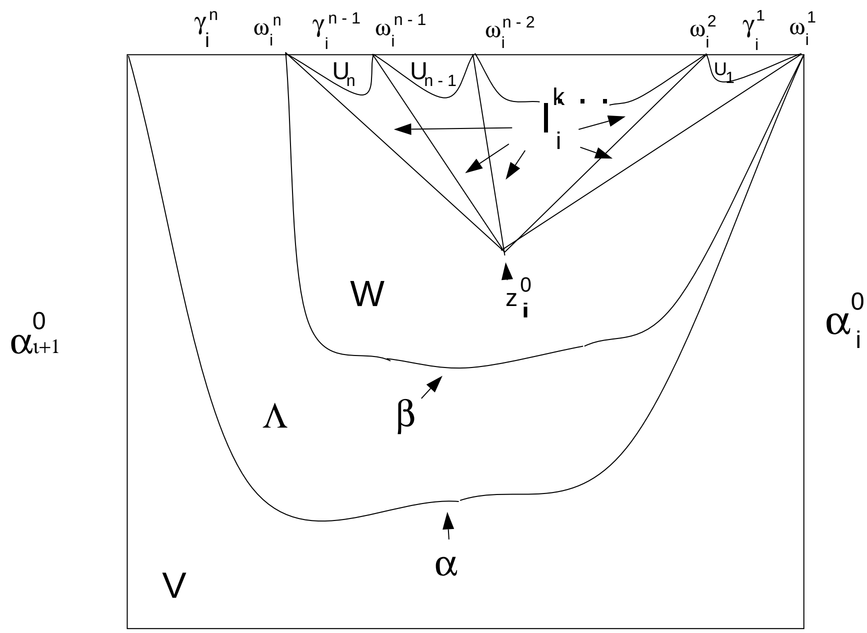

FIGURE 1. The region Δ_i^0

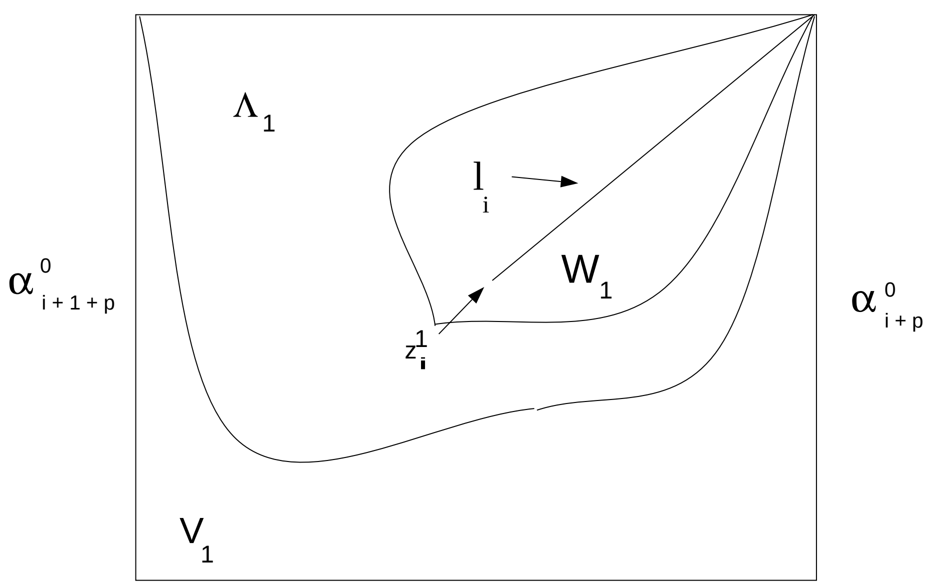

FIGURE 2. The region Δ_i^1, here $\overline{V}_1 = g(\overline{V})$

on the interior of U_k and coinciding on γ_i^k and β_k with R and Ψ, respectively. Then we set:

$$\mathbf{F} := \phi_k \text{ on } U_k, \text{ respectively.}$$

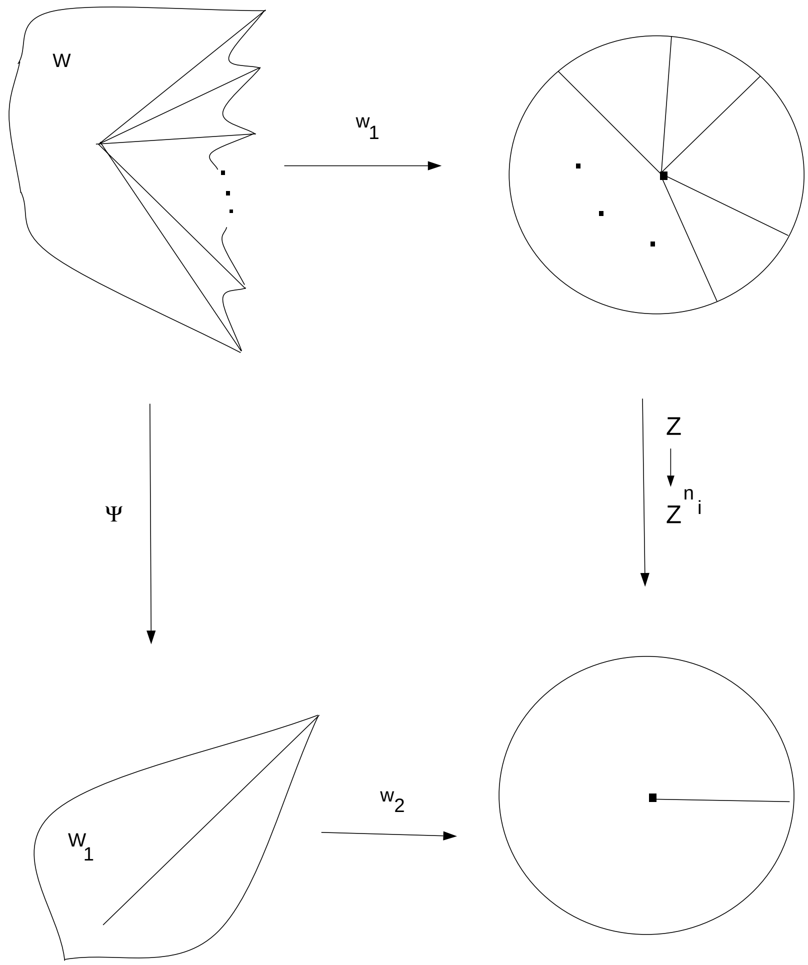

FIGURE 3. The homeomorphisms w_1 and w_2 are quasiconformal

(d) Finally consider Λ. The boundary of Λ consists of $\gamma_{n_i} \cup \alpha \cup \beta$, where $\alpha \subset \partial V$ and $\beta \subset \partial W$. The map $R|_{\gamma_{k_i}} \cup g|_\alpha \cup \Psi|_\beta$ maps $\partial\Lambda$ onto $\partial\Lambda_1 = \partial\{\Delta^1_{i+p} \backslash W_1 \backslash V_1\}$. Let Φ be a map mapping Λ onto Λ_1, coinciding with R, g and Ψ on γ_{k_i}, α and β, respectively and which is a quasiconformal homeomorphism on the interior of Λ. Then we set:

$$\mathbf{F}|_{\overline{\Lambda}} = \Phi.$$

Thus by induction we constructed a map from $\Delta \backslash D^0_t$ onto Δ. Now consider the ring C^1_t. Then $\mathbf{F}$ is a one-to-one map of the circle $\{|z| = t^d\}$ onto the circle $\{|z| = t^{d^2}\}$. Let $\delta(z) = t^d z$ be a map from D^0_t onto D^1_t and H be a quasiconformal automorphism of C^1_t such that

$$H \circ \mathbf{F} \circ H^{-1}|_{\{|z|=t^d\}} = \delta|_{\{|z|=t^d\}}.$$

Then the union $\bigcup_i \{\bigcup_n \delta^n(H(\alpha^1_i))\}$ forms n arcs $\beta_1, \ldots, \beta_n$ such that $H(\alpha^1_i) \subset \beta_i, \beta_i \cap \beta_j = \varnothing$ for $i \neq j$ and $\delta(\beta_i) = \beta_{i+p}$.

Now, we glue D_t^0 with δ to $\Delta\backslash D_t^1$ with $\mathbf{F}$ by H to produce a topological disk D and a branched covering $\Phi : D \to D$. Construct an invariant conformal structure μ on $\{D_t^0 \cup \{\Delta\backslash D_t^1\}\}/H$ by Φ starting with the standard one on $\delta(D_t^1)$. Since $\mathbf{F}$ is holomorphic on $\Delta\backslash D_t^0$ the structure μ has a bounded distortion. The Measurable Riemann Mapping Theorem gives a conformal map $\phi : (D, \mu) \to (\Delta, \sigma_0)$, where σ_0 is the standard conformal structure on Δ. Then $B = \phi \circ \Phi \circ \phi^{-1}$ is a Blaschke map and the arcs $\gamma_i = \phi(\beta_i \cup \{z \in r_i, r^d < |z|\})$ go from the attractive point to the periodic cycle $\phi(X)$. By construction the arcs project onto a closed Jordan curve $\gamma' \subset S_\Delta \subset S_B$, and the geodesic γ is homotopic to γ' as desired. ■

Remark 4.1. In the theorem above we constructed the Blaschke map possibly having critical points of local degree bigger than 2. However, quasiconformal surgery shows that for any structurally stable Blaschke map B, any periodic cycle of B having a combinatorial number is geodesically accessible. Thus, in Theorem 4.1 we may take B to be structurally stable.

Now we complete the proof of Theorem C. Let B be the Blaschke model for $R^k : D \to D$, where $k = L(D)$ and h is the conjugating conformal map. Let $X = (x_1, \ldots, x_n) \subset \partial\Delta$ be a periodic point satisfying $\lim_{r\to1} h(rx_i) = x$ for any i. Then X has a combinatorial number $\frac{m}{n} = \frac{-p}{q}$ or in other words $\frac{m}{n} = \frac{q-p}{q}$, where $\frac{p}{q}$ is the combinatorial rotation number of x. Further, the assumption (2) of the theorem and Lemma 1 allow us to assume that $S_\Delta \subset S_B$ is a $(d-1)$-punctured torus, where $d = d_{i_0}$. Therefore Proposition 3.1, Remark 3.1, and Theorem A complete the proof of Theorem C.

Corollary C. By Theorem A one can assume that the Julia set of R is connected.

Assume that all fixed points $x \notin D$ are repelling. Then by a result of Erëmenko and Levin [EL] all fixed points of R are accessible from the domain D. Thus there exists at least one fixed point, say x, admitting a non-trivial combinatorial number. Then Lemma 1 and Theorem C completes this case.

Now, let x be an attractive fixed point and let D_1 be a corresponding attractive invariant domain. Then ∂D_1 is a quasicircle. Further, let $Y = (y_1, y_2) \subset \partial D_1$ be a periodic cycle of period two. Then Y admits the trivial combinatorial rotation number with respect to $D \cup D_1$.

Let B and B_1 be the Blaschke models for $R : D \to D$ and $R : D_1 \to D_1$ and h_1 and h_2 be the corresponding conformal maps. Then the points $X = (x_1, x_2)$ and $X' = (x_1', x_2') \in \partial\Delta$ satisfying $\lim_{r\to1} h_1(rx_i) = \lim_{r\to1} h_2(rx_i') = y_i$ are periodic for B and B_1, respectively and have combinatorial numbers $= \frac{1}{2}$. Then again Theorem C and Remark 3.1 completes this case together with Corollary C. ■

Remark 4.2. It is easy to see that the above argument also works in the parabolic case except when $\deg(R|_{D_1}) = 2$.

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