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A COMBINATION THEOREM FOR COVERING CORRESPONDENCES AND AN APPLICATION TO MATING POLYNOMIAL MAPS WITH KLEINIAN GROUPS

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Abstract. The simplest version of the Maskit-Klein combination theorems concerns the action of a free product of two finite subgroups of $PSL(2,\mathbb{C})$ on the Riemann sphere $\hat{\mathbb{C}}$, when these subgroups have fundamental domains whose interiors together cover $\hat{\mathbb{C}}$. We prove an analogous combination theorem for covering correspondences of rational maps, making use of Douady and Hubbard's Straightening Theorem for polynomial-like maps to describe the structure of the limit sets. We apply our theorem to construct holomorphic correspondences which are matings of polynomial maps with Hecke groups $C_p * C_q$, and we show how it may also be applied to the analysis of separable correspondences.

1. Covering correspondences and transversals for rational maps

For any rational map $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree q we define

• Cov^Q to be the $(q : q)$ correspondence (multi-valued map) $z \to w$ where w runs through all values such that

$$Q(w) = Q(z),$$

and

• Gal^Q to be the $(q-1 : q-1)$ correspondence $z \to w$ where

$$\frac{Q(w) - Q(z)}{w - z} = 0.$$

We shall use the same notation for both the graph of a correspondence and the multi-valued map defined by that graph. Thus $Cov^Q = Gal^Q \cup I$, where I denotes the graph of the identity. The intersection $Gal^Q \cap I$ consists precisely of the points (z, z) such that z is a critical point of Q. Recall that Q has $2q-2$ critical points, counted with multiplicity. The reader should be warned of a small change in terminology from our earlier usage [1, 2, 3] in which Gal^Q was taken to be the correspondence sending z to all $w \neq z$ satisfying $Q(w) = Q(z)$. Our new Gal^Q has graph the *closure* of that given by the old definition.

In general, by a *holomorphic correspondence* on $\hat{\mathbb{C}}$ we shall mean any multi-valued map $z \to w$ defined by a polynomial relation $p(z, w) = 0$. Examples are rational maps ($(n : 1)$ correspondences), their inverses ($(1 : n)$ correspondences),

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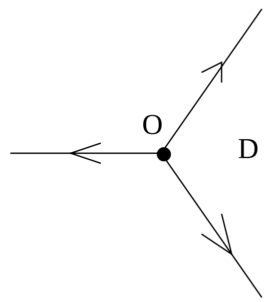

FIGURE 1.

and n-generator subgroups of $PSL(2,\mathbb{C})$ ($(n:n)$ correspondences). Covering correspondences of rational maps are a natural generalisation of *finite* subgroups of $PSL(2,\mathbb{C})$.

1.1. **Example.** $Q(z)=z^3$. This function has double critical points at 0 and ∞. Here

$$Cov^Q : z \to \{z, e^{2\pi i/3}z, e^{4\pi i/3}z\}$$

and

$$Gal^Q : z \to \{e^{2\pi i/3}z, e^{4\pi i/3}z\}.$$

Thus

$$Cov^Q = \{I, \rho, \rho^2\} = C_3,$$

the cyclic group of order 3 generated by $\rho : z \to e^{2\pi i/3}z$, and Gal^Q consists of the non-identity elements $\{\rho, \rho^2\}$. A fundamental domain D for the action of ρ on the Riemann sphere $\hat{\mathbb{C}}$ is illustrated in Figure 1. We write $Gal^Q(D)$ for the set of *all* images w of points $z \in D$ under Gal^Q. Different authors have slightly different notions of a 'fundamental domain' but note that whatever definition is chosen the interiors of D and $Gal^Q(D)$ are disjoint and their closures together cover $\hat{\mathbb{C}}$.

1.2. **Example.** $Q(z)=z^3-3z$. This has a double critical point at ∞ and simple critical points at ± 1. Now

$$Cov^Q : z \to w \quad \text{where} \quad w^3-3w = z^3-3z$$

and

$$Gal^Q : z \to w \quad \text{where} \quad w^2+wz+z^2 = 3.$$

There is no longer an action of the cyclic group of order 3, but there is an analogue of a *fundamental domain*, namely any *transversal* for $Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, that is to say any maximal subset of $\hat{\mathbb{C}}$ on which Q is injective. An example of such a transversal D_Q is illustrated in Figure 2. Here D_Q is the open right hand region together with the part of its boundary lying in the upper half-plane (the curve running from 1 to ∞, including both of these points).

Note that for any rational map Q and transversal D_Q the sets D_Q and $Gal^Q(D_Q)$ together cover $\hat{\mathbb{C}}$ and their intersection is the set of critical points of Q which lie in D_Q (necessarily on the boundary). In our example the critical point 1 lies in D_Q: its images under Gal^Q are itself and the corresponding co-critical point -2

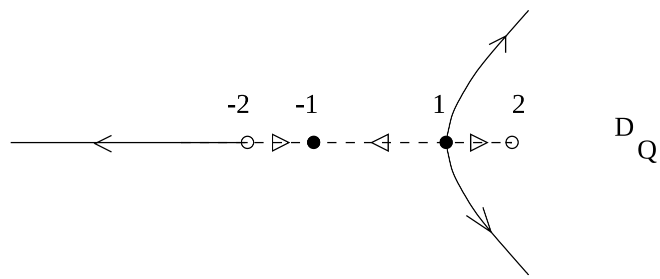

FIGURE 2.

(the distinct image of 1 under Gal^Q). The critical point ∞ also lies in D_Q, but has only one image under Gal^Q, namely itself. The critical point -1 lies outside D_Q: its images under Gal^Q are itself and the corresponding co-critical point 2 (which lies in the interior of D_Q). In order to divide $Gal^Q(D_Q)$ in this example into two 'tiles', each mapping bijectively onto D_Q but allowed to intersect at critical points, we must choose a curve along which to cut between the critical points -1 and 1: the part of the tile boundary from -1 to -2 is an image of this curve under Gal^Q.

Up to pre- and post-composition by Möbius transformations every degree 3 rational function with exactly two distinct critical points is equivalent to Example 1.1, and every degree 3 rational function with exactly three distinct critical points is equivalent to Example 1.2. Any other degree 3 rational function has four distinct critical points: its equivalence class is determined by their cross-ratio.

Lemma 1. *Let D_Q be a transversal of a rational map Q on $\hat{\mathbb{C}}$ and let D_Q° and $\overline{D_Q}$ denote the interior and closure of D_Q respectively. Then*

(i) *the restriction of Gal^Q to domain $Gal^Q(D_Q)$ and range D_Q is a single-valued (though not in general continuous) surjection;*

(ii) *$Gal^Q(D_Q^\circ)$ is open and $Gal^Q : Gal^Q(D_Q^\circ) \to D_Q^\circ$ is a complex analytic map of degree $q-1$ (where q is the degree of Q);*

(iii) *if $\overline{D_Q^\circ} = \overline{D_Q}$, then $Gal^Q(D_Q^\circ)$ is disjoint from $\overline{D_Q}$.*

Proof. Part (i) follows at once from the definition of a transversal. For part (ii) observe that D_Q° cannot contain any critical point of Q and thus every point $z \in D_Q^\circ$ has an open disc neighbourhood U such that $Gal^Q(U)$ is a disjoint union of open discs, together branch-covering U with degree $(k-1)$ under the complex analytic map $Gal^Q : Gal^Q(U) \to U$, with ramification points at critical points of Q. Thus $Gal^Q(D_Q^\circ)$ is open and $Gal^Q : Gal^Q(D_Q^\circ) \to D_Q^\circ$ is complex analytic. For (iii) observe that $Gal^Q(D_Q^\circ)$ is disjoint from D_Q° (since D_Q is a transversal), and as it is open it is disjoint from $\overline{D_Q^\circ} = \overline{D_Q}$. □

We remark that $Gal^Q : Gal^Q(D_Q) \to D_Q$ is strictly $(k-1)$-to-one except over co-critical points of Q in D_Q (where the notion of *co-critical point* is taken to include any *critical* points of Q of multiplicity > 1). In the case illustrated in Figure 2, $Gal^Q(D_Q) = (\hat{\mathbb{C}} - D_Q) \cup \{1\} \cup \{\infty\}$ and the single-valued map $Gal^Q(D_Q) \to D_Q$ defined by Gal^Q is two-to-one except over ∞ and 2. This map is continuous except on $(-\infty, -2) \subset \mathbb{R}$. The set $Gal^Q(D_Q^\circ)$ is the region $\hat{\mathbb{C}} - \overline{D_Q}$ with $[-\infty, -2]$ removed, and restricted to $Gal^Q(D_Q^\circ)$ the map defined by Gal^Q is complex analytic.

2. Klein's combination theorem for groups

The basic situation for Klein's combination theorem [7] is that of a pair of finitely generated discrete subgroups H and K of $PSL(2,\mathbb{C})$ having fundamental domains D_H and D_K with interiors together covering the Riemann sphere $\hat{\mathbb{C}}$. For our generalisation to correspondences we shall only need to consider *finite* subgroups H and K of $PSL(2,\mathbb{C})$, which of course are necessarily discrete. We start by defining our terms.

Definitions. Let G be any finitely generated Kleinian group, that is to say any finitely generated discrete subgroup of $PSL(2,\mathbb{C})$.

• The *ordinary set* (or *discontinuity set*) $\Omega(G)$ of G is the set of all points $z \in \hat{\mathbb{C}}$ having a neighbourhood U such that $U \cap g(U) \neq \emptyset$ for only finitely many $g \in G$.

• The *free ordinary set* $\Omega'(G)$ is the subset of $\Omega(G)$ of points z having a neighbourhood U such that $U \cap g(U) = \emptyset$ for all $g \in G - \{I\}$. (The free ordinary set is obtained from the ordinary set by puncturing it at the discrete set of points which have non-trivial stabiliser.)

• The *limit set* $\Lambda(G)$ is the set of all points z such that there exists $z_0 \in \hat{\mathbb{C}}$ and distinct $g_n \in G$ with $\lim_{n\to\infty} g_n(z_0) = z$. (The limit set is the complement in $\hat{\mathbb{C}}$ of the ordinary set.)

• A *transversal* T for the action of G on $\Omega = \Omega(G)$ is a maximal subset of Ω on which the orbit quotient map $\Omega \to \Omega/G$ is injective.

• A *fundamental set* S for the action of G on $\Omega' = \Omega'(G)$ is a subset S of Ω' such that $g(S) \cap h(S) = \emptyset$ when g and h are any two distinct elements of G, and $\bigcup_{g\in G} g(S) = \Omega'$. (Equivalently S is a transversal for the action of G on Ω'.)

In Example 1.1 above the ordinary set Ω for the action of C_3 is the whole of $\hat{\mathbb{C}}$ and the free ordinary set Ω' is $\hat{\mathbb{C}} - \{0,\infty\}$. For the set D illustrated in Figure 1 to be a transversal for the action of G on Ω it must contain both 0 and ∞, and exactly one out of each pair of points identified under C_3 on the remaining part of the boundary. To make it into a fundamental set D' for the action of C_3 on Ω' we excise 0 and ∞.

Theorem 1. *Let H and K be non-trivial finite subgroups of $PSL(2,\mathbb{C})$, let D_H and D_K be transversals for the actions of H and K respectively on $\hat{\mathbb{C}}$, having interiors D_H°, D_K° with $\overline{D_H^\circ} = \overline{D_H}$ and $\overline{D_K^\circ} = \overline{D_K}$, and suppose that $D_H^\circ \cup D_K^\circ = \hat{\mathbb{C}}$. Let D'_H and D'_K denote the fundamental sets for the actions of H and K on $\Omega'(H)$ and $\Omega'(K)$ obtained by removing from D_H and D_K those points which have non-trivial stabilisers under the respective groups. Let Δ denote $D_H \cap D_K$ and Δ' denote $D'_H \cap D'_K$. Then:*

• *The free product $G = H * K$ acts freely on $\Omega(G,\Delta') = \bigcup_{g\in G} g(\Delta')$, and Δ' is a fundamental set for this action.*

• *$\Omega(G,\Delta') = \Omega'(G)$, the free ordinary set for the action of G on $\hat{\mathbb{C}}$.*

• *G acts properly discontinuously on $\Omega(G,\Delta) = \bigcup_{g\in G} g(\Delta)$, and Δ is a transversal for this action.*

• *$\Omega(G,\Delta) = \Omega(G)$, the ordinary set for the action of G on $\hat{\mathbb{C}}$.*

• *$\Lambda(G)(= \hat{\mathbb{C}} - \Omega(G,\Delta))$ is a Cantor set except in the special case that H and K are both cyclic of order 2, in which case $\Lambda(G)$ consists of exactly two points.*

This is essentially the classical theorem of Klein [7], in the special case of finite groups. Proofs of one form or another of this theorem can be found in standard

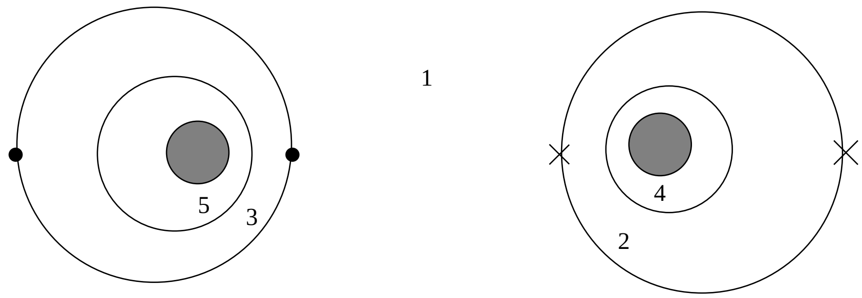

FIGURE 3.

texts on Kleinian groups. The exact wording above is motivated by the analogous statement for covering correspondences of rational maps which we shall prove in the next section, and which contains Theorem 1 as a particular family of cases. More general combination theorems for Kleinian groups are proved in [8, 9, 10, 11]. Before we move on to correspondences we pause to consider some elementary examples illustrating Theorem 1.

2.1. **Example.** $H \cong C_2;\ K \cong C_2$.

In Figure 3, $H = \{I, \sigma\}$ and $K = \{I, \rho\}$, where σ is the involution which fixes the points marked by crosses, and ρ is the involution which fixes the points marked by solid circles. The region exterior to the largest circle on the right is D_H, a fundamental domain for H, and the region exterior to the largest circle on the left is D_K, a fundamental domain for K. The intersection $\Delta = D_H \cap D_K$ is labelled '1'. The other regions represented in the figure are $\sigma(\Delta)$ (labelled '2'), $\rho(\Delta)$ (labelled '3'), $\sigma\rho(\Delta)$ (labelled '4'), and $\rho\sigma(\Delta)$ (labelled '5'). Only the first few 'layers' are shown. The central (grey) discs, $\sigma\rho\sigma(D_H)$ on the right hand side and $\rho\sigma\rho(D_K)$ on the left hand side, are in reality further subdivided into infinitely many annuli ($\sigma\rho\sigma(\Delta)$, $\rho\sigma\rho(\Delta)$, and so on).

2.2. **Example.** $H \cong C_2;\ K \cong C_3$.

In Figure 4, $H = \{I, \sigma\}$ and $K = \{I, \rho, \rho^2\}$, where σ is the involution which interchanges the right- and left-hand half-planes, fixing the origin and infinity, and ρ is the rotation of order three which fixes the points marked with a solid circle. In the figure D_H is the left-hand half-plane (bounded by the imaginary axis, marked L) and D_K is the region exterior to the curve marked M. Their intersection Δ is labelled '1', $\sigma(\Delta)$ is labelled '2'. The other regions represented in the figure are $\rho(\Delta)$ (labelled '3a'), $\rho^2(\Delta)$ (labelled '3b'), $\sigma\rho(\Delta)$ (labelled '4a'), $\sigma\rho^2(\Delta)$ (labelled '4b'), $\rho\sigma(\Delta)$ (labelled '5a'), and $\rho^2\sigma(\Delta)$ (labelled '5b'). Once again only the first few 'layers' are shown. The central (grey) discs $\sigma\rho\sigma(D_H)$ and $\sigma\rho^2\sigma(D_H)$ on the right hand side, and the central (grey) regions $\rho\sigma\rho(D_K) \cup \rho\sigma\rho^2(D_K)$ and $\rho^2\sigma\rho(D_K) \cup \rho^2\sigma\rho^2(D_K)$ on the left hand side, are of course further subdivided into infinitely many further regions.

3. A COMBINATION THEOREM FOR COVERING CORRESPONDENCES

In this section we present a generalisation of Theorem 1 in which the roles of the finite groups H and K of orders $|H|$ and $|K|$ are taken by the *covering correspondences* of rational maps P and Q of degrees p and q, and the roles of fundamental

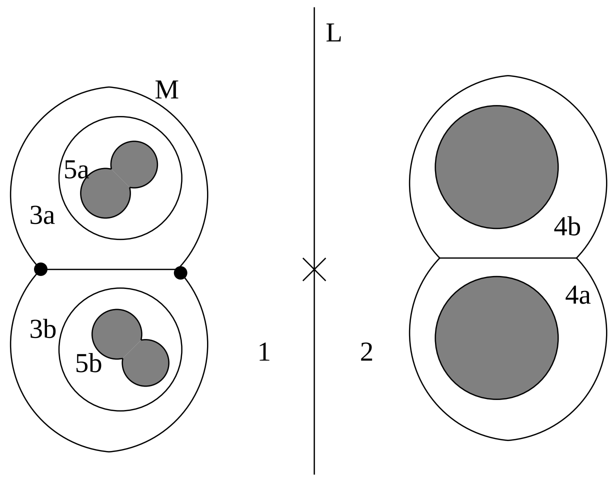

FIGURE 4.

domains for H and K are taken by *transversals* D_P and D_Q for P and Q. The correspondences Cov^P, Cov^Q, Gal^P and Gal^Q are multi-valued maps, a generic point having $p, q, p-1$ and $q-1$ images respectively. Each of these individual correspondences has all its grand orbits finite, but when we combine them by composition we obtain correspondences with (generically) infinite grand orbits. We shall be concerned with the dynamics of such combinations. Note that for us the dynamics (under composition) of the rational maps P and Q themselves is irrelevant.

We first establish some terminology. Let $f : z \to w$ be a holomorphic correspondence on the Riemann sphere, defined by a polynomial relation $p(z, w) = 0$.

Definitions.

• The *regular set* $\Omega(f)$ of f is the set of all points $z \in \hat{\mathbb{C}}$ which have a neighbourhood U such that U has only finitely many distinct returns under (mixed) iteration of f and f^{-1}. Formally, we require [3] that there exist a positive integer N such that

$$\bigcup_{|e|<\infty} f^e \cap (U \times U) \subset \bigcup_{|e|<N} f^e$$

(as graphs) where e runs through all finite sequences $e_1, \ldots, e_n$ with each $e_j = \pm 1$, and where the length n of e is denoted $|e|$.

• The *free regular set* $\Omega'(f)$ is the subset of $\Omega(f)$ consisting of points z with the property that every point on the grand orbit of z has a neighbourhood U such that the only branch of (mixed) iteration of f and f^{-1} which returns some point of U to U is the identity, in other words

$$\bigcup_{|e|<\infty} f^e \cap (U \times U) = I \cap (U \times U).$$

The free regular set is obtained from the regular set by puncturing it at the discrete set of points which have grand orbits containing singular points - that is to say points which have image under f or f^{-1} smaller in cardinality than the corresponding image of some neighbour.

• The *global limit set* $\Lambda(f)$ is defined to be the complement in $\hat{\mathbb{C}}$ of $\Omega(f)$.

• A *transversal* T for the action of f on $\Omega = \Omega(f)$ is a maximal subset of Ω on which the grand orbit quotient map $\Omega \to \Omega/f$ is injective.

• A *fundamental set* S for the action of f on $\Omega' = \Omega'(f)$ is a transversal for the action of f on Ω'.

In Example 1.2 the regular set Ω is the whole of $\hat{\mathbb{C}}$ and the free regular set Ω' is $\hat{\mathbb{C}} - \{\pm 1, \pm 2, \infty\}$. The set D_Q described in this example is a transversal for the action of Cov^Q on Ω. To make D_Q into a fundamental set D'_Q for the action of Cov^Q on Ω' we must remove the points $\infty, +1$ and $+2$.

• If P and Q are rational maps we denote by $Cov^P \circ Cov^Q$ the composite of Cov^Q followed by Cov^P (generically a $(pq : pq)$ correspondence), and we denote by $Cov^P * Cov^Q$ the equivalence relation on $\hat{\mathbb{C}}$ generated by Cov^P and Cov^Q.

• We call a sequence of points $z_0, \dots, z_n$ in U a *reduced orbit* linking z_0 to z_n if alternate pairs (z_i, z_{i+1}) lie in Cov^P and Cov^Q and no $z_{i+1} = z_i$.

Note that since $Cov^P \circ Cov^P = Cov^P$ and $Cov^Q \circ Cov^Q = Cov^Q$ a pair of points $(z, w) \in \hat{\mathbb{C}} \times \hat{\mathbb{C}}$ lies in $Cov^P * Cov^Q$ if and only if there exists a reduced orbit linking them. Observe also that the inverse of a reduced orbit is a reduced orbit, and that we can compose reduced orbits by concatenation followed by reduction.

• We say that $Cov^P * Cov^Q$ *acts freely* on a subset $U \subset \hat{\mathbb{C}}$ if U is invariant under both Cov^P and Cov^Q, each $z \in U$ has p distinct images under Cov^P and q distinct images under Cov^Q (where p and q are the degrees of P and Q), and the only reduced orbit linking any point $z \in U$ to itself is the trivial one.

Theorem 2. *Let P and Q be rational maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$, let D_P, D_Q be transversals for P and Q respectively having interiors D_P°, D_Q° with $\overline{D_P^\circ} = \overline{D_P}$ and $\overline{D_Q^\circ} = \overline{D_Q}$, and suppose that $D_P^\circ \cup D_Q^\circ = \hat{\mathbb{C}}$. Let D'_P and D'_Q denote the sets obtained by removing from D_P and D_Q all points on grand orbits under $Cov^P * Cov^Q$ of critical points of P and Q. Let Δ denote $D_P \cap D_Q$ and Δ' denote $D'_P \cap D'_Q$. Then:*

• *$\mathcal{F} = Cov^P * Cov^Q$ acts freely on $\Omega(\mathcal{F}, \Delta')$, the union of grand orbits of points of Δ' under $\mathcal{F}$, and Δ' is a fundamental set for this action.*

• *$\Omega(\mathcal{F}, \Delta') \subset \Omega'(\mathcal{F})$, the free regular set for the action of $\mathcal{F}$ on $\hat{\mathbb{C}}$.*

• *Δ is a transversal for the action of $\mathcal{F}$ on the union $\Omega(\mathcal{F}, \Delta)$ of grand orbits of points of Δ under $\mathcal{F}$.*

• *$\Omega(\mathcal{F}, \Delta) \subset \Omega(\mathcal{F})$, the regular set of $\mathcal{F}$.*

• *$\hat{\mathbb{C}} - \Omega(\mathcal{F}, \Delta)$ is the disjoint union of $\Lambda_+(\mathcal{G}) = \bigcap_0^\infty \mathcal{G}^n(D_Q)$ and $\Lambda_-(\mathcal{G}) = \bigcap_0^\infty \mathcal{G}^{-n}(D_P)$, where $\mathcal{G} = Gal^P \circ Gal^Q$ and $\mathcal{G}^n(D_Q)$ denotes the set of all images of points of D_Q under the composition of n iterates of the multi-valued map $\mathcal{G}$.*

• *if $\mathcal{G}(D_Q^\circ)$ and $\mathcal{G}^{-1}(D_P^\circ)$ are topological discs, then Λ_+ and Λ_- are homeomorphic to the filled Julia sets $K(R_+)$ and $K(R_-)$ of polynomial maps R_+ and R_-, each of degree $(p-1)(q-1)$. The restrictions of $\mathcal{G}^{-1}$ to domain and range Λ_+, and of $\mathcal{G}$ to domain and range Λ_-, are both single-valued maps and are conjugate to the maps R_+ on $K(R_+)$ and R_- on $K(R_-)$ respectively, conformally on interiors.*

Before proving the theorem we illustrate in Figure 5 some similarities to (and differences from) the group case. This figure is not intended to be a precise plot for an actual pair of rational functions P and Q but rather to represent the various regions schematically. In the figure, P is a rational function of degree 4, having a triple critical point c_0, a simple critical point c_1 and a double critical point c_2 (all marked by crosses). The region D_P exterior to the closed curve L is a transversal

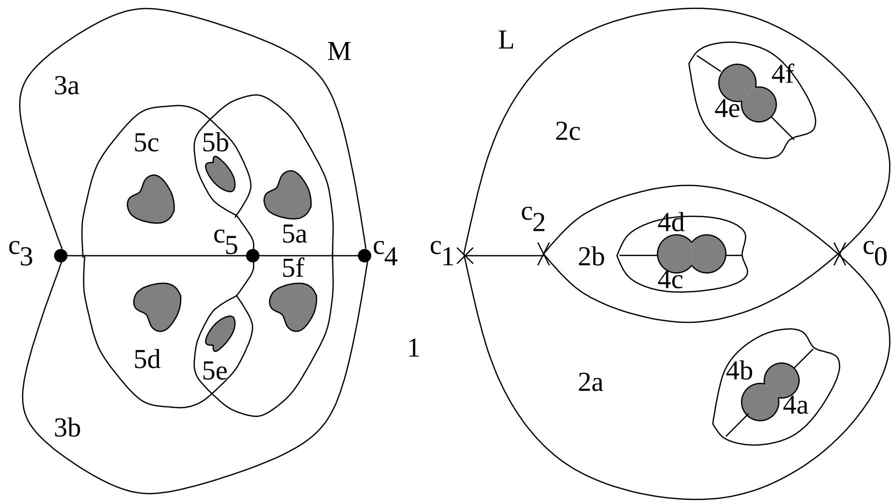

FIGURE 5.

for P. The function Q is of degree 3, having one double critical point c_3 and two simple critical points c_4 and c_5 (all marked by solid circles). The region D_Q exterior to the closed curve M is a transversal for Q. The intersection $\Delta = D_P \cap D_Q$ is labelled '1'. The set $Gal^P(\Delta)$ is made up of three 'tiles' (copies of Δ) labelled '2a', '2b', and '2c' in the figure; $Gal^Q(\Delta)$ is made up of the two 'tiles' labelled '3a' and '3b'; $Gal^P \circ Gal^Q(\Delta)$ is made up of the six tiles labelled '4a' to '4f'; and $Gal^Q \circ Gal^P(\Delta)$ is made up of the six tiles labelled '5a' to '5f'. Only the first few 'layers' are shown. The six inner (grey) regions on the left (which together make up $Gal^Q \circ Gal^P \circ Gal^Q(D_Q)$) are in reality further subdivided, as are the six inner (grey) regions on the right (which together make up $Gal^P \circ Gal^Q \circ Gal^P(D_P)$). The tiling is closely analogous to tilings we have seen in the group case, for example that in Figure 4. However there are differences: for example in Figure 5 the set $Gal^Q \circ Gal^P(D_P)$ (which consists of the tiles numbered '5' together with the grey regions they surround) is a topological disc, double-covering the disc $Gal^P(D_P)$ (the region interior to L) with ramification point c_5, whereas in Figure 4 the set $Gal^Q \circ Gal^P(D_P)$ has two components, each single-covering $Gal^P(D_P)$. If for sufficiently large n all critical points are outside $(Gal^Q \circ Gal^P)^n(D_P) \cup (Gal^P \circ Gal^Q)^n(D_Q)$ (as they are in Figure 5) the limit set will be a Cantor set, but this will not always be the case, as we shall see in Section 4.

We now proceed to the proof of Theorem 2.

Proof. Since D_P is a transversal to P, writing D_P^c for $\hat{\mathbb{C}} - D_P$ we have

$$Gal^P(D_P) = D_P^c \cup (Gal^P(D_P) \cap D_P).$$

But $D_Q = D_P^c \cup \Delta$ and $Gal^P(D_P) \cap D_P \subset \partial D_P \subset D_Q$ so

$$D_Q = Gal^P(D_P) \cup \Delta.$$

By inserting into this expression the analogous equality for D_P we find that

$$D_Q = (Gal^P \circ Gal^Q)(D_Q) \cup Gal^P(\Delta) \cup \Delta.$$

Similarly

$$D_P = (Gal^Q \circ Gal^P)(D_P) \cup Gal^Q(\Delta) \cup \Delta.$$

Writing $\mathcal{G}$ for the composition $Gal^P \circ Gal^Q$, so $\mathcal{G}^{-1} = Gal^Q \circ Gal^P$, we set

$$\mathcal{A}_{2n} = \mathcal{G}^n(\Delta), \quad \mathcal{A}_{2n+1} = \mathcal{G}^n \circ Gal^P(\Delta),$$

$$\mathcal{B}_{2n} = \mathcal{G}^{-n}(\Delta), \quad \mathcal{B}_{2n+1} = \mathcal{G}^{-n} \circ Gal^Q(\Delta)$$

(so $\mathcal{A}_0 = \mathcal{B}_0 = \Delta$). The expressions above for D_Q and D_P generalise inductively to

$$D_Q = \mathcal{G}^n(D_Q) \cup \bigcup_{m=0}^{2n-1} \mathcal{A}_m, \quad D_P = \mathcal{G}^{-n}(D_P) \cup \bigcup_{m=0}^{2n-1} \mathcal{B}_m$$

and (by definition) the union of grand orbits of points of Δ is

$$\Omega(\mathcal{F}, \Delta) = \Delta \cup \bigcup_{n \geq 1} \mathcal{A}_n \cup \bigcup_{n \geq 1} \mathcal{B}_n.$$

We remark that the restricted maps $Gal^P : \mathcal{A}_{n+1} \to \mathcal{B}_n$ and $Gal^Q : \mathcal{B}_{n+1} \to \mathcal{A}_n$ are single-valued surjections for all $n \geq 0$, by Lemma 1(i), since $\mathcal{B}_n \subset D_P$ and $\mathcal{A}_n \subset D_Q$.

Clearly no $\overline{\mathcal{A}_m}$ with $m > 0$ can meet any $\overline{\mathcal{B}_n}$ with $n > 0$ as the complements D_P^c of D_P and D_Q^c of D_Q have disjoint closures. We shall need to ascertain how far there can be non-empty intersections amongst the $\overline{\mathcal{A}_n}$ or the $\overline{\mathcal{B}_n}$. Observe that

$$Gal^P(\overline{D_Q^c}) \cap \overline{D_P} = \emptyset$$

by Lemma 1(iii), since $\overline{D_Q^c} \subset D_P^\circ$. From this we deduce firstly that

$$Gal^P(\overline{Gal^Q(D_Q)}) \subset (D_P^c)^\circ$$

and so

$$\overline{\mathcal{G}(D_Q)} \subset \overline{D_P^c} \subset D_Q^\circ$$

which we shall need later, when we apply the 'Straightening Theorem' of Douady and Hubbard. We deduce secondly that for any $n \geq 2$ the set $\overline{\mathcal{A}_0} = \overline{\Delta}$ is disjoint from $\overline{\mathcal{A}_n}$, since the former is contained in $\overline{D_P}$ and the latter in $Gal^P(\overline{D_Q^c})$ (note that $Gal^P(\overline{D_Q^c})$ is closed, being the inverse image of $\overline{D_Q^c}$ under a continuous map, since $\overline{D_Q^c} \subset D_P^\circ$). Similarly $\overline{\mathcal{B}_0} \cap \overline{\mathcal{B}_n} = \emptyset$ for any $n \geq 2$. Next, since $\overline{D_P^c} \subset D_Q^\circ$, we have $Gal^Q(\overline{D_P^c}) \subset Gal^Q(D_Q^\circ)$ and thus, for all $n \geq 2$, $\overline{\mathcal{B}_n} \subset Gal^Q(D_Q^\circ)$ and so

$$\overline{\mathcal{B}_n} \cap \overline{\mathcal{B}_1} = \overline{\mathcal{B}_n} \cap \overline{\mathcal{B}_1} \cap Gal^Q(D_Q^\circ).$$

But for any $n \geq 3$ this set is empty since there is a map (a single-valued restriction of Gal^Q) from it into $\overline{\mathcal{A}_{n-1}} \cap \overline{\mathcal{A}_0}$, which we have just shown to be empty. Similarly $\overline{\mathcal{A}_n} \cap \overline{\mathcal{A}_1} = \emptyset$ for any $n \geq 3$. It now follows by induction that for any m, n with $m - n \geq 2$ the intersections $\overline{\mathcal{A}_m} \cap \overline{\mathcal{A}_n}$ and $\overline{\mathcal{B}_m} \cap \overline{\mathcal{B}_n}$ are empty.

Our next objective is to show that each point $z \in \Omega(\mathcal{F}, \Delta)$ is linked to a *unique* point of Δ by a *unique* reduced orbit. By definition z is mapped to a point of Δ by some sequence of iterates of Gal^P and Gal^Q, and hence linked to that point by a reduced orbit. Our task is to prove uniqueness. Suppose firstly that z lies in just one $\mathcal{A}_m$ or $\mathcal{B}_m$, say $z \in \mathcal{A}_{2n} = \mathcal{G}^n(\Delta)$ for definiteness. Then any two reduced orbits from points $z_0, z_0' \in \Delta$ to z must have successive points in $\Delta = \mathcal{A}_0, \mathcal{B}_1, \mathcal{A}_2, \mathcal{B}_3, \ldots, \mathcal{B}_{2n-1}, \mathcal{A}_{2n}$. But consider the reverse orbits: since the maps from $\mathcal{A}_m$ to $\mathcal{B}_{m-1}$ and $\mathcal{B}_m$ to $\mathcal{A}_{m-1}$ defined by Gal^P and Gal^Q are single-valued, the two reverse orbits are identical. Hence z is indeed linked to a *unique* point $z_0 \in \Delta$, by a *unique* reduced orbit. A similar argument applies to any z which is in just one $\mathcal{A}_{2n+1}, \mathcal{B}_{2n}$ or $\mathcal{B}_{2n+1}$. Next we turn our attention to the only other possibility for z, that it lies in two successive $\mathcal{A}_n$ or two successive $\mathcal{B}_n$. First observe that $\mathcal{A}_0 = \Delta$ can only meet $\mathcal{A}_1 = Gal^P(\Delta)$ at a fixed point of Gal^P in D_P (necessarily on the boundary). The map Gal^Q restricted to $\mathcal{B}_1 \cup \mathcal{B}_2 \to \mathcal{A}_0 \cup \mathcal{A}_1$ sends any point in $\mathcal{B}_1 \cap \mathcal{B}_2$ to a point in $\mathcal{A}_0 \cap \mathcal{A}_1$, that is to say a fixed point of Gal^P. Thus the maps $Gal^Q : \mathcal{B}_1 \to \Delta$ and $Gal^P \circ Gal^Q : \mathcal{B}_2 \to \Delta$ agree on $\mathcal{B}_1 \cap \mathcal{B}_2$. Similarly $Gal^Q \circ Gal^P : \mathcal{A}_2 \cup \mathcal{A}_3 \to \mathcal{A}_0 \cup \mathcal{A}_1$ sends any point of $\mathcal{A}_2 \cap \mathcal{A}_3$ to a fixed point of Gal^P - in other words the maps $Gal^Q \circ Gal^P : \mathcal{A}_2 \to \Delta$ and $Gal^P \circ Gal^Q \circ Gal^P : \mathcal{A}_3 \to \Delta$ agree on $\mathcal{A}_2 \cap \mathcal{A}_3$. A similar argument applies to any point in the intersection of any successive pair of $\mathcal{A}_n$'s or $\mathcal{B}_n$'s. Thus each point of $\Omega(\mathcal{F}, \Delta)$ is indeed on the grand orbit under $\mathcal{F}$ of a unique point of Δ and is linked to that point by a unique reduced orbit. Moreover it also follows that between any two points z and w on the grand orbit of $z_0 \in \Delta$ there is a unique reduced orbit, for if there were two such orbits, concatenation with the reduced orbit from z_0 to z (followed if necessary by reduction) would give two different reduced orbits from z_0 to w.

Now consider the set $\Omega(\mathcal{F}, \Delta')$, where all grand orbits of critical points have been removed. Each $z \in \Omega(\mathcal{F}, \Delta')$ has exactly p images under Cov^P and q images under Cov^Q, and together with our observation concerning unique reduced orbits this completes the proof that $Cov^P * Cov^Q$ acts freely on $\Omega(\mathcal{F}, \Delta')$. As each grand orbit contains exactly one point of Δ', the latter is a fundamental set for the action. Moreover since every point z of Δ' is interior to either $\Delta' \cup Gal^P(\Delta')$ or to $\Delta' \cup Gal^Q(\Delta')$, and the projections from $\Omega(\mathcal{F}, \Delta')$ to these sets are regular covers (being complex analytic, and having no critical points), $\Omega(\mathcal{F}, \Delta')$ is contained in $\Omega'(\mathcal{F})$, the free regular set for the action of $\mathcal{F}$ on $\hat{\mathbb{C}}$.

Returning to the general situation where critical orbits are present, and using our single-valued projections from $\Omega(\mathcal{F}, \Delta)$ onto Δ, we can subdivide each $\mathcal{A}_n$ and $\mathcal{B}_n$ into finitely many 'tiles' (copies of Δ, possibly intersecting at boundaries). See Figure 5 for a schematic example. Note that we may have to make choices of cuts between critical points in order to define some parts of the boundaries (as in Example 1.2, Figure 2). Since each $\overline{\mathcal{A}_n}$ and $\overline{\mathcal{B}_n}$ can only meet two others, the tiling is locally finite and the grand orbits of critical points are discrete. Hence for any $z \in \Omega(\mathcal{F}, \Delta)$ on a critical grand orbit there exists a neighbourhood U of z such that any branch of $\mathcal{F}$ which returns some point of U to U fixes z, and there are only finitely many such branches. For points z not on critical grand orbits we already know that there is only one such branch, the identity. Thus $\Omega(\mathcal{F}, \Delta) \subset \Omega(\mathcal{F})$, the regular set of $\mathcal{F}$. We have already established that each point of $\Omega(\mathcal{F}, \Delta)$ has a unique point of Δ in its grand orbit, that is to say Δ is a transversal to the action.

It remains to prove the statements concerning the structure of Λ_+ and Λ_-. Observe firstly that

$$\mathcal{G}^{-1} : \mathcal{G}(D_Q^\circ) \to D_Q^\circ$$

is a (single-valued) complex analytic map of degree $(p-1)(q-1)$, being the composite of restrictions of Gal^P and Gal^Q which are (single-valued) complex analytic by Lemma 1(ii). Moreover this restriction of $\mathcal{G}^{-1}$ has the property that the closure of its domain is contained in the interior of its image, from our observation earlier in the proof that

$$\overline{\mathcal{G}(D_Q)} \subset D_Q^\circ.$$

Thus if $\mathcal{G}(D_Q^\circ)$ is a topological disc, then our restriction of $\mathcal{G}^{-1}$ is a *polynomial-like mapping* in the sense of Douady and Hubbard [6] and we may deduce from their *Straightening Theorem* that on a neighbourhood of Λ_- the map $\mathcal{G}^{-1}$ is *hybrid equivalent* to a polynomial map R_- of degree $(p-1)(q-1)$. In particular Λ_- is homeomorphic to the filled Julia set $K(R_-)$ of R_- and $\mathcal{G}^{-1}$ restricted to Λ_- is conjugate to R_- restricted to $K(R_-)$, conformally on the interior. The same reasoning proves the analogous results for Λ_+. □

Comments on Theorem 2.

1. Although there is a unique *reduced orbit* linking any two points z and w on any grand orbit in $\Omega(\mathcal{F}, \Delta)$, it is only if the grand orbit is non-critical that this implies that the only branch of $\mathcal{F}$ fixing z is the identity. Any critical point c of P is fixed by a branch of Gal^P as well as by the identity, and any point z on the grand orbit of c is fixed by first applying a sequence of iterations which carries z to c, then applying a branch of Gal^P which fixes c, and finally applying the reverse sequence of iterations to carry c back to z. The *finite* set of branches of $\mathcal{F}$ fixing a point z on a critical grand orbit, described in the proof above, consists of concatenations of such sequences.
2. The orbit space $\Omega(\mathcal{F}, \Delta)/\mathcal{F}$ is an orbifold, with a finite set of cone points (corresponding to grand orbits of $\mathcal{F}$ containing singular points). Deleting the cone points one obtains the punctured Riemann surface $\Omega(\mathcal{F}, \Delta')/\mathcal{F}$ ([3]).
3. In contrast to the group case the inclusions $\Omega(\mathcal{F}, \Delta') \subset \Omega'(\mathcal{F})$ and $\Omega(\mathcal{F}, \Delta) \subset \Omega(\mathcal{F})$ need not be equalities. Equivalently the inclusion $\Lambda(\mathcal{F}) \subset \Lambda_+(\mathcal{G}) \cup \Lambda_-(\mathcal{G})$ need not be an equality (see the examples and comment in section 4.1).
4. If $p = 2$ and $Gal^Q(D_Q^\circ)$ is a disc, then $\mathcal{G}(D_Q^\circ)$ is a disc, since Gal^P is a bijection.
5. If $p = 2$ the involution Gal^P conjugates $\mathcal{G}$ to $\mathcal{G}^{-1}$. In particular Gal^P maps Λ_- homeomorphically onto Λ_+ and conjugates the action of $\mathcal{G}$ on Λ_- to that of $\mathcal{G}^{-1}$ on Λ_+. For p and q both greater than 2, the best we can say is that Gal^P and Gal^Q define holomorphic maps $f_0 : \Lambda_- \to \Lambda_+$ and $f_1 : \Lambda_+ \to \Lambda_-$ of degrees $p-1$ and $q-1$ respectively, and that on suitable neighbourhoods of Λ_- and Λ_+ the composition $f_1 \circ f_0$ is hybrid equivalent to a polynomial map R_- and $f_0 \circ f_1$ is hybrid equivalent to a polynomial map R_+. It is not clear whether R_- and R_+ themselves need be compositions in general, nor whether Λ_+ need be homeomorphic to Λ_-.
6. If $\mathcal{G}^{-1}(D_P^\circ)$ is a disjoint union of discs (as it is in the group case), then $\mathcal{G}$ restricted to it is a *generalised polynomial mapping* in a suitable sense and one can draw appropriate conclusions about the structure of Λ_-. For example if $\mathcal{G}^{-1}(D_P^\circ)$ consists of $(p-1)(q-1)$ disjoint discs (as it does in the group case if D_P° is a disc) one can deduce that Λ_- is necessarily a Cantor set.

4. Matings of Hecke groups with polynomial maps

4.1. **Mating the modular group $C_2 * C_3$ with quadratic maps.** Matings of quadratic maps with the modular group were first constructed (modulo a technical difficulty) in [1]. These are quadratic correspondences which on a topological disc contained in the Riemann sphere are conjugate to the modular group $PSL(2, \mathbb{Z}) (\cong C_2 * C_3)$, acting on complex upper half-space by $z \to z+1, z \to z/(z+1)$, and on the complement of this disc have branches conjugate to $q_c : z \to z^2 + c$

and its inverse, acting on the filled Julia set $K(q_c)$ of q_c. Below we repeat the construction (modulo the same technical difficulty), making use of Theorem 2 to simplify the description, and we show how representations of $C_2 * C_3$ with limit sets which are Cantor sets may also be mated with quadratic maps, now without any technical problem.

Let P be a rational map of degree two, and Q be a rational map of degree three with one double critical point and two simple ones. Without loss of generality we may take:

$$P(z) = z^2,$$

$$Q(z) = \left(\frac{az+1}{z+1}\right)^3 - 3k\left(\frac{az+1}{z+1}\right).$$

Then P has critical points at 0 and ∞, and $Gal^P(z) = -z$. The map Q has a double critical point at $c_0 = -1$ and single critical points at c_1, c_2, where

$$c_1 = \frac{\sqrt{k}-1}{a-\sqrt{k}} \text{ and } c_2 = \frac{-\sqrt{k}-1}{a+\sqrt{k}}.$$

The correspondence Gal^Q is given by

$$z \to w \text{ where } \left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw+1}{w+1}\right) + \left(\frac{aw+1}{w+1}\right)^2 = 3k$$

and the composite $\mathcal{G} = Gal^P \circ Gal^Q$ is given by

$$z \to w \text{ where } \left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3k,$$

which is also the polynomial relation considered in [1]. For an appropriate set of values of the parameters a and k we can find transversals D_P and D_Q for P and Q satisfying the hypotheses of Theorem 2. An example is illustrated in Figure 6. Here a and $\sqrt{k}$ are real, and D_Q is as drawn in Figure 2 (but with the coordinate changed by a Möbius transformation, so that the point ∞ of Figure 2 has become the point $c_0 = -1$ of Figure 6). In this figure D_P is the left-hand half-plane, bounded by the imaginary axis L and D_Q is the region exterior to the heart-shaped curve M. The critical point 0 of P is marked with a cross (the other critical point of P is ∞); the critical points of Q are marked with solid circles.

By Theorem 2 the Riemann sphere is now partitioned into two completely invariant sets, $\Omega(\mathcal{F}, \Delta)$ and $\Lambda(\mathcal{G}) = \Lambda_+(\mathcal{G}) \sqcup \Lambda_-(\mathcal{G})$. If the values of a and k are such that the critical point c_2 is inside $\mathcal{G}^{-n}(D_P)$ for all $n \geq 0$, then $\Lambda_-(\mathcal{G})$ is connected and so is $\Lambda_+(\mathcal{G}) = Gal^P(\Lambda_-(\mathcal{G}))$, whence $\Omega(\mathcal{F}, \Delta)$ is a topological annulus. The situation is illustrated by Figure 7. In this computer plot the red lines are the images of $L(= \partial D_P)$ (the imaginary axis) and the blue lines are the images of $M(= \partial D_Q)$. Both Λ_+ and Λ_- (plotted in black) are homeomorphic to connected filled Julia sets of quadratic maps.

There are now two ways to obtain an action of $C_2 * C_3$: either we perturb the parameter k until it takes the value 1, in which case there is a conformal homeomorphism from $\Omega(\mathcal{F}, \Delta)$ to the upper half-plane $\mathcal{H}$, conjugating the action of the correspondence $\mathcal{F}$ on Ω to that of $PSL(2, \mathbb{Z})(\cong C_2 * C_3)$ on $\mathcal{H}$ (Method 1 below), or we keep the value of k unchanged and identify an action of $C_2 * C_3$ on a suitable modification of $\Omega(\mathcal{F}, \Delta)$, obtained by making cuts (Method 2 below).

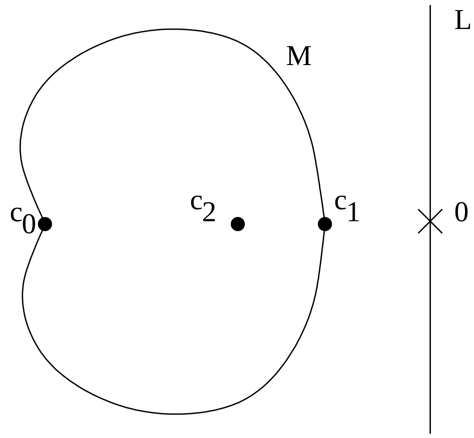

FIGURE 6.

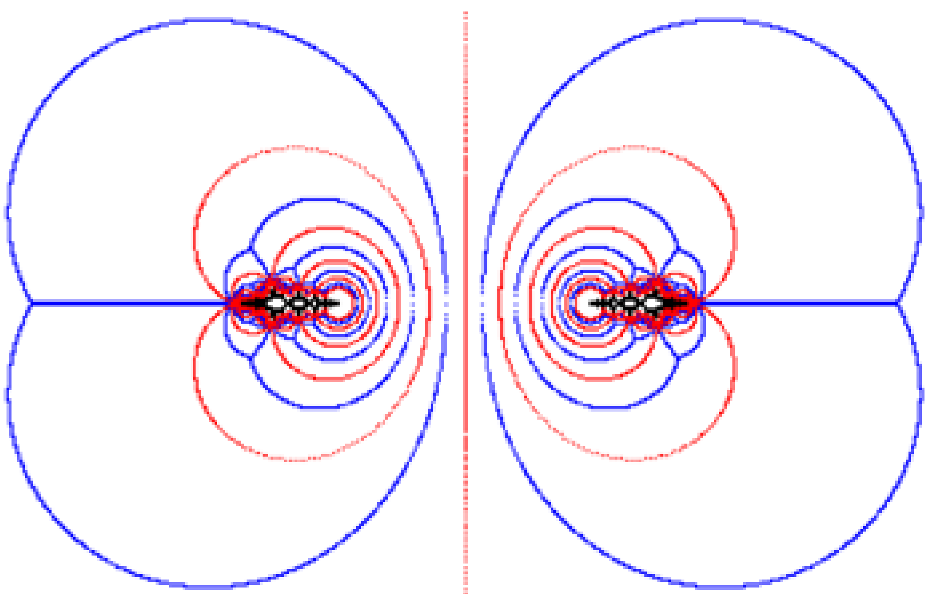

FIGURE 7. $a = 3.42; k = 0.8$

Method 1 ([1]). Adjust the parameter k to the value 1. The critical point c_1 of Q now moves to the origin which, we recall, is a critical point of P. Since $0 \in \partial D_P$ and $c_1 \in \partial D_Q$ it is no longer possible for the interiors of D_P and D_Q to together cover $\hat{\mathbb{C}}$ but for values of the parameter a in an appropriate set we can arrange that $D_P^\circ \cup D_Q^\circ = \hat{\mathbb{C}} - \{0\}$ [1]. As c_1 has become 0, the sets Λ_+ and Λ_- now touch at the origin, and $\Omega(\mathcal{F}, \Delta)$ has become a (topological) disc. This disc contains just one critical point of P (at $z = \infty$), and just one critical point of Q (at $z = -1$). As these critical points have multiplicities $p - 1 = 1$ and $q - 1 = 2$ respectively it follows that Cov^P and Cov^Q act as cyclic groups C_2 and C_3 on $\Omega(\mathcal{F}, \Delta)$ and

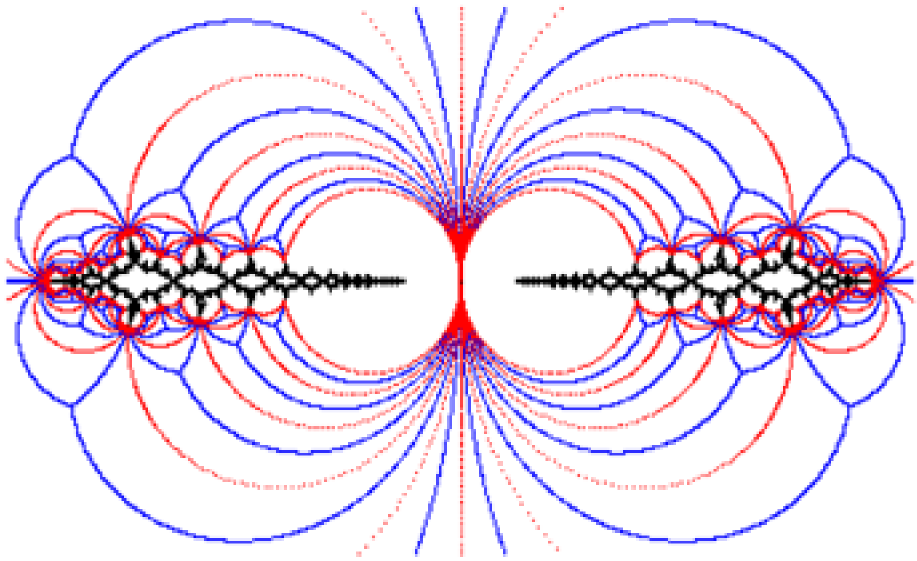

FIGURE 8. $a = 4.3; k = 1$

that we have a conformal bijection from $\Omega(\mathcal{F}, \Delta)$ to the complex upper half-plane conjugating the action of $\mathcal{F}$ to that of $PSL(2, \mathbb{Z})$. An example with $k = 1$ is plotted in Figure 8. Here we have zoomed in to twice the scale of Figure 7. See [1] for more computer plots of examples of matings obtained by setting $k = 1$.

There are two drawbacks to this method. Firstly, by setting k to be 1 we have violated the conditions of Theorem 2 in that the interiors of D_P and D_Q together only cover the *punctured* sphere $\hat{\mathbb{C}} - \{0\}$. While this causes no difficulty with the proof that the action of the correspondence $\mathcal{F}$ on $\Omega(\mathcal{F}, \Delta)$ is conjugate to that of $PSL(2, \mathbb{Z})$ on the upper half-plane, it means that $\mathcal{G} : \mathcal{G}^{-1}(D_P^\circ) \to D_P^\circ$ is no longer strictly *polynomial-like* in the Douady-Hubbard sense: although $\mathcal{G}^{-1}(D_P^\circ) \subset D_P^\circ$, the *closure* of the first set is not contained in the second since the origin is on the boundary of both. This is the technical difficulty alluded to earlier: it remains a conjecture in the case $k = 1$ that Λ_+ and Λ_- are homeomorphic to the filled Julia set $K(q_c)$ of a quadratic map q_c and that the appropriate restrictions of $\mathcal{G}$ and $\mathcal{G}^{-1}$ are topologically conjugate to the action of q_c on K_c. The second drawback is that we have to adjust the value of k to become 1 and we only obtain the standard action of $C_2 * C_3$ (as $PSL(2, \mathbb{Z})$ on the upper half-plane). However we shall now see that there is an action of $C_2 * C_3$ implicit within the correspondence without any alteration to k, and that this action is that of a faithful discrete representation of $C_2 * C_3$ on $\hat{\mathbb{C}}$ having a Cantor set as its limit set.

Method 2. We assume that the conditions of Theorem 2 are satisfied, so that $D_P^\circ \cup D_Q^\circ = \hat{\mathbb{C}}$, and we assume also that Λ_+ and Λ_- are each connected. The set $\Omega = \Omega(\mathcal{F}, \Delta)$ is now an annulus. Consider any curve l which crosses Δ from the critical point 0 of P to the critical point c_1 of Q, and let $\mathcal{C}$ denote the union of the grand orbits of points of l under $\mathcal{F}$. The set $\mathcal{C}$ is illustrated in green in

Figures 9 and 10. It consists of an arc joining Λ_+ to Λ_- together with a countable union of line segments linking the points $\mathcal{G}^n(c_1), n \geq 1$ to Λ_+ and linking the points $\mathcal{G}^{-n}(c_1'), n \geq 0$ to Λ_-, where c_1' denotes the *co-critical* point of Q corresponding to the critical point c_1. Cutting Ω along $\mathcal{C}$ yields a (topological) disc Ω^{cut}. We claim there is a unique (up to Möbius conjugacy) faithful discrete representation r of $C_2 * C_3$ in $PSL(2,\mathbb{C})$, with connected regular set $\Omega_r \subset \hat{\mathbb{C}}$, such that there is an invariant (topological) disc $\mathcal{D} \subset \Omega_r$ and a conformal bijection $\Omega^{cut} \to \mathcal{D}$ conjugating $\mathcal{F}$ on Ω^{cut} to $C_2 * C_3$ on $\mathcal{D}$. Thus we may regard $\mathcal{F}$ as a *mating* of this representation with the relevant quadratic map q_c.

To see why such a representation r should exist we first consider the Fuchsian case (a and k real), illustrated by Figure 9. Here we take l to be the segment of the real axis from the critical point 0 of P to the critical point c_1 of Q. When we cut along l and its images, Ω becomes a disc Ω^{cut} conformally homeomorphic to the upper half-plane via a bijection which carries the action of $\mathcal{F}$ to a (Fuchsian) action of $C_2 * C_3$, that is to say a faithful discrete action with limit set (a Cantor set) contained in the boundary $\hat{\mathbb{R}}$ of the upper half-plane. This is self-evident: opening up the green cuts by halving angles at their end points is precisely what is needed to counteract the effect of the critical point c_1 and resolve the action of Cov^Q on the boundary of Ω^{cut} into that of a cyclic group C_3 of conformal bijections. As P and Q each have a single critical point in Ω^{cut} (of multiplicity $p-1$ and $q-1$ respectively), Cov^P and Cov^Q act on Ω^{cut} as cyclic groups of conformal bijections conjugate to iterates of elliptic transformations of the upper half-plane.

Now to generalise the situation to the non-Fuchsian case (illustrated in Figure 10), we join the origin to c_1 (which is no longer necessarily real) by any curve l in Δ which descends to a simple curve in the quotient orbifold $\Omega/\mathcal{F}$. When we cut along l and its images under $\mathcal{F}$, the actions of Cov^P and Cov^Q on Ω become actions of cyclic groups of conformal bijections on Ω^{cut} just as before. Every representation r of $C_2 * C_3$ in $PSL(2,\mathbb{C})$ is equipped with an involution χ conjugating the generators of C_2 and C_3 to their inverses (on the Poincaré ball χ is rotation through π about the common orthogonal to the fixed axes of C_2 and C_3). The orbifold $\Omega/\mathcal{F}$ is conformally isomorphic to $\Omega_r/\langle C_2 * C_3, \chi\rangle$ for a unique r (up to conjugacy), since by varying r we run through the whole moduli space of complex structures on the topological type of this orbifold. Cutting Ω open along $\mathcal{C}$ to give Ω^{cut}, and cutting Ω_r along the corresponding curves to yield two discs $\mathcal{D}$ and $\chi(\mathcal{D})$, we obtain the desired conjugacy from the action of $\mathcal{F}$ on Ω^{cut} to that of $C_2 * C_3$ on $\mathcal{D}$. It can be shown by techniques of quasiconformal surgery that within the family of correspondences we are considering are matings of *every* quadratic map having a connected filled Julia set with *every* discrete faithful representation of $C_2 * C_3$ having a connected regular set [5].

Comments.

1. As dynamically they partition the sphere into *three* disjoint regions $\Lambda_- \cong K(R_-)$, $\Lambda_+ \cong K(R_+)$, and $\Omega(\mathcal{F}, \Delta) \cong \Omega_r$ one might also regard the correspondences above as matings of R_- (on $K(R_-)$) with R_+ (on $K(R_+)$) along an action of $C_2 * C_3$ (on Ω_r). This is a legitimate viewpoint, but it must be remembered that R_- and R_+ are by no means independent. The sets Λ_- and Λ_+ are not completely invariant, but only backwards invariant and forwards invariant respectively. Indeed, since P has degree two in the construction considered above, $R_-|K(R_-)$ is conjugate to $R_+|K(R_+)$ (see Comment 5 at the end of Section 3).

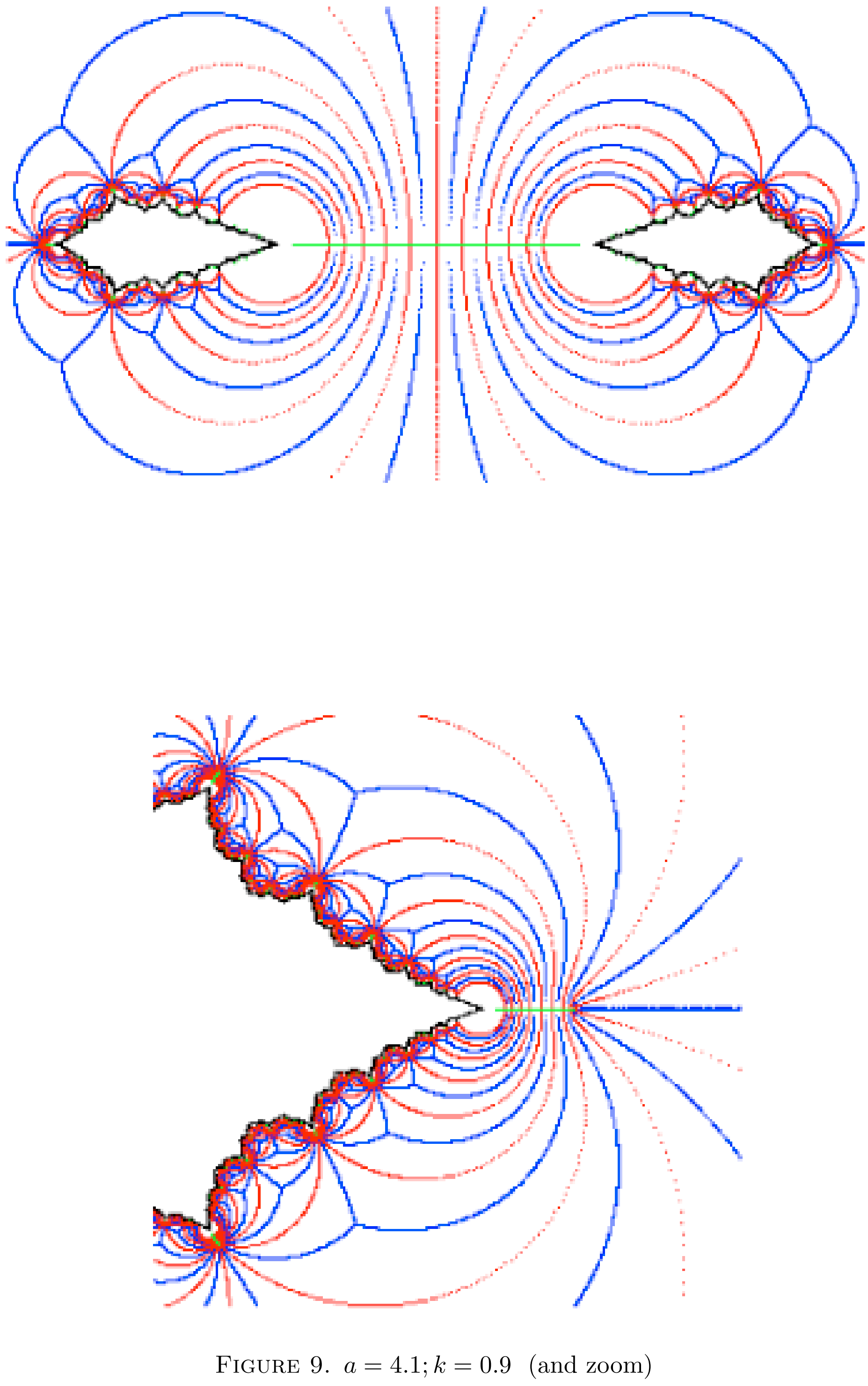

FIGURE 9. $a = 4.1; k = 0.9$ (and zoom)

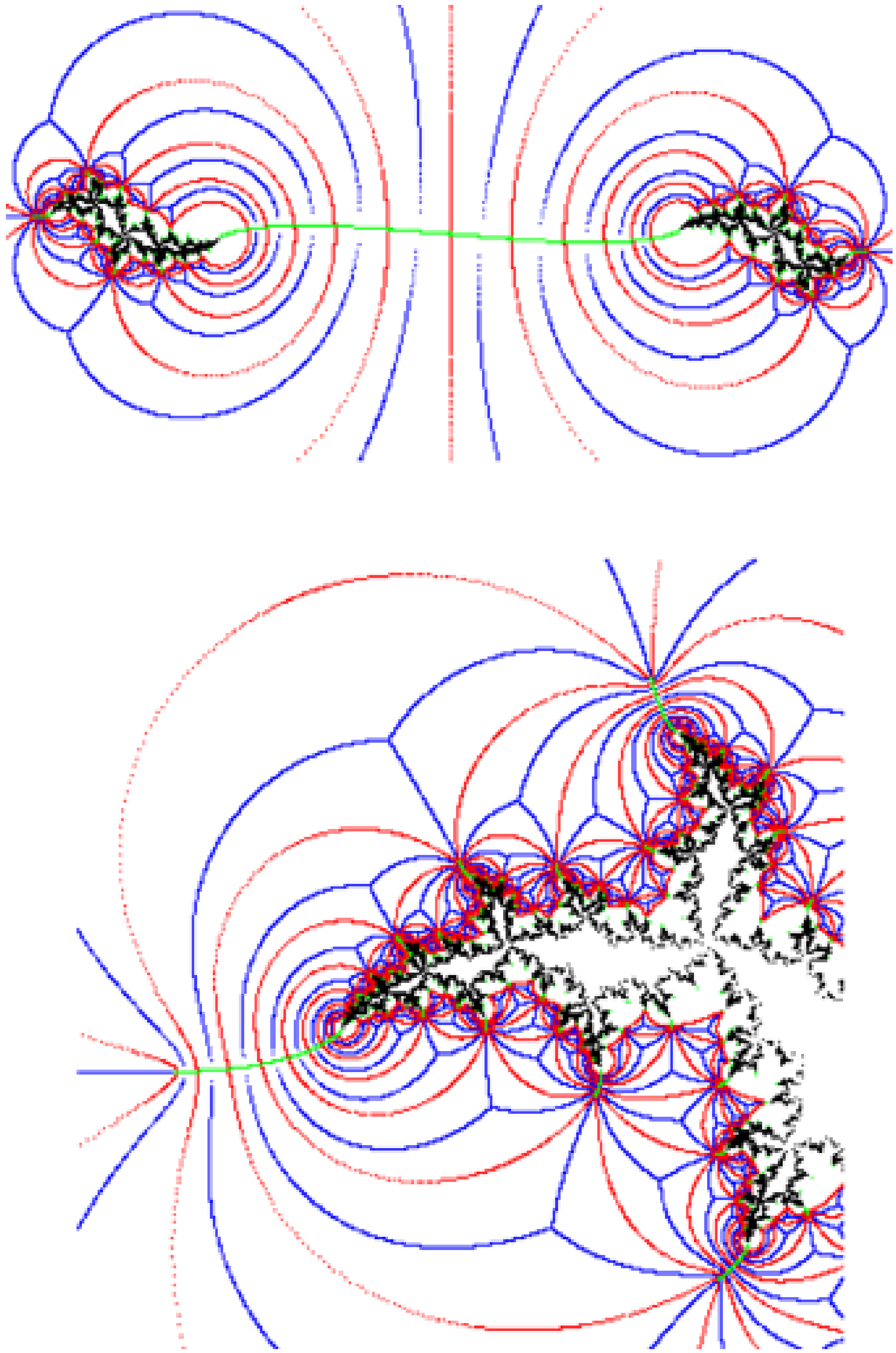

FIGURE 10. $a = 3.55 + 0.22i; k = 0.8$ (and zoom)

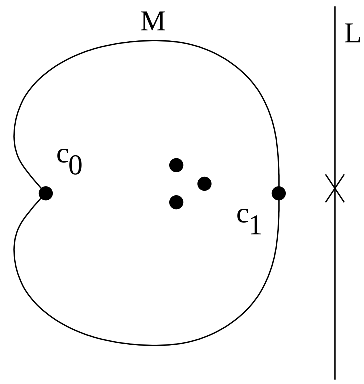

FIGURE 11.

2. It seems inevitable that some sort of construction involving cuts is needed for any tractable description of the matings of Method 2. There is no single involution on the regular set Ω_r of the representation of $C_2 * C_3$ having as its quotient a space homeomorphic to $\Omega = \Omega(\mathcal{F}, \Delta)$: in the description above we cut Ω_r along infinitely many curve segments (which join together the components of the Cantor set $\mathbb{C} - \Omega_r$) to divide it into two discs, $\mathcal{D}$ and $\chi\mathcal{D}$, and then obtain a copy of Ω by applying to one of these discs a boundary identification which uses a different conjugate of χ on each segment. To obtain a 'cut-free' description would involve passing to the universal cover of Ω_r and then quotienting by a group which is not finitely generated.
3. The region $\Omega = \Omega(\mathcal{F}, \Delta)$ may be only a part of the *regular set* $\Omega(\mathcal{F})$ as defined in the previous section. When the map q_c has an attracting (but not superattracting) periodic orbit, or a neutral periodic orbit of multiplier $e^{2\pi i\alpha}$ with α rational, the basin of this orbit is contained in $\Omega(\mathcal{F})$. Thus in this case $\Omega(\mathcal{F})$ contains the interior of $\Lambda_+ \cup \Lambda_-$ as well as $\Omega(\mathcal{F}, \Delta)$.

4.2. **Mating the group $C_2 * C_q$ with polynomial maps of degree $q - 1$.** For $q > 3$ we proceed in a similar way as for the case $q = 3$. We need rational maps P and Q of degree 2 and q respectively, with critical points and transversals as depicted in Figure 11. Here, as in Figure 6, D_P is the left-hand half-plane, bounded by the imaginary axis L and D_Q is the region exterior to the heart-shaped curve M. The function P is taken to be $z \to z^2$, with critical points 0 and ∞. We require the function Q to have a critical point c_0 of multiplicity $q - 1$, we require the transversal D_Q to be such that c_0 lies on its boundary, we require that Q also has a simple critical point c_1 on the boundary of D_Q, and we require that all the other $q - 2$ critical points of Q (counted with multiplicity) lie in the *interior* of the 'heart' $\hat{\mathbb{C}} - D_Q$.

Any coordinate change on the *domain* of Q alters Cov^Q by a conjugacy and any coordinate change on the *range* preserves Cov^Q. Since c_0 has multiplicity $q - 1$ we may choose coordinates for the domain and range of Q for which this map becomes polynomial. In order that $\Lambda_-(\mathcal{G})$ be connected we need all the critical points of Q which lie in the 'heart' $\hat{\mathbb{C}} - D_Q$ to remain inside $\mathcal{G}^{-n}(D_P)$ for all n. The simplest maps Q to consider are those with just 3 critical points, of multiplicity $q - 1$,1 and

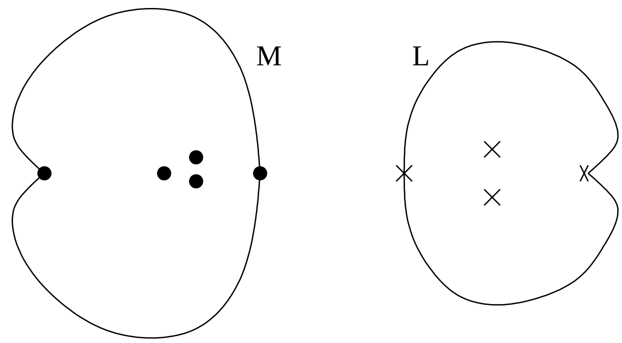

Figure 12.

$q-2$ respectively. These are the maps obtained by precomposing

$$Q_0(z) = z\left(1+\frac{z}{q-1}\right)^{q-1}$$

by a Möbius transformation, since Q_0 has a multiplicity $q-1$ critical point at ∞ (being a polynomial), a simple critical point at $z=(1-q)/q$ and a multiplicity $q-2$ critical point at $z=1-q$. Setting $P(z)=z^2$ and $Q(z)=Q_0(Mz)$, where M is the Möbius transformation sending $c_0 \to \infty$, $c_1 \to (1-q)/q$, and $c_2 \to 1-q$, we obtain a two-parameter family of correspondences $Cov^P * Cov^Q$ with the same pattern of critical points as illustrated earlier in Figure 6, except that now c_2 is of multiplicity $q-2$. By the same methods as in the $q=3$ case, within this two-parameter family there are matings of $C_q * C_2$ with polynomials of the form $z \to z^{q-1}+c$ having connected filled Julia sets. (The particular form $z \to z^{q-1}+c$ follows from choosing Q to have a multiplicity $q-2$ critical point: by choosing Q to have different patterns of critical points we can mate $C_q * C_2$ with other types of polynomial of degree $q-1$.)

4.3. **Mating the group $C_p * C_q$ with polynomial maps of degree $(p-1)(q-1)$.** Now we need rational maps P and Q of degree p and q respectively, with critical points and transversals arranged as depicted in Figure 12. Here D_P is the region outside the curve L and D_Q is that outside M. P has a multiplicity $p-1$ critical point on L, a single (simple) critical point elsewhere on L, and its other critical points lie in the interior of the 'heart' $\hat{\mathbb{C}} - D_P$. Similarly Q has a multiplicity $q-1$ critical point on L, a single (simple) critical point elsewhere on M, and its other critical points lie in the interior of the 'heart' $\hat{\mathbb{C}} - D_Q$.

Since Gal^P is not a bijection it no longer follows automatically that if $Gal^Q(D_Q^\circ)$ is a disc, then its image $Gal^P \circ Gal^Q(D_Q^\circ)$ need also be a disc, but this will be the case if the latter contains $p-2$ critical points of P (counted with multiplicity). To apply Theorem 2 we shall require this and the corresponding condition that $Gal^Q \circ Gal^P(D_P^\circ)$ contain $q-2$ critical points of Q. The easiest way to achieve this scenario is to restrict consideration to maps P and Q obtained by pre-composing P_0 and Q_0 by Möbius transformations, where

$$P_0(z) = z\left(1+\frac{z}{p-1}\right)^{p-1}$$

and

$$Q_0(z) = z\left(1 + \frac{z}{q-1}\right)^{q-1}.$$

Then P and Q have critical points arranged as in Figure 12, but now inside the 'heart' $\hat{\mathbb{C}} - D_P$ we have just one critical point of P (of multiplicity $p-2$), and inside the other 'heart' $\hat{\mathbb{C}} - D_Q$ we have just one critical point of Q (of multiplicity $q-2$). When these 'interior' critical points lie in $\Lambda_-(\mathcal{G}) \cup \Lambda_+(\mathcal{G})$ rather than $\Omega(\mathcal{F}, \Delta)$ we have a mating of $C_p * C_q$ with polynomials R_+ and R_- of degree $(p-1)(q-1)$, having filled Julia sets homeomorphic to Λ_+ and Λ_- respectively.

Comment. In the case $p, q > 2$ the correspondence Gal^P is of type $(p-2 : p-2)$ when restricted to $\Lambda_- \to \Lambda_-$ and $(p-1 : 1)$ as a map $\Lambda_- \to \Lambda_+$. Similarly Gal^Q is $(q-2 : q-2)$ restricted to $\Lambda_+ \to \Lambda_+$ and $(q-1 : 1)$ as a map $\Lambda_+ \to \Lambda_-$. In the case $p = 2$ the $(p-1 : 1)$ map becomes a homeomorphism. In the matings described above we observe that $K(R_-)$ has a $(q-1)$-fold cover homeomorphic to $K(R_+)$ and that $K(R_+)$ has a $(p-1)$-fold cover homeomorphic to $K(R_-)$. As we mentioned at the end of Section 3, it is an interesting question as to whether in these circumstances $K(R_+)$ is necessarily homeomorphic to $K(R_-)$. In the example below the presence of an additional symmetry provides such a homeomorphism.

Example. Take $p = q = 3$. Let

$$P(z) = \left(\frac{z - 1/3}{z+1}\right)\left(\frac{z + 1/3}{z+1}\right)^2 \text{ and } Q(z) = P(-z).$$

Then P has a double critical point at $z = -1$ and simple critical points at $z = 0$ and $z = -1/3$, and Q has a double critical point at $z = +1$ and simple critical points at $z = 0$ and $z = +1/3$. An elementary calculation shows that Gal^P maps $z = -1/3$ to $z = +1/3$ and Gal^Q maps $z = +1/3$ back to $z = -1/3$. It follows that in this example $Cov^P * Cov^Q$ is a mating of $C_3 * C_3$ with $z \to z^4$. In fact this example has an identical limit set to that obtained from the mating of $C_3 * C_2$ with $z \to z^2$ obtained by setting $a = 5$ in the family [1]

$$z \to w \text{ where } \left(\frac{az+1}{z+1}\right)^2 + \left(\frac{az+1}{z+1}\right)\left(\frac{aw-1}{w-1}\right) + \left(\frac{aw-1}{w-1}\right)^2 = 3,$$

the mating above of $C_3 * C_3$ with $z \to z^4$ being an index two subcorrespondence of the mating [1] of $C_3 * C_2$ with $z \to z^2$. More generally, given any rational maps P of degree 2 and Q of degree q such that $Cov^P * Cov^Q$ is a mating of $C_2 * C_q$ with a polynomial R of degree $q-1$, if we let σ denote the involution Gal^P and S denote the rational function $S(z) = Q(\sigma z)$, then $Cov^S * Cov^Q$ is an index two subcorrespondence of $Cov^P * Cov^Q$ and is a mating of an index two subgroup $C_q * C_q$ of $C_2 * C_q$ with the degree $(q-1)^2$ polynomial $R \circ R$.

5. Separable correspondences

An $(r : r)$ correspondence $f : z \to w$ on the Riemann sphere is called *separable* if it can be defined by a relation of the form $Q_+(z) = Q_-(w)$ (where Q_+ and Q_- are rational functions), and *reversible* if there exists an involution σ such that $z \to w$ if and only if $\sigma w \to \sigma z$ [3]. If f is both separable and reversible the reversing involution σ conjugates Cov^{Q_+} to Cov^{Q_-} and therefore descends to an involution $\underline{\sigma}$ such that $Q_- = \underline{\sigma} Q_+ \sigma$. Thus, writing R for Q_+ to simplify notation, $f : z \to w$

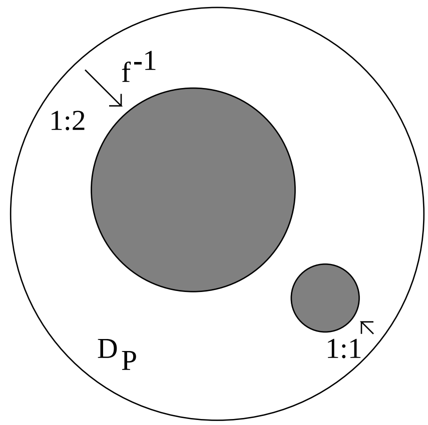

FIGURE 13.

is *reversible separable* if and only if there exist a rational map R and involutions $\sigma, \underline{\sigma}$ such that

$$z \to w \;\Leftrightarrow\; \underline{\sigma}R(z) = R(\sigma w).$$

Given such an $(r : r)$ correspondence f, consider the degree $2r$ rational map $Q = S \circ R$ obtained by composing with R a rational map S of degree two having $Gal^S = \underline{\sigma}$, and let P be a rational map of degree two having $Gal^P = \sigma$. Now $\mathcal{G} = Gal^P \circ Gal^Q$ is the union of f and the $(r-1 : r-1)$ correspondence $Gal^P \circ Gal^R$.

Suppose that we can find transversals D_P, D_Q which have interiors D_P°, D_Q° with $\overline{D_P^\circ} = \overline{D_P}$ and $\overline{D_Q^\circ} = \overline{D_Q}$, and such that D_P° and D_Q° together cover the Riemann sphere $\hat{\mathbb{C}}$. Then, by Theorem 2, $\hat{\mathbb{C}}$ is partitioned into a regular set $\Omega(\mathcal{F}, \Delta)$ and its complement $\Lambda_- \sqcup \Lambda_+$, where $\Lambda_- = \bigcap \mathcal{G}^{-n}(D_P)$ and $\Lambda_+ = \bigcap \mathcal{G}^n(D_Q)$. Consider $\mathcal{G}^{-1} : D_P^\circ \to D_P^\circ$. The image $\mathcal{G}^{-1}(D_P^\circ)$ is the *disjoint* union of $f^{-1}(D_P^\circ)$ and $(Gal^P \circ Gal^R)^{-1}(D_P^\circ)$ (since $\mathcal{G}^{-1}$ is a $(1 : 2r-1)$ correspondence on D_P°). If each of these sets is a topological disc, then $\mathcal{G}$ is a generalised polynomial-like mapping in an appropriate sense and we can obtain a description of Λ_- and Λ_+. For example with $r = 2$, if $f^{-1}(D_P^\circ)$ is a disc the correspondence $\mathcal{G}^{-1}$ restricted to D_P° decomposes as indicated in Figure 13. Now $\bigcap f^{-n}(D_P)$ (the limit set under unidirectional iteration of f^{-1}) is homeomorphic to a filled quadratic Julia set, and $\bigcap \mathcal{G}^{-n}(D_P) \sqcup \bigcap \mathcal{G}^n(D_Q)$, which is also the *global* limit set of f under mixed forward and backward iteration, is the closure of a countable infinite union of copies of this quadratic Julia set. See [2, 3] for examples.

We remark that the perturbations of circle-packing representations of $C_2 * C_4$ discussed in [4] fit into the same framework. These $(2 : 2)$ reversible separable correspondences f can be regarded as subcorrespondences of $(3 : 3)$ correspondences of the form $Gal^P \circ Gal^Q$ with $Q = S \circ R$, where P, R and S are degree 2 rational maps, just as above.

The technique outlined in this section for analysing reversible separable correspondences can be applied to separable correspondences in general. Any separable correspondence can be lifted to a *reversible* separable correspondence on the trivial double cover of the sphere (one sheet for forward iteration of the correspondence, and one for backward iteration) [3]. See [2] for computer plots of examples.

6. Conclusion

There is a well-established construction which 'mates' two Fuchsian groups which are isomorphic as abstract groups and have limit sets each the whole of $\mathbb{R} \cup \infty$, to construct a *quasifuchsian* Kleinian group with limit set a quasi-circle. Another well-established construction is to 'mate' two polynomial maps by fitting together copies of their actions on their filled Julia sets to construct an action of a rational map. The examples above are an indication that a more general kind of construction is possible, involving 'xenotransplants' between the two best known species of one-dimensional holomorphic dynamical system - rational maps and Kleinian groups. The key tool above is Klein's combination theorem in its most elementary form. More complicated arrangements involving the same basic building blocks should be possible by considering more sophisticated forms of the Maskit-Klein combination theorems and covering correspondences of rational functions P and Q which are built up as compositions of rational functions of lower degrees in different ways.

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