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THE ROLE OF THE AHLFORS FIVE ISLANDS THEOREM IN COMPLEX DYNAMICS

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Abstract. The Ahlfors five islands theorem has become an important tool in complex dynamics. We discuss its role there, describing how it can be used to deal with a variety of problems. This includes questions concerning the Hausdorff dimension of Julia sets, the existence of singleton components of Julia sets, and the existence of repelling periodic points. We point out that for many applications a simplified version of the Ahlfors five islands theorem suffices, and we give an elementary proof of this version.

1. Introduction

We shall discuss the role of the Ahlfors theory of covering surfaces in complex dynamics. That the Ahlfors theory may be potentially useful in complex dynamics had been realised as early as 1939 by Töpfer [49, p. 69]. It turned out, however, that his way to apply it was not quite correct; cf. [3, p. 34]. The first successful application of the Ahlfors theory of covering surfaces to complex dynamics was then given by Baker [2] in 1968, who used the five islands theorem – one of the main results of the Ahlfors theory – to prove that repelling periodic points are dense in the Julia set of an entire function; see §6.2 below. Since then the Ahlfors five islands theorem has found various other applications in complex dynamics; see [8, 11, 13, 14, 15, 22, 25, 26, 46, 47]. Here we discuss some of them. We point out that for many applications a primitive version of the Ahlfors five islands theorem suffices, and we supply a simple proof of this primitive version. This provides a more elementary approach to certain results in complex dynamics.

To state the Ahlfors five islands theorem, let $D_1, \ldots, D_5$ be Jordan domains on the Riemann sphere $\widehat{\mathbb{C}}$ with pairwise disjoint closures. Let $D \subset \widehat{\mathbb{C}}$ be a domain and denote by $\mathcal{F}\left(D, \{D_j\}_{j=1}^5\right)$ the family of all meromorphic functions $f : D \to \widehat{\mathbb{C}}$ with the property that no subdomain of D is mapped conformally onto one of the domains D_j by f. (If there is such a subdomain, then it is called a *simple island* over D_j.)

We state three versions of the Ahlfors five islands theorem.

Theorem A.1. $\mathcal{F}\left(D, \{D_j\}_{j=1}^5\right)$ *is normal.*

Theorem A.2. $\mathcal{F}\left(\mathbb{C}, \{D_j\}_{j=1}^5\right)$ *contains only the constant functions.*

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Theorem A.3. *If $f \in \mathcal{F}\left(D\backslash\{\xi\}, \{D_j\}_{j=1}^5\right)$ for some $\xi \in D$, then f has a meromorphic extension to D.*

These three results are of course closely related, see §§2–3 below. They were proved by Ahlfors using his theory of covering surfaces, see [1], [30, Chapter 5], [42, Chapter XIII] or [50, Chapter VI]. A new proof was given in [16]. (Actually [16] was only concerned with Theorems A.1 and A.2, but we shall see in §3 that Theorem A.3 can easily be deduced from them.) The proof in [16] breaks into two parts. In the first part it is shown that the conclusion of Theorem A.2 holds if the D_j are small disks, and in the second part it is shown how the case of general Jordan domains D_j can be reduced to the case of small disks D_j.

While the second part uses quasiconformal mappings, and in particular the existence theorem for solutions of the Beltrami equation, the first part is much more elementary. It uses only a rescaling lemma for normal families, see Lemma 1 below, whose proof is short and elementary. The rescaling lemma also allows us to deduce Theorems A.1 and A.3 from Theorem A.2.

The rescaling lemma and the methods of [16] thus lead to a simple proof of weak versions of the Ahlfors five islands theorem where the domains D_j are replaced by small disks. To state these versions formally, we use the notation $D(a, r) := \{z \in \mathbb{C} : |z-a| < r\}$ for $a \in \mathbb{C}$ and $r > 0$. In the following, let $a_1, \dots, a_5 \in \mathbb{C}$ be distinct.

Theorem B.1. *There exists $\varepsilon > 0$ such that $\mathcal{F}\left(D, \{D(a_j, \varepsilon)\}_{j=1}^5\right)$ is normal.*

Theorem B.2. *There exists $\varepsilon > 0$ such that $\mathcal{F}\left(\mathbb{C}, \{D(a_j, \varepsilon)\}_{j=1}^5\right)$ contains only the constant functions.*

Theorem B.3. *There exists $\varepsilon > 0$ such that if $f \in \mathcal{F}\left(D\backslash\{\xi\}, \{D(a_j, \varepsilon)\}_{j=1}^5\right)$ for some $\xi \in D$, then f has a meromorphic extension to D.*

For completeness we shall give a proof of these results in §§2–3, and we shall indicate in §4 how Theorems A.1–A.3 can be deduced from them.

We note that the number "five" in Theorems A.1–A.3 and B.1–B.3 can be replaced by "three" for families of holomorphic functions (provided $\infty \notin \overline{D_j}$ in Theorems A.1–A.3).

2. A proof of Theorems B.1 and B.2

We denote the spherical derivative of a meromorphic function f by $f^{\#}$. The rescaling lemma referred to in the introduction is the following.

Lemma 1. *Let $D \subset \mathbb{C}$ be a domain and let $\mathcal{F}$ be a family of functions meromorphic in D. If $\mathcal{F}$ is not normal, then there exists a sequence (z_k) in D, a sequence (ρ_k) of positive real numbers, a sequence (f_k) in $\mathcal{F}$, a point $z_0 \in D$ and a non-constant meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ such that $z_k \to z_0$, $\rho_k \to 0$ and $f_k(z_k + \rho_k z) \to f(z)$ locally uniformly in $\mathbb{C}$. Moreover, f can be chosen such that $f^{\#}(z) \le 1 = f^{\#}(0)$ for all $z \in \mathbb{C}$.*

This lemma is due to Zalcman [51]. The corresponding result for normal *functions* had been proved earlier by Lohwater and Pommerenke [38]. For a discussion of the various applications of this lemma we refer to [44, Chapter 4] and [52]. For a proof of Lemma 1 we also refer, besides the papers mentioned, to [16, §4].

The statement about $f^{\#}$ is often not included in the formulation of the result, but follows immediately from the proof. While we could do without this statement for the present purposes by suitably modifying our arguments, the boundedness of $f^{\#}$ is essential for some applications; see [16, §3] and [52].

The proof of Theorem B.1 also requires the following result.

Lemma 2. *Let $a_1, \dots, a_5 \in \widehat{\mathbb{C}}$ be distinct and let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a non-constant meromorphic function. Then there exists $j \in \{1, \dots, 5\}$ such that f has a simple a_j-point.*

This result was proved by Nevanlinna using his theory on the distribution of values, see [41, p. 102] or [42, §X.3]. A different proof was given by Robinson [43]. For a proof of Lemma 2 based on Lemma 1 we refer to [16, §3].

Proof of Theorem B.1. We assume that the conclusion is false. Applying Lemma 1 to $\mathcal{F} = \mathcal{F}\left(D, \{D(a_j, \varepsilon)\}_{j=1}^5\right)$ we obtain a meromorphic function $f_\varepsilon : \mathbb{C} \to \widehat{\mathbb{C}}$ with $f_\varepsilon^{\#}(z) \le 1 = f_\varepsilon^{\#}(0)$ for all $z \in \mathbb{C}$. It is easy to see that $f_\varepsilon \in \mathcal{F}\left(\mathbb{C}, \{D(a_j, \varepsilon')\}_{j=1}^5\right)$ if $\varepsilon' > \varepsilon$. By Marty's theorem, $\{f_\varepsilon\}_{\varepsilon>0}$ is normal. Thus there exists a sequence (ε_k) tending to zero such that $f_{\varepsilon_k} \to f$ for some meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$. Since $f_\varepsilon^{\#}(0) = 1$ for all $\varepsilon > 0$ we have $f^{\#}(0) = 1$ so that f is non-constant. Moreover, we see that f has no simple a_j-points for $j \in \{1, \dots, 5\}$, contradicting Lemma 2. □

Proof of Theorem B.2. We note that if $f : \mathbb{C} \to \widehat{\mathbb{C}}$ is non-constant and meromorphic, then $\{f(nz)\}_{n\in\mathbb{N}}$ is not normal at 0. Thus Theorem B.2 follows immediately from Theorem B.1. □

We note that Lemma 1 can in turn be used to deduce Theorem B.1 from Theorem B.2. Similarly Theorem A.1 follows from Theorem A.2 and vice versa.

3. A proof of Theorem B.3

We shall deduce Theorem B.3 from Theorem B.1 and Theorem B.2. In order to do this, we also need the following result. Here and in the following we use the notation $\Delta(r) := \{z \in \mathbb{C} : |z| > r\}$ for $r > 0$.

Lemma 3. *Let f be meromorphic in $\Delta(r)$ for some $r > 0$. If f has an essential singularity at ∞, then*

$$\limsup_{|z|\to\infty} |z| f^{\#}(z) \ge \frac{1}{2}.$$

This result is due to Lehto [36]. Earlier, Lehto and Virtanen [37] had shown that $\limsup_{|z|\to\infty} |z| f^{\#}(z) \ge k$ for some absolute constant $k > 0$. This weaker result would suffice for our purposes.

Proof of Theorem B.3. We assume that $f \in \mathcal{F}\left(D\backslash\{\xi\}, \{D(a_j, \varepsilon)\}_{j=1}^5\right)$ has an essential singularity at ξ. Without loss of generality we may assume that $D = \Delta(1) \cup \{\infty\}$ and $\xi = \infty$. By Lemma 3 there exists a sequence (c_n) in $\Delta(1)$ such that $c_n \to \infty$ and $|c_n| f^{\#}(c_n) \ge 1/4$. We put $r_n := 1/|c_n|$ and define $g_n : \Delta(r_n) \to \widehat{\mathbb{C}}$ by $g_n(z) := f(c_n z)$. Then $r_n \to 0$ and $g_n \in \mathcal{F}\left(\Delta(r_n), \{D(a_j, \varepsilon)\}_{j=1}^5\right)$. By Theorem B.1 the g_n form a normal family and we may thus assume that $g_n \to g$ for some $g : \mathbb{C}\backslash\{0\} \to \widehat{\mathbb{C}}$. Since $g_n^{\#}(1) = c_n f^{\#}(c_n) \ge 1/4$ we have $g^{\#}(1) \ge 1/4$ so that g is non-constant. We also have $g \in \mathcal{F}\left(\mathbb{C}\backslash\{0\}, \{D(a_j, \varepsilon')\}_{j=1}^5\right)$ if $\varepsilon' > \varepsilon$. We define $h : \mathbb{C} \to \widehat{\mathbb{C}}$ by $h(z) = g(e^z)$. Then h is non-constant and meromorphic. Moreover, if h had a simple island V over some $D(a_j, \varepsilon')$, then $U := \exp V$ would be a simple island of g over $D(a_j, \varepsilon')$. (Note that if h is univalent in V, then exp is univalent in V and g is univalent in U.) Thus h has no simple islands over any $D(a_j, \varepsilon')$ which means that $h \in \mathcal{F}\left(\mathbb{C}, \{D(a_j, \varepsilon')\}_{j=1}^5\right)$, contradicting Theorem B.2 for sufficiently small ε'. $\square$

We remark that the same argument can be used to deduce Theorem A.3 from Theorems A.1 and A.2. Similarly, the argument can be used to deduce the great Picard theorem from the little Picard theorem and Montel's theorem. (This is usually done by a different method, cf., e.g., [23, pp. 59–60], [24, pp. 300–301] or [44, p. 60], but I have not been able to modify the argument given there for the present case. Andreas Sauer has (independently) found another way to deduce (a version of) Theorem A.3 from Theorem A.2. Although some of the underlying ideas such as the use of Lemma 3 are similar, his method is somewhat different from ours.)

There are many other cases where the above method can be used to obtain results for functions with isolated singularities from results about normal families and functions in the plane. The context is as follows: a heuristic principle attributed to Bloch says that the family of functions meromorphic in domain and having a certain property P is normal if there is no non-constant function meromorphic in the plane which has property P. Examples are Theorem A.1 versus Theorem A.2, or Montel's theorem versus the little Picard theorem, and many others, but there are also some counterexamples; see [44, Chapter 4] or [52] for a thorough discussion of Bloch's principle. Zalcman [51] introduced Lemma 1 in order to make the heuristic principle rigorous for certain properties.

A modification of the heuristic principle says that for a property P as above there should not be a meromorphic function having the property P in the neighborhood of an essential singularity. (The results corresponding to the previous examples are Theorem A.3 and the great Picard theorem, of course.) Minda [40] gives a discussion of this modification of the heuristic principle, and he shows that for holomorphic families the modified heuristic principle holds whenever Zalcman's formalization of the heuristic principle applies. However, Minda also points out that there are meromorphic families where Zalcman's rigorous version of the heuristic principle applies, but where the modified heuristic principle does not hold.

The above reasoning shows that the modified heuristic principle holds if besides Zalcman's hypotheses the following condition is satisfied: *if g satisfies P in $\mathbb{C}\backslash\{0\}$, then $g \circ \exp$ satisfies P in $\mathbb{C}$.* This additional condition can be compared with one of the hypotheses of Zalcman which says that if g has property P in a domain D, and if $\phi(z) = \alpha z + \beta$, $\alpha \neq 0$, then $g \circ \phi$ has property P in $\phi^{-1}(D)$.

4. A sketch of the proof of Theorems A.1, A.2 and A.3

As already mentioned in §§2–3, Theorems A.1 and A.3 can be deduced from Theorem A.2 using Lemmas 1 and 3. Thus it suffices to prove Theorem A.2.

Proof of Theorem A.2. Let $f \in \mathcal{F}\left(\mathbb{C}, \{D_j\}_{j=1}^5\right)$. Let $a_1, \dots, a_5 \in \mathbb{C}$ be distinct and choose $\varepsilon > 0$ according to Theorem B.2. Now there exists a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ with $\phi(D_j) \subset D(a_j, \varepsilon)$ for $j \in \{1, \dots, 5\}$, and the quasiregular map $\phi \circ f$ can be factored as $\phi \circ f = g \circ \psi$ with a meromorphic function $g : \mathbb{C} \to \widehat{\mathbb{C}}$ and a quasiconformal map $\psi : \mathbb{C} \to \mathbb{C}$. It follows that $g \in \mathcal{F}\left(\mathbb{C}, \{D(a_j, \varepsilon)\}_{j=1}^5\right)$, and thus g is constant by Theorem B.2. Hence f is constant. $\square$

5. Complex dynamics

For sets X, Y and a map $f : X \to Y$ we define the iterates f^n of f by $f^0(x) = x$ and $f^n(x) = f(f^{n-1}(x))$ for $n \in \mathbb{N}$. Note that $f^n(x)$ is defined only for those $x \in X$ for which all $f^j(x)$ with $j \le n-1$ are defined and for which in addition $f^{n-1}(x) \in X$. Thus in general f^n is defined only on a subset of X. A point $x \in X$ is called a *periodic* point of *period* n of f if $f^n(x) = x$, but $f^k(x) \neq x$ for $1 \le k \le n-1$. A set $Z \subset X$ is called *forward invariant* if $f(Z) \subset Z$ and *backward invariant* if $f^{-1}(Z) \subset Z$. A set is called *completely invariant* if it is both forward and backward invariant.

We now consider functions f for which there exists a countable compact set $A(f) \subset \widehat{\mathbb{C}}$ such that f is meromorphic in $\widehat{\mathbb{C}} \backslash A(f)$, but not in any larger subset of $\widehat{\mathbb{C}}$. We denote the class of all such functions which are not Möbius transformations or constants by $\mathcal{M}$. The iteration of functions in the class $\mathcal{M}$ seems to have been considered first by Bolsch [19] and Herring [32].

We recall some of the basic definitions and results. First we note that if $f \in \mathcal{M}$, then $f^n \in \mathcal{M}$ for all $n \in \mathbb{N}$, with $A(f^n) = A(f^{n-1}) \cup f^{-1}(A(f^{n-1}))$. If $A(f) = \emptyset$, then f is rational. The iteration theory of rational functions is well developed, see [12, 21, 39, 48] for an introduction. The case of transcendental meromorphic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$, that is, $A(f) = \{\infty\}$, is treated in [15, 33]. This includes the case of entire functions $f : \mathbb{C} \to \mathbb{C}$.

For $f \in \mathcal{M}$, let $B(f) = \overline{\bigcup_{n=1}^{\infty} A(f^n)}$ and $D(f) = \widehat{\mathbb{C}} \backslash B(f)$. Then all iterates of f are defined in $D(f)$. (The case $D(f) = \emptyset$ is possible.) The *Fatou set* $F(f)$ is defined as the set of all $z \in D(f)$ such that the family $\{f^n\}_{n \in \mathbb{N}}$ is normal at z. The *Julia set* $J(f)$ is defined as the complement of $F(f)$; that is, $J(f) = \widehat{\mathbb{C}} \backslash F(f)$. We consider four cases, depending on the cardinality $|B(f)|$ of $B(f)$.

(i) $B(f) = \emptyset$. Then f is rational.
(ii) $|B(f)| = 1$. We may assume that $B(f) = \{\infty\}$. Then f is transcendental entire.
(iii) $|B(f)| = 2$. We may assume that $B(f) = \{0, \infty\}$. Then f is a holomorphic self-map of the punctured plane $\mathbb{C}^* := \mathbb{C} \backslash \{0\}$.
(iv) $|B(f)| > 2$. Then $|B(f)| = \infty$ by the great Picard theorem.

In case (iv) we have $F(f) = D(f)$ and $J(f) = B(f)$ by Montel's theorem. So in this case the Fatou set is simply the largest open set where all iterates are defined. In cases (i)–(iii) the situation is quite the opposite one: it is clear where the iterates are defined; what matters is where they are normal. In all cases it turns out that $J(f)$ is a perfect set; that is, $J(f)$ is non-empty, compact, and without isolated points. In particular, this implies that $J(f)$ is uncountable.

We also mention that $F(f)$ is completely invariant. (With a slightly extended definition of complete invariance this property also holds for $J(f)$.) Finally we note that $F(f) = F(f^n)$ and $J(f) = J(f^n)$ for all $n \in \mathbb{N}$.

6. Applications of the five islands theorem in complex dynamics

6.1. **Some consequences of the five islands theorem.**

We begin with a simple corollary of Theorems A.1 and A.3, respectively B.1 and B.3.

Proposition A.1. *Let $D_1, \dots, D_5 \subset \widehat{\mathbb{C}}$ be Jordan domains with pairwise disjoint closures. If $f \in \mathcal{M}$ and if $D \subset \widehat{\mathbb{C}}$ is a domain such that $D \cap J(f) \neq \emptyset$, then there* *exists* $\nu \in \{1,\dots,5\}$, $n \in \mathbb{N}$ *and a domain* $U \subset D$ *such that* $f^n : U \to D_\nu$ *is conformal.*

Proposition B.1. *Let* $a_1,\dots,a_5 \in \mathbb{C}$ *be distinct. Then there exists* $\varepsilon > 0$ *with the following property: if* $f \in \mathcal{M}$ *and if* $D \subset \widehat{\mathbb{C}}$ *is a domain such that* $D \cap J(f) \neq \emptyset$, *then there exists* $\nu \in \{1,\dots,5\}$, $n \in \mathbb{N}$ *and a domain* $U \subset D$ *such that* $f^n : U \to D(a_\nu, \varepsilon)$ *is conformal.*

These results follow immediately from Theorems A.1 and B.1 in cases (i)–(iii). In case (iv) D contains an essential singularity of some f^n, and thus the results follow from Theorems A.3 and B.3 in this case.

Proposition A.2. *Let* $f \in \mathcal{M}$ *and let* $D_1,\dots,D_5 \subset \widehat{\mathbb{C}}$ *be Jordan domains with pairwise disjoint closures. Let* $V_1,\dots,V_5$ *be domains satisfying* $V_j \cap J(f) \neq \emptyset$ *and* $V_j \subset D_j$ *for* $j \in \{1,\dots,5\}$. *Then there exist* $\mu \in \{1,\dots,5\}$, $n \in \mathbb{N}$ *and a domain* $U \subset V_\mu$ *such that* $f^n : U \to D_\mu$ *is conformal.*

Proposition B.2. *Let* $f \in \mathcal{M}$, $a_1,\dots,a_5 \in \mathbb{C}$ *and* ε *as in Proposition* B.1. *Let* $0 < \delta < \varepsilon$ *and let* $V_1,\dots,V_5$ *be domains satisfying* $V_j \cap J(f) \neq \emptyset$ *and* $V_j \subset D(a_j,\delta)$ *for* $j \in \{1,\dots,5\}$. *Then there exist* $\mu \in \{1,\dots,5\}$, $n \in \mathbb{N}$ *and a domain* $U \subset V_\mu$ *such that* $f^n : U \to D(a_\mu,\delta)$ *is conformal.*

Proof of Proposition B.2. Proposition B.1 implies that for each $j \in \{1,\dots,5\}$ there exist $\nu(j) \in \{1,\dots,5\}$, $n(j) \in \mathbb{N}$ and a domain $U_j \in V_j$ such that $f^{n(j)} : U_j \to D(a_{\nu(j)},\delta)$ is conformal. Now the map $\nu : \{1,\dots,5\} \to \{1,\dots,5\}$ defined this way has a periodic point, say $\nu^p(\mu) = \mu$. The conclusion follows with this value of μ and $n := n(\mu)n(\nu(\mu))\cdots n(\nu^{p-1}(\mu))$ for some $U \subset U_\mu$. □

The proof of Proposition A.2 is analogous, using Proposition A.1 instead of Proposition B.1.

The point of Proposition A.2 and B.2 is that one can find a domain D such that a proper subdomain U of D is mapped conformally onto D by some iterate of f. For some applications it is important that one can even find a domain D which has two such subdomains.

Proposition B.3. *Let* $f \in \mathcal{M}$, $a_1,\dots,a_5 \in J(f) \cap \mathbb{C}$, *and* ε *as in Proposition* B.1. *Let* $0 < \gamma < \delta < \varepsilon$. *Then there exist* $\nu \in \{1,\dots,5\}$, $n \in \mathbb{N}$ *and domains* $U_1, U_2 \subset D(a_\nu,\gamma)$ *with* $U_1 \cap U_2 = \emptyset$ *such that* $f^n : U_m \to D(a_\nu,\delta)$ *is conformal for* $m \in \{1,2\}$.

Proof of Proposition B.3. As $J(f)$ is perfect there exist for each $j \in \{1,\dots,5\}$ six distinct points $b_{j,1},\dots,b_{j,6} \in J(f) \cap D(a_j,\gamma)$. Choose $\eta > 0$ such that the disks $D(b_{j,k},\eta)$ are disjoint and contained in $D(a_j,\gamma)$. Proposition B.2 implies that for each $k \in \{1,\dots,6\}$ there exist $\mu(k) \in \{1,\dots,5\}$, $n(k) \in \mathbb{N}$ and domains $V_k \subset D(b_{\mu(k),k},\eta) \subset D(a_{\mu(k)},\gamma)$ such that $f^{n(k)} : V_k \to D(a_{\mu(k)},\delta)$ is conformal.

There exists $k_1 \neq k_2$ with $\mu(k_1) = \mu(k_2)$. With $\nu := \mu(k_1) = \mu(k_2)$ and $n := n(k_1)n(k_2)$ we then find domains $U_m \subset V_{k_m} \subset D(a_\nu,\gamma)$ with the required properties. □

There is a corresponding consequence of Proposition A.2, the proof being analogous. Since we do not need this result here, we omit it.

6.2. **The density of repelling periodic points in the Julia set.** Let $f \in \mathcal{M}$ and let z_0 be a periodic point of period n of f. Then $\lambda := (f^n)'(z_0)$ is called the *multiplier* of z_0, with a slight modification if $z_0 = \infty$. The periodic point z_0 is called *repelling*, *indifferent*, or *attracting* depending on whether $|\lambda| > 1$, $|\lambda| = 1$ or $|\lambda| < 1$.

A basic result in complex dynamics is the following.

Theorem 1. *Let $f \in \mathcal{M}$. Then $J(f)$ is the closure of the set of repelling periodic points.*

Proof. We note that repelling periodic points are easily seen to be in $J(f)$ and thus we only have to show that if $D \subset \widehat{\mathbb{C}}$ is a domain intersecting $J(f)$, then D contains a repelling periodic point.

As $J(f)$ is perfect, $D \cap J(f)$ contains five points $a_1, \dots, a_5 \in \mathbb{C}$. Let ε be as in Proposition B.1, and choose δ such that $0 < \delta < \varepsilon$ and $D(a_j, \delta) \subset D$ for $j \in \{1, \dots, 5\}$. By Proposition B.2 there exists $\mu \in \{1, \dots, 5\}$, $n \in \mathbb{N}$ and a domain U with $\overline{U} \subset D(a_\mu, \delta)$ such that $f^n : U \to D(a_\mu, \delta)$ is conformal. This implies that the branch of the inverse function of f^n which maps $D(a_\mu, \delta)$ onto U has an attracting fixed point $z_0 \in D(a_\mu, \delta) \subset D$. It follows that z_0 is a repelling periodic point of f. $\square$

Remarks. For rational functions, Theorem 1 is due to Fatou [28, §30, p. 69] and Julia [34, p. 99, p. 118]. For entire functions the result was first proved by Baker [2], with essentially the above proof (using Theorem A.1 instead of Theorem B.1). The result was extended to meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ in [8, Theorem 1], and to class $\mathcal{M}$ in [18, 19, 32]. The argument in [8] used Theorem A.3.

Even though the proof of Theorems B.1–B.3 is considerably simpler than that of Theorems A.1–A.3, and thus the above proof of Theorem 1 is more elementary than the proofs in [2] and [8], the above approach is not the shortest way towards Theorem 1 – and it is included here mainly for completeness.

In fact, Schwick [45] has given a proof of Theorem 1 for entire functions which does not use any covering theorems like the ones used here. Instead Schwick's proof uses Lemmas 1 and 2 directly. His proof has subsequently been simplified by Bargmann [10] and by Berteloot and Duval [17]. Their arguments still have Lemma 1 as the main tool, but they do not require Lemma 2. Bolsch [18, 19] has used Lemma 3 to carry over Schwick's argument to class $\mathcal{M}$.

6.3. **Existence of periodic points of given period.** The following result, which confirmed a conjecture of Baker [31, Problem 2.20], was proved in [14].

Theorem 2. *Let f be an entire transcendental function and let $n \geq 2$. Then f has infinitely many repelling periodic points of period n.*

One of the tools used in the proof is Theorem A.1. (Actually this result is used for holomorphic families, so "five" can be replaced by "three", but this is not essential.) We do not attempt to sketch a proof of Theorem 2, but we point out that instead of Theorem A.1 the weaker Theorem B.1 suffices for the argument given in [14]. In fact, what is needed in [14, §4] is that if $r > 0$, then there exists $N \in \mathbb{N}$ such that the conclusion of Theorem A.1 holds for $D_j = D(2\pi ijN, r)$. This is the case, however, if and only if it holds for $D_j = D(2\pi ij, r/N)$, and Theorem B.1 implies that this is true for sufficiently large N.

It is shown in [15, p. 161] how Theorem A.3 can be used to obtain the conclusion of Theorem 2 for transcendental meromorphic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$ having more than two poles. Again it turns out that Theorem B.3 suffices for the argument given there.

6.4. **Hausdorff dimension of Julia sets.** We denote by $\dim X$ the Hausdorff dimension of a set $X \subset \widehat{\mathbb{C}}$. The following result is due to Stallard [47].

Theorem 3. *Let* $f \in \mathcal{M}$. *Then* $\dim J(f) > 0$.

Actually Stallard was only concerned with meromorphic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$, but her argument (which is the one given below) extends to the case that $f \in \mathcal{M}$. For rational functions, Theorem 3 is due to Garber [29]. For transcendental entire functions and analytic self-maps of the punctured plane, i.e., in cases (ii) and (iii), it follows from results of Baker ([4, Corollary to Theorem 1] and [6, Theorem 1]) and Keen [35, Theorem 3.1] that $J(f)$ contains non-degenerate continua, and thus $\dim J(f) \geq 1$ in these cases.

Proof of Theorem 3. We only sketch the argument, concentrating on the part where the Ahlfors five islands theorem comes into play, and referring to [47] for further details. We take five points $a_1, \dots, a_5 \in J(f) \cap \mathbb{C}$ and apply Proposition B.3 with $\gamma = \delta/96$ to obtain ν, n, U_1 and U_2. We put $a := a_\nu$ and $W := D(a, \delta/2)$. For $m \in \{1, 2\}$ we define $V_m := U_m \cap (f^n)^{-1}(W)$ and denote the branch of $(f^n)^{-1}$ that maps W onto V_m by ϕ_m. Because ϕ_m extends univalently to $D(a, \delta)$ the Koebe distortion theorem yields $|\phi_m'(z)| \leq 12|\phi_m'(a)|$ for $z \in W$. By Schwarz's lemma, applied to the function $z \mapsto (\phi_m(a + \delta z/2) - \phi_m(a))/2\gamma$, we have $\delta|\phi_m'(a)|/4\gamma \leq 1$ and thus $|\phi_m'(z)| \leq 12 \cdot 4\gamma/\delta = \frac{1}{2}$ for $z \in W$. This implies that $|\phi_m(z_1) - \phi_m(z_2)| \leq \frac{1}{2}|z_1 - z_2|$ for $z_1, z_2 \in W$. On the other hand, it is clear that there exist $c_1, c_2 > 0$ such that $|\phi_m(z_1) - \phi_m(z_2)| \geq c_m|z_1 - z_2|$ for $z_1, z_2 \in W$ and $m \in \{1, 2\}$.

The functions ϕ_1, ϕ_2 are thus contractions, and they form an iterated function scheme (also called iterated function system) as defined in [27, §9.1]. It follows from the theory of iterated function schemes that there exists a unique non-empty compact set $K \subset W$ which is invariant for ϕ_1 and ϕ_2; see [27, Theorem 9.1]. Moreover, if s is defined by $c_1^s + c_2^s = 1$, then $\dim K \geq s$ by [27, Proposition 9.7]. Finally, it is not difficult to show that $K \subset J(f)$; see [47] and §6.5 below. Thus $\dim J(f) \geq s > 0$. ∎

6.5. **Conjugacy to the shift map.** Let

$$\Sigma = \{(x_1, x_2, x_3, \dots) : x_n \in \{0, 1\} \text{ for all } n \in \mathbb{N}\},$$

equipped with the product topology of discrete topologies. Let $S : \Sigma \to \Sigma$ be the shift map defined by $S((x_1, x_2, x_3, \dots)) = (x_2, x_3, x_4, \dots)$. Christensen and Fischer [22] have shown the following result. (Actually Christensen and Fischer stated their result only for transcendental entire functions, but the proof extends to functions in $\mathcal{M}$.)

Theorem 4. *Let* $f \in \mathcal{M}$ *and let* $D \subset \widehat{\mathbb{C}}$ *a domain with* $D \cap J(f) \neq \emptyset$. *Then there exist* $n \in \mathbb{N}$, *a compact, invariant set* $K \subset J(f) \cap D$ *and a homeomorphism* $P : \Sigma \to K$ *such that* $P \circ S = f^n \circ P$.

The homeomorphism P thus conjugates $f^n|_K$ to the shift S. Hence shift-invariant measures on Σ give rise to f^n-invariant measures on $J(f)$. The construction of certain invariant measures was in fact the main motivation in [22]. The lower bound for the Hausdorff dimension given in [27] is also obtained by considering certain measures, and thus it is not surprising that the underlying idea in Theorems 3 and 4 is essentially the same.

Proof of Theorem 4. We proceed as in the proof of Theorem 3 and we see that if W, K, ϕ_1 and ϕ_2 are defined as there, and if $x = (x_1, x_2, x_3, \dots) \in \Sigma$ and $z \in W$, then $P(x) := \lim_{k\to\infty}(\phi_{x_1+1} \circ \phi_{x_2+1} \circ \dots \circ \phi_{x_k+1})(z)$ exists, is independent of z, and is contained in K. One checks that the function $P : \Sigma \to K$ defined this way has all the required properties. $\square$

6.6. **Singleton components of Julia sets.** Domínguez [25, 26] has used the Ahlfors theory of covering surfaces to show that under certain conditions the Julia set of transcendental entire or meromorphic functions contains singleton components. One of her results is as follows [25, Theorem 8.1].

Theorem 5. *Let f be an entire transcendental function. Suppose that $F(f)$ has a multiply-connected component. Then $J(f)$ has singleton components, and such components are dense in $J(f)$.*

Proof. Let D be a multiply-connected component of $F(f)$. Then $f^n|_D \to \infty$. It follows from results of Baker [4, 5] that D is bounded and that if σ is a curve which is not homotopic to zero in D, then for large $n \in \mathbb{N}$ the winding number of the curve $f^n(\sigma)$ with respect to the origin is non-zero.

These results imply that if $r > 0$, then $f^{-1}(D(0, r))$ consists of infinitely many components $D_1, D_2, \dots$ which are all bounded. By the maximum principle, each D_j is simply-connected and in fact a Jordan domain. We choose r such that $\overline{D} \subset D(0, r)$. Let now $j \in \mathbb{N}$. As $f : D_j \to D(0, r)$ is a proper map we see that D_j contains a domain W_j such that $f : W_j \to D$ is a proper map. The Riemann-Hurwitz formula (see, e.g., [48, p. 7]) implies that W_j is multiply-connected. Since $F(f)$ is completely invariant we see that W_j is also a component of $F(f)$. We can thus find a domain $V_j \subset \mathbb{C}$ such that V_j intersects $J(f)$, but does not intersect the unbounded component of $\mathbb{C}\backslash W_j$.

We may assume that $D_1, \dots, D_5$ have pairwise disjoint closures. Applying Proposition A.2 we obtain $\mu \in \{1, \dots, 5\}$, $n \in \mathbb{N}$ and $U \subset V_\mu$ such that $f^n : U \to D_\mu$ is conformal. Let ϕ be the branch of $(f^n)^{-1}$ which maps D_μ onto U. Then ϕ has an attracting fixed point $z_0 \in U \subset V_\mu$ and $\phi^k(z) \to z_0$ as $k \to \infty$, uniformly for $z \in D_\mu$.

Furthermore, $\phi^k(W_\mu)$ is a component of $F(f)$ for each $k \in \mathbb{N}$, and z_0 is contained in a bounded component of the complement of $\phi^k(W_\mu)$. This implies that $\{z_0\}$ is a singleton component of $J(f)$. $\square$

The point z_0 constructed in the above proof is a repelling periodic point of f. Moreover, z_0 is a *buried* point of $J(f)$; that is, there is no component G of $F(f)$ with $z_0 \in \partial G$. While it is also shown in [25] that $J(f)$ has buried singleton components, the argument used there does not seem to give whether there are repelling periodic points which are singleton components. On the other hand, it is shown in [7, Theorem G] that there exists an entire function f for which every periodic point is repelling and forms a buried singleton component of $J(f)$.

7. Concluding remarks

7.1. In the proof of Theorem 5 we have used Proposition A.2. We thus require the Ahlfors five islands theorem in the form given in Theorem A.1. I have not seen how to obtain Theorem 5 from the weaker Theorem B.1.

Theorems 1–4, however, do not require the Ahlfors five islands theorem in its strong form given by Theorems A.1–A.3, but the weaker Theorems B.1–B.3 suffice.

7.2. In this paper we have concentrated on applications of the Ahlfors five islands theorem in complex dynamics. Actually the Ahlfors five islands theorem is only a special case of a more general result called "Scheibensatz" by Ahlfors [1, p. 190]. To state this result, let $D \subset \widehat{\mathbb{C}}$ be a domain and $f : D \to \widehat{\mathbb{C}}$ be meromorphic. We say that f has an *island* (*of multiplicity* $\mu \in \mathbb{N}$) over a Jordan domain $D_0 \subset \widehat{\mathbb{C}}$ if there exists a simply-connected domain $U \subset D$ such that $f : U \to D_0$ is a proper map (of degree μ). Let now $q \in \mathbb{N}$ and $\mu_1, \dots, \mu_q \in \mathbb{N}$, and let $D_1, \dots, D_q$ be Jordan domains on $\widehat{\mathbb{C}}$ with pairwise disjoint closures. Let $\mathcal{F}(D, \{(D_j, \mu_j)\}_{j=1}^q)$ be the family of all meromorphic functions $f : D \to \widehat{\mathbb{C}}$ which have no islands of multiplicity less than μ_j over D_j, for $j \in \{1, \dots, q\}$. Ahlfors's Scheibensatz says that if

$$\sum_{j=1}^{q} \left(1 - \frac{1}{\mu_j}\right) > 2,$$

then the conclusion of Theorems A.1–A.3 holds with $\mathcal{F}(\cdot, \{D_j\}_{j=1}^5)$ replaced by $\mathcal{F}(\cdot, \{(D_j, \mu_j)\}_{j=1}^q)$. The five islands theorem is the special case $q = 5$ and $\mu_1 = \dots = \mu_5 = 2$.

The method used in §§2–4 (except for §3 it is the one used in [16]) applies in the more general situation of the Scheibensatz, and thus also yields a proof of this result. (This was already pointed out in [16, §5.1].) Again the situation is particularly simple in the case where the D_j are small disks so that we obtain short and elementary proofs of the corresponding generalizations of Theorems B.1–B.3.

Another important special case of the Scheibensatz is the case $q = 3$, with μ_j so large that the hypothesis of the Scheibensatz is satisfied. (The choice $\mu_j = 4$ suffices.) The result corresponding to Theorem A.2 now says that a non-constant meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ has an island (of multiplicity less than 4) over at least one of three Jordan domain D_1, D_2, D_3 with pairwise disjoint closures. This result has been used by Domínguez [26, Theorems A and B] to extend Theorem 5 to a large class of meromorphic functions.

This version of the Scheibensatz has also been used in [11, §4] to give a new proof of the result that an entire transcendental function has infinitely many periodic points of period n for all $n \geq 2$. While this result is weaker than Theorem 2, it suffices to prove Baker's conjecture [31, Problem 2.20]. For this latter application the version where the D_j are small disks (i.e., the result corresponding to Theorem B.2) suffices. This is not the case for Domínguez's application.

7.3. The Ahlfors theory not only yields the five islands theorem and the Scheibensatz in the form stated here, but it gives stronger, quantitative forms of these results by giving lower bounds for the number of islands of a function in subdomains of its domain of definition. I do not see how to obtain such quantitative estimates with the method of [16].

While these quantitative aspects are irrelevant for the applications considered here, this more precise theory of Ahlfors is essential for other applications. In fact, Bolsch ([19, Chapter 3] and [20]) has used the Ahlfors theory to prove that if $f \in \mathcal{M}$, then periodic components of $F(f)$ have connectivity $1, 2$, or ∞. For invariant components of meromorphic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$ this had been proved before in [9, Theorem 3.1] by a different method.

7.4. This survey of applications of the Ahlfors theory in complex dynamics is incomplete. Among further applications we only mention the work of Stallard [46] on the measure of Julia sets and that of Bedford, Lyubich and Smillie [13, §7] on polynomial diffeomorphisms of $\mathbb{C}^2$. There is no doubt that the Ahlfors theory will continue to have interesting applications in complex dynamics.

References

[1] L. V. Ahlfors, Zur Theorie der Überlagerungsflächen, *Acta Math.* 65 (1935), 157–194, and *Collected Papers*, Birkhäuser, Boston, Basel, Stuttgart, 1982, Vol. I, pp. 214–251. CMP 98:13

[2] I. N. Baker, Repulsive fixpoints of entire functions, *Math. Z.* 104 (1968), 252–256. MR **37:**1599

[3] ———, Completely invariant domains of entire functions, in *Mathematical Essays Dedicated to A. J. Macintyre*, edited by H. Shankar, Ohio University Press, Athens, Ohio, 1970, 33–35. MR **42:**6227

[4] ———, The domains of normality of an entire function, *Ann. Acad. Sci. Fenn. Ser. A I Math.* 1 (1975), 277–283. MR **53:**5867

[5] ———, Wandering domains in the iteration of entire functions, *Proc. London Math. Soc.* (3) 49 (1984), 563–576. MR **86d:**58066

[6] ———, Wandering domains for maps of the punctured plane, *Ann. Acad. Sci. Fenn. Ser. A I Math.* 12 (1987), 191–198. MR **89g:**30046

[7] I. N. Baker and P. Domínguez, Some connectedness properties of Julia sets, preprint, 1998.

[8] I. N. Baker, J. Kotus and Y. Lü, Iterates of meromorphic functions. I, *Ergodic Theory Dynamical Systems* 11 (1991), 241–248. MR **92m:**58113

[9] ———, Iterates of meromorphic functions. III: Preperiodic domains, *Ergodic Theory Dynamical Systems* 11 (1991), 603–618. MR **92m:**58115

[10] D. Bargmann, Simple proofs of some fundamental properties of the Julia set, *Ergodic Theory Dynamical Systems* 19 (1999), 553–558. CMP 99:14

[11] D. Bargmann and W. Bergweiler, Periodic points and normal families, to appear in Proc. Amer. Math. Soc.

[12] A. F. Beardon, *Iteration of rational functions*, Springer, New York, Berlin, Heidelberg, 1991. MR **92j:**30026

[13] E. Bedford, M. Lyubich and J. Smillie, Polynomial diffeomorphisms of $\mathbb{C}^2$. IV: The measure of maximal entropy and laminar currents, *Invent. Math.* 112 (1993), 77–125. MR **94g:**32035

[14] W. Bergweiler, Periodic points of entire functions: proof of a conjecture of Baker, *Complex Variables Theory Appl.* 17 (1991), 57–72. MR **92m:**30048

[15] ———, Iteration of meromorphic functions, *Bull. Amer. Math. Soc. (N.S.)* 29 (1993), 151–188. MR **94c:**30033

[16] ———, A new proof of the Ahlfors five islands theorem, *J. Anal. Math.* 76 (1998), 337–347. MR **99m:**30032

[17] F. Berteloot and J. Duval, Une démonstration directe de la densité des cycles répulsifs dans l'ensemble de Julia, preprint, 1998.

[18] A. Bolsch, Repulsive periodic points of meromorphic functions, *Complex Variables Theory Appl.* 31 (1996), 75–79. MR **98c:**30033

[19] ———, *Iteration of meromorphic functions with countably many singularities*, Dissertation, Berlin, 1997.

[20] ———, Periodic Fatou components of meromorphic functions, *Bull. London Math. Soc.* 31 (1999), 543–555. CMP 99:16

[21] L. Carleson and T. W. Gamelin, *Complex Dynamics*, Springer, New York, Berlin, Heidelberg, 1993. MR **94h:**30033

[22] J. P. R. Christensen and P. Fischer, Ergodic invariant probability measures and entire functions, *Acta Math. Hungar.* 73 (1996), 213–218. MR **97h:**30041

[23] C.-T. Chuang, *Normal families of meromorphic functions*, World Scientific, Singapore, 1993. MR **95d:**30065

[24] J. B. Conway, *Functions of one complex variable. I*, 2^{nd} ed., Springer, New York, Berlin, Heidelberg, 1978. MR **80c:**30003

[25] P. Domínguez, Connectedness properties of Julia sets of transcendental entire functions, *Complex Variables Theory Appl.* 32 (1997), 199–215. MR **98f:**30027

[26] ———, Dynamics of transcendental meromorphic functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.* 23 (1998), 225–250. MR **99b:**30031

[27] K. Falconer, *Fractal geometry*, John Wiley & Sons, Chichester, 1997.

[28] P. Fatou, Sur les équations fonctionelles, *Bull. Soc. Math. France* 47 (1919), 161–271; 48 (1920), 33–94, 208–314. MR **58:**11343

[29] V. Garber, On the iteration of rational functions, *Math. Proc. Cambridge Philos. Soc.* 84 (1978), 497–505. MR **29:**1337

[30] W. K. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964. MR **29:**1337

[31] ———, *Research problems in function theory*, Athlone Press, London, 1967. MR **36:**359

[32] M. E. Herring, *An extension of the Julia-Fatou theory of iteration*, Ph.D. thesis, London, 1995.

[33] X.-H. Hua and C.-C. Yang, *Dynamics of transcendental functions*, Gordon and Breach, Australia, 1998. MR **99j:**58169

[34] G. Julia, Sur l'itération des fonctions rationelles, *J. Math. Pures Appl.* (7) 4 (1918), 47–245, and *Œuvres de Gaston Julia*, Gauthier-Villars, Paris, 1968, Vol. I.

[35] L. Keen, Dynamics of holomorphic self-maps of $\mathbb{C}^*$, in *Holomorphic Functions and Moduli I*, edited by D. Drasin, C. J. Earle, F. W. Gehring, I. Kra and A. Marden, Springer, New York, Berlin, Heidelberg, 1988. MR **90e:**58075

[36] O. Lehto, The spherical derivative of meromorphic functions in the neighbourhood of an isolated singularity, *Comment. Math. Helv.* 33 (1959), 196–205. MR **21:**5732

[37] O. Lehto and K. I. Virtanen, On the behaviour of meromorphic functions in the neighbourhood of an isolated singularity, *Ann. Acad. Sci. Fenn. Ser. A*, no. 240 (1957). MR **19:**404a

[38] A. J. Lohwater and Ch. Pommerenke, On normal meromorphic functions, *Ann. Acad. Sci. Fenn. Ser. A*, no. 550 (1973). MR **49:**3146

[39] J. Milnor, *Dynamics in one complex variable*, Vieweg, Braunschweig, Wiesbaden, 1999. CMP 2000:03

[40] D. Minda, A heuristic principle for a nonessential isolated singularity, *Proc. Amer. Math. Soc.* 93 (1985), 443–447. MR **86g:**30040

[41] R. Nevanlinna, *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*, Gauthiers-Villars, Paris, 1929. MR **54:**5468

[42] ———, *Eindeutige analytische Funktionen*, Springer, Berlin, Göttingen, Heidelberg, 1953. MR **15:**208c

[43] R. M. Robinson, A generalization of Picard's and related theorems, *Duke Math. J.* 5 (1939), 118–132.

[44] J. L. Schiff, *Normal families*, Springer, New York, Berlin, Heidelberg, 1993. MR **94f:**30046

[45] W. Schwick, Repelling periodic points in the Julia set, *Bull. London Math. Soc.* 29 (1997), 314–316. MR **97m:**30029

[46] G. M. Stallard, Entire functions with Julia sets of zero measure, *Math. Proc. Cambridge Philos. Soc.* 108 (1990), 551–557. MR **91g:**30037

[47] ———, The Hausdorff dimension of Julia sets of meromorphic functions, *J. London Math. Soc.* (2) 49 (1994), 281–295. MR **95b:**58127

[48] N. Steinmetz, *Rational iteration*, Walter de Gruyter, Berlin, 1993. MR **94h:**30035

[49] H. Töpfer, Über die Iteration der ganzen transzendenten Funktionen, insbesondere von $\sin z$ und $\cos z$, *Math. Ann.* 117 (1939), 65–84. MR **1:**211e

[50] M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1959. MR **22:**5712

[51] L. Zalcman, A heuristic principle in complex function theory, *Amer. Math. Monthly* 82 (1975), 813–817. MR **52:**757

[52] ———, Normal families: new perspectives, *Bull. Amer. Math. Soc. (N.S.)* 35 (1998), 215–230. MR **99g:**30048

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