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QUASICONFORMAL STABILITY OF KLEINIAN GROUPS AND AN EMBEDDING OF A SPACE OF FLAT CONFORMAL STRUCTURES

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ABSTRACT. We show the quasiconformal stability for torsion-free convex cocompact Kleinian groups acting on higher dimensional hyperbolic spaces. As an application, we prove an embedding theorem of a space of flat conformal structures on a certain class of compact manifolds.

1. INTRODUCTION

One of the interesting classes of conformally flat manifolds is that obtained as quotient spaces associated to Kleinian groups. Here a flat conformal structure C on a manifold M is a conformal structure which is locally flat; more precisely speaking, for any given point $p \in M$, there is a Riemannian metric in C which is flat on a neighborhood of p. There have been some interesting and important studies of this class of conformally flat manifolds from the viewpoint of Riemannian geometry, particularly in [23] and [17]. These results suggest that there is a close relation between the curvature of conformally flat Riemannian manifolds and the critical exponents of Kleinian groups. This observation seems to be quite useful for the investigation of conformally flat manifolds from the viewpoint of Kleinian group theory, and vice versa (examples can be found in [17], [7]). In this paper we show a stability theorem for Kleinian groups by means of geometry of conformally flat manifolds. We also present an application of the stability result to the study of the moduli space of flat conformal structures.

Before explaining our results, let us recall some basic notions in Kleinian group theory. By Kleinian groups we simply mean discrete subgroups of the conformal transformation group $\mathrm{Conf}(S^n)$ of S^n with the standard conformal structure. As is well-known, a conformal transformation of the standard sphere can be regarded as the extension of an isometry of hyperbolic space $\mathbb{H}^{n+1}$ to its ideal boundary $\partial\mathbb{H}^{n+1} = S^n$. Thus, having a Kleinian group Γ, we can think of its action on the union $\mathbb{H}^{n+1} \cup S^n$. Fix a point $x \in \mathbb{H}^{n+1}$. The set of accumulation points of the orbit Γx is called the *limit set* of a Kleinian group Γ. We denote the set by $\Lambda(\Gamma)$. Since Γ is discrete and its action on $\mathbb{H}^{n+1}$ is properly discontinuous, $\Lambda(\Gamma)$ lies in the boundary S^n, and does not depend on the choice of $x \in \mathbb{H}^{n+1}$. The complement of $\Lambda(\Gamma)$ in S^n is called the *domain of discontinuity* of Γ and is denoted by $\Omega(\Gamma)$. (Sometimes we denote it as $\Omega^n(\Gamma)$ in order to specify the dimension of the sphere

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on which Γ acts.) Γ acts on $\Omega(\Gamma)$ properly discontinuously. A Kleinian group Γ is called *convex cocompact* if the quotient space $(\Omega(\Gamma) \cup \mathbb{H}^{n+1})/\Gamma$ is compact. Note that if $\Omega(\Gamma)/\Gamma$ is a manifold, this is the case if Γ is torsion-free, then there is a natural conformal structure coming from the standard one on S^n and it is obviously conformally flat. We call conformally flat manifolds obtained in this way *Kleinian manifolds* (or *manifolds with uniformized flat conformal structure*). Flat conformal structures on these manifolds are also called *Kleinian structures* (or *uniformized flat conformal structures*). If Γ is convex cocompact, we shall say $\Omega(\Gamma)/\Gamma$ is a convex cocompact Kleinian manifold. Further facts on Kleinian groups which will be needed in this paper are summarized in [7, §2].

The first theorem in this paper is the following quasiconformal stability of convex cocompact Kleinian groups. We denote by $\mathrm{Hom}(G, \mathrm{Conf}(S^n))$ the space of all (not necessarily faithful nor discrete) representations equipped with the topology defined by the pointwise convergence of maps into the Lie group $\mathrm{Conf}(S^n)$.

Theorem 1. *Let Γ be a torsion-free convex cocompact Kleinian group given as the image of a faithful discrete representation $\rho : G \longrightarrow \mathrm{Conf}(S^n)$, $n \geq 2$. Then there is a neighborhood U of ρ in $\mathrm{Hom}(G, \mathrm{Conf}(S^n))$ such that every $\rho' \in U$ is faithful discrete and $\rho'(G)$ is convex cocompact. For such a ρ', there is a quasiconformal mapping $\varphi : S^n \longrightarrow S^n$ which conjugates $\rho'(G)$ to $\rho(G)$, namely $\varphi\rho(G)\varphi^{-1} = \rho'(G)$. Moreover, φ can be taken so that it is smooth outside the limit set, and can be extended to a conjugation defined on $\mathbb{H}^{n+1} \cup S^n$.*

In case $n = 2$, the theorem has been known ([12]) and it is true even for the groups with torsion ([14]). Their proof was achieved by comparing the dimension of representation space and that of the space of quasiconformal deformation of geometrically finite groups. In contrast to their case, we cannot say anything about the dimension of $\mathrm{Hom}(G, \mathrm{Conf}(S^n))$ in general, and our approach is different from that in [12] and [14]. The proof of Theorem 1 is roughly as follows. Let $\mathcal{FC}(M)$ be the set of all flat conformal structures on M. We denote by $\mathcal{M}(M)$ the moduli space $\mathcal{FC}(M)/\mathrm{Diff}(M)$ of flat conformal structures on M. The topology of the space $\mathcal{FC}(M)$ comes from C^∞ topology of the space of Riemannian metrics, and $\mathcal{M}(M)$ is equipped with the quotient topology. It is well-known that there is a connection between the representation space $\mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^n))$ and the moduli space $\mathcal{M}(M)$, which is given by so-called holonomy theorem. (See the next section for the detail.) On the other hand, for any convex cocompact Kleinian group, we can associate a conformally flat Riemannian manifold with positive scalar curvature as we will see in the proof. Through the holonomy theorem, together with results from [23] and [7], we can interpret the obvious openness of the set of metrics with positive scalar curvature as the quasiconformal stability of convex cocompact Kleinian groups. This leads us to Theorem 1.

From the viewpoint of conformal geometry, Theorem 1 asserts the openness of convex cocompact Kleinian structures in the moduli space. Moreover, the following is true.

Theorem 2. *Let M be a compact connected conformally flat manifold with $n = \dim M \geq 3$. Suppose M satisfies:*

(1) *M admits a flat conformal structure C with an injective developing map.*

(2) $\mathrm{vcd}\,\pi_1(M) < n$*, where* $\mathrm{vcd}\,G$ *is the virtual cohomological dimension of G.*

Let $\mathcal{M}_0$ be the connected component of the moduli space containing the element represented by C in (1). *Then every element in $\mathcal{M}_0$ is an equivalence class* *consisting of flat conformal structures with injective developing maps and the image of* $\pi_1(M)$ *under the associated holonomy representations are convex cocompact Kleinian groups. In particular,* $\mathcal{M}_0$ *consists of convex cocompact Kleinian structures.*

For example, if C contains a metric with positive scalar curvature, then C satisfies the assumption (1) by a result in [23]. (See also Theorem 2.5 in this paper. We will use this fact also in the proof of Theorem 1.) Under the assumption (1), the virtual cohomological dimension of $\pi_1(M)$ is less than or equal to n. Thus what we exclude by the assumption (2) is the equality case $\operatorname{vcd}\pi_1(M) = n$. This excluded case contains compact manifolds admitting a hyperbolic structure in which case the conclusion of Theorem 2 seems to be false. Also taking an n-torus as M, we see $\operatorname{vcd}\pi_1(M) = n$ and it does not admit convex cocompact Kleinian structure though all of its flat conformal structures have injective developing maps and come from geometrically finite Kleinian groups.

As a consequence of Theorem 2, we obtain an embedding of a certain branched covering space of $\mathcal{M}_0$. Let $\operatorname{Diff}_C(M)$ be a group of diffeomorphisms which has a lift to the universal covering $\tilde{M}$ of M commuting with the covering transformations. Define $\mathcal{T}_C(M)$ by

$$\mathcal{T}_C(M) = \mathcal{F}C(M)/\operatorname{Diff}_C(M),$$

and equip $\mathcal{T}_C(M)$ with the quotient topology. Then we have

Theorem 3. *Suppose* M *satisfies assumptions* (1) *and* (2) *in Theorem 2. Denote by* $\mathcal{T}_0$ *the connected component of* $\mathcal{T}_C(M)$ *corresponding to* $\mathcal{M}_0$*. Then the map* hol'' *turns out to be an embedding of* $\mathcal{T}_0$ *onto an open subset of* $R(\pi_1(M), \operatorname{Conf}(S^n))$*, where* $R(\pi_1(M), \operatorname{Conf}(S^n))$ *is the quotient of* $\operatorname{Hom}(\pi_1(M), \operatorname{Conf}(S^n))$ *by the conjugate action of* $\operatorname{Conf}(S^n)$*.*

The map hol'' above comes from the map that assigns the holonomy representation to a developing map. See section 2 for the definition.

Another consequence of Theorem 1 is the following continuity of the critical exponent regarded as a function defined on a certain subset of the representation space.

Theorem 4. *Let* $C(G, \operatorname{Conf}(S^n))$ *be a subset of* $\operatorname{Hom}(G, \operatorname{Conf}(S^n))$ *consisting of faithful discrete representations whose images are convex cocompact, and assume* G *is torsion-free. Then*

(1) $C(G, \operatorname{Conf}(S^n))$ *is open in* $\operatorname{Hom}(G, \operatorname{Conf}(S^n))$*, and*

(2) *the critical exponent regarded as a function on* $C(G, \operatorname{Conf}(S^n))$ *is continuous.*

Though Theorem 4 has been proved by Bourdon [3] in his thesis, we give a different proof of this result. The second part is a consequence of the uniform continuity of the critical exponent, regarded as a function on $\mathcal{T}_0$, with respect to the Teichmüller distance defined in [8] (see Theorem 4.2). Here, we should mention that the real analyticity of the critical exponent has already been established in [3] (see also [1] for a result on a restricted class of Kleinian groups acting on S^2).

As a consequence of Theorem 4, we see that Nayatani's metric associated to a Kleinian group varies continuously on $C(G, \operatorname{Conf}(S^n))$ ([9, Theorem 2.1]) and this suggests a possibility of a compactification of $\mathcal{T}_0$ in Theorem 2 by means of Nayatani's metric. On the other hand, in view of Theorem 3, for a manifold satisfying the assumptions in Theorem 2, we have a compactification of the space $\mathcal{T}_0$ by means of the compactification of the space of faithful discrete representations due to Morgan and Shalen ([16], [15]). It might be possible to give a new interpretation of this compactification in terms of the deformation of flat conformal structures or Nayatani's metrics, though such a description in terms of hyperbolic structures has been given in [2] and [22].

Most of the necessary facts for our proof are found in sections 2–4 of [7] and the rest about the holonomy theorem is explained in section 2 of this paper. In section 3 we recall the critical exponent, Patterson-Sullivan measure, and a metric constructed by Nayatani associated to a Kleinian group. In section 4 we prove Theorems 1, 2, 3, and 4.

2. The developing maps

The purpose of this section is to prepare basic facts on developing maps associated to flat conformal structures. We start with the definition of developing maps and a description of deformation spaces of flat conformal structures by means of developing maps. Theorems 2.3 and 2.5 play an important role in the proofs of our theorems.

In this section, the manifolds under consideration are smooth and of dimension greater than 2. Though Theorem 1 is stated for the case dimension greater than or equal to 2, this restriction does not affect our proof of Theorem 1.

Recall that every point of a conformally flat manifold (M, C) has a neighborhood conformal to an open subset of the standard sphere S^n by definition. Thus, if M is simply connected and of dimension n, we can construct a smooth immersion from M into S^n by piecing these local conformal diffeomorphisms into S^n together. This follows from Liouville's theorem about conformal transformations on S^n $(n \geq 3)$ and a standard monodromy argument. This immersion is called a *developing map*. For a given flat conformal structure C on M, the developing map associated to C is unique up to composition with conformal transformations of S^n. For a manifold M which may not be simply connected, by lifting a flat conformal structure on M to the universal covering space $\tilde{M}$ of M, we can construct a developing map Ψ defined on $\tilde{M}$. We call Ψ also a *developing map of* M. In this case, the fundamental group $\pi_1(M)$ of M acts on $\tilde{M}$ as the deck transformation group and this action preserves the flat conformal structure on $\tilde{M}$. By the uniqueness of the developing map, $\Psi \circ \gamma = \xi \circ \Psi$ holds for some conformal transformation ξ, where $\gamma \in \pi_1(M)$. It is easy to see that this gives rise to a representation $\rho : \pi_1(M) \longrightarrow \mathrm{Conf}(S^n)$. This representation is called the *holonomy representation* of Ψ.

Let $\widetilde{\mathcal{FC}}(M)$ be the space of all developing maps of M and topologize this space by compact-C^∞ topology of maps from $\tilde{M}$ to S^n. Let $\widetilde{\mathrm{Diff}_C}(M)$ be the group consisting of the lifts of elements in $\mathrm{Diff}(M)$ to the universal covering space that commute with all the deck transformations. The topology of this group is given by the compact-C^∞ topology. Clearly this group $\widetilde{\mathrm{Diff}_C}(M)$ is nonempty and acts continuously on $\widetilde{\mathcal{FC}}(M)$ by composition on the right. Let us denote by $\mathrm{Diff}_C(M)$ the image of $\widetilde{\mathrm{Diff}_C}(M)$ under the map from $\widetilde{\mathrm{Diff}_C}(M)$ to $\mathrm{Diff}(M)$ induced from the covering projection. In other words, $\mathrm{Diff}_C(M)$ is the group consisting of diffeomorphisms having a lift to $\tilde{M}$ that commutes with all the deck transformations.

By using the notations above, we can describe deformation spaces of flat conformal structures in the following way.

Definition 2.1. Let $\widetilde{\mathcal{F}C}(M)$ and $\mathrm{Diff}_C(M)$ be as above. Let

$$\mathcal{F}C(M) = \mathrm{Conf}(S^n)\backslash\widetilde{\mathcal{F}C}(M),$$
$$\mathcal{T}_C(M) = \mathcal{F}C(M)/\,\mathrm{Diff}_C(M),$$

where the action of $\mathrm{Conf}(S^n)$ on $\widetilde{\mathcal{F}C}(M)$ is defined by composition on the left. Then $\mathcal{F}C(M)$ can be viewed as the space of all flat conformal structures because of the uniqueness property of developing maps. Thus the action of $\varphi \in \mathrm{Diff}_C(M)$ on $\mathcal{F}C(M)$ is defined by the pull-back of a flat conformal structure by φ. The topology of $\mathcal{F}C(M)$ and $\mathcal{T}_C(M)$ is given by the quotient topology.

Remark 1. The space $\mathcal{F}C(M)$ is homeomorphic to

$$\{\text{conformally flat } C^\infty \text{ Riemannian metrics}\}/C^\infty_+(M)$$

with the topology induced from the compact-C^∞ topology on the space of C^∞ Riemannian metrics. Here $C^\infty_+(M)$ is the space of positive C^∞ functions and its action on the space of Riemannian metrics is defined by the multiplication.

Recall that there is a right action of $\widetilde{\mathrm{Diff}_C(M)}$ on $\widetilde{\mathcal{F}C}(M)$. It is easy to see that, passing to the quotient, this right action of $\widetilde{\mathrm{Diff}_C}(M)$ gives rise to that of $\mathrm{Diff}_C(M)$ on $\mathcal{F}C(M)$ in the definition above.

Let us consider the map $\mathrm{hol} : \widetilde{\mathcal{F}C}(M) \longrightarrow \mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^n))$ that assigns the holonomy representation of Ψ to $\Psi \in \widetilde{\mathcal{F}C}(M)$. The continuity of this map hol is clear. Here we define the topology of $\mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^n))$ by the pointwise convergence of representations with respect to the topology of $\mathrm{Conf}(S^n)$ as a Lie group. If $\pi_1(M)$ is finitely generated, this topology coincides with that given by uniform convergence on the finite set of generators. By definition, the action of $\widetilde{\mathrm{Diff}_C}(M)$ on $\widetilde{\mathcal{F}C}(M)$ preserves the holonomy representation. In other words, the map hol is constant on each orbit of $\widetilde{\mathrm{Diff}_C}(M)$. Let us define the space $\tilde{\mathcal{T}}_C(M)$ by $\widetilde{\mathcal{F}C}(M)/\widetilde{\mathrm{Diff}_C}(M)$. Then hol induces the map $\mathrm{hol}' : \tilde{\mathcal{T}}_C(M) \longrightarrow \mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^n))$, and this map hol' is clearly continuous. This map hol' satisfies $\mathrm{hol}'(\eta \circ \Phi) = \eta\,\mathrm{hol}'(\Phi)\eta^{-1}$ and $\mathcal{T}_C(M)$ is obtained as $\mathrm{Conf}(S^n)\backslash\tilde{\mathcal{T}}_C(M)$. Thus we have a map $\mathrm{hol}'' : \mathcal{T}_C(M) \longrightarrow R(\pi_1(M), \mathrm{Conf}(S^n))$ induced from hol', where $R(\pi_1(M), \mathrm{Conf}(S^n)) = \mathrm{Conf}(S^n)\backslash\,\mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^n))$ and the action of $\mathrm{Conf}(S^n)$ is defined by $\eta\cdot\rho = \eta\rho\eta^{-1}$ for $\eta \in \mathrm{Conf}(S^n)$, $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^n))$.

Now assume M is compact. The following lemma asserts that there is a map from a neighborhood of a given holonomy representation in $\mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^n))$ into the space $\widetilde{\mathcal{F}C}(M)$ and it is the inverse map of hol on the image.

Lemma 2.2 ([4, 1.7.2])**.** (1) *Let $\rho \in \mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^n))$ be the holonomy representation of a developing map Φ of a compact manifold M. Then there is a neighborhood V of ρ and a continuous map $D : V \longrightarrow \widetilde{\mathcal{F}C}(M)$ which maps ρ to Φ and $\mathrm{hol}\circ D$ is the identity map on V.*

(2) *Let Φ be a developing map which is not surjective, and ρ its holonomy representation. Suppose there is a round sphere $S^k \subset S^n$, $k < n$, which is preserved by the action of $\rho(\pi_1(M))$. Let V be as above and $\rho' \in V \cap \mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^k))$. Then we can take D in* (1) *so that $(D(\rho') \circ \Phi^{-1})(S^k \cap \mathrm{Im}\,\Phi) \subset S^k$.*

Proof. The statement of (1) is exactly the same as [4, 1.7.2] and (2) can be shown by a slight modification of the proof. However, since we will use (2) in what follows, we present the proof here.

Since Φ is not surjective, Φ is a covering map onto its image ([10]) and the image must miss some point $p \in S^k$ (see [13, §5]). We may assume S^k corresponds to k-dimensional subspace in $\mathbb{R}^n \cong S^n \setminus \{p\}$.

Let $\{U_i^r\}_{i=1,\dots,k,\ r=1,\dots,k}$ be a collection of open subsets of M such that

(1) each U_i^r is simply connected,

(2) $\{U_i^r\}_{i=1,\dots,k}$ is an open covering of M for each fixed r,

(3) $U_i^{s+1} \subset U_i^s$ for each fixed i.

Fix a connected component of the lifts of U_i^r and denote it by V_i^r. Since each U_i^r is simply connected, the covering projection $\pi : \tilde{M} \longrightarrow M$ is a homeomorphism on V_i^r. Take $\rho' \in \mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^k))$ close to ρ. We construct a ρ'-equivariant map $\Phi' : \tilde{M} \longrightarrow S^n$ by induction.

First construct Φ' on $\pi^{-1}U_1^1$. Set

$$\Phi'|_{V_1^1} = \Phi|_{V_1^1}$$

and extend it so that Φ' becomes ρ'-equivariant. Note that this Φ' defined on $\pi^{-1}U_1^1$ continuously depends on the representation.

Assume Φ' is defined on $\pi^{-1}U_1^s \cup \cdots \cup \pi^{-1}U_s^s$. We extend this map to $\pi^{-1}U_1^{s+1} \cup \cdots \cup \pi^{-1}U_{s+1}^{s+1}$. Since Φ' is already defined on $\pi^{-1}U_1^{s+1} \cup \cdots \cup \pi^{-1}U_s^{s+1}$, we have only to define a map on V_{s+1}^{s+1} so that it coincides with Φ' on $V_{s+1}^{s+1} \cap (\pi^{-1}U_1^{s+1} \cup \cdots \cup \pi^{-1}U_s^{s+1})$. Then we obtain the desired map, extending it by ρ'-equivariance. Take open subsets W_0, W_1, and W_2 as follows:

(1) $W_1 = V_{s+1}^{s+1} \cap (\pi^{-1}U_1^s \cup \cdots \cup \pi^{-1}U_s^s)$,

(2) $W_2 \supset \bar{W}_1$ and $\bar{W}_2 \subset (\pi^{-1}U_1^s \cup \cdots \cup \pi^{-1}U_s^s) \cap W_0$.

Take a smooth partition of unity $\{\lambda_1, \lambda_2\}$ subordinate to $\{W_2, \tilde{M} \setminus W_1\}$ so that $\lambda_1 = 1$ on W_1. Then we can define a map Φ'_{s+1} on V_{s+1}^{s+1} by

$$\Phi'|_{V_{s+1}^{s+1}} = \lambda_1 \Phi' + \lambda_2 \Phi,$$

where addition is taken by regarding $S^n \setminus \{p\}$ as $\mathbb{R}^n$. Since Φ misses p, this makes sense and since S^k corresponds to the k-dimensional subspace of $\mathbb{R}^n$, $\Phi'|_{V_{s+1}^{s+1}}$ maps $\Phi^{-1}(S^k \cap \mathrm{Im}\,\Phi)$ to S^k. Moreover, $\Phi|_{V_{s+1}^{s+1}}$ misses p again. It is clear by definition that $\Phi'_{s+1} = \Phi'$ on W_1. Therefore we can extend this map to $\pi^{-1}U_1^{s+1} \cup \cdots \cup \pi^{-1}U_{s+1}^{s+1}$ by means of equivariance. It is also clear that if Φ' on $\pi^{-1}U_1^s \cup \cdots \cup \pi^{-1}U_s^s$ depends continuously on the variation of representation, then the same is true for Φ' extended to $\pi^{-1}U_1^{s+1} \cup \cdots \cup \pi^{-1}U_{s+1}^{s+1}$.

Proceeding in this way, we get a desired equivariant map at the kth step. If ρ' is close to ρ, then our map Φ' is C^∞ close to Φ on each compact subset. Thus it is locally diffeomorphic on a compact fundamental set and hence it is locally diffeomorphic everywhere because of the equivariance. This completes the proof. □

If two sufficiently close developing maps Φ, Φ' have the same holonomy representation, then there is an equivariant map $\tilde{i} : \tilde{M} \longrightarrow \tilde{M}$ such that $\Phi' = \Phi \circ \tilde{i}$. (See [4, 1.7.4].) Combining this fact with Lemma 2.2, we get the following theorem. Slightly different treatment can be found in [6].

Theorem 2.3 (see [4, 1.7.1]). *Let Ψ be a developing map of a compact manifold M. Then there is a neighborhood $V \subset \widetilde{\mathcal{FC}}(M)$ of Ψ homeomorphic to $V_1 \times V_2$, where $V_1 \subset \mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^n))$ is a neighborhood of the holonomy representation of Ψ and V_2 is a neighborhood of the identity in $\mathrm{Diff}_C(M)$.*

Theorem 2.3, originally due to Lok, is called the holonomy theorem and asserts that the map hol induces a local homeomorphism

$$\mathrm{hol}' : \tilde{\mathcal{T}}_C(M) \longrightarrow \mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^n)).$$

On the other hand, the map $\mathrm{hol}'' : \mathcal{T}_C \longrightarrow R(\pi_1(M), \mathrm{Conf}(S^n))$ induced from hol' need not be a homeomorphism. But it is easy to see the following holds.

Lemma 2.4. *If M is compact, then* hol'' *is a continuous open map.*

Remark 2. The results stated above should be found in slightly different form in [4] and [6]. The difference is the choice of the group acting on $\mathcal{FC}(M)$. In those papers cited above, the group is chosen as the group of diffeomorphisms isotopic to the identity. However, the essential point is that the action of the chosen group preserves the holonomy representation and it is easy to see the proofs are applicable to our case. The reason we have chosen $\mathrm{Diff}_C(M)$ is to make the statement of Theorem 3 simpler.

In general, it is not easy to see how the developing map of a given flat conformal structure behaves. But for compact conformally flat manifolds, there is an important theorem due to Schoen and Yau ([23]). We state it below in a form convenient to our purpose.

Theorem 2.5 ([23, Theorem 4.5])**.** *Let (M, C) be a compact conformally flat manifold with a flat conformal structure C containing a metric with positive scalar curvature. Then the developing map Φ of C is injective. In particular, the holonomy representation ρ of Φ is a faithful discrete representation. Moreover, the image of the universal covering $\tilde{M}$ under Φ coincides with the domain of discontinuity $\Omega(\rho(\pi_1(M)))$ of $\rho(\pi_1(M))$ and (M, C) can be recovered as the Kleinian manifold $\Omega(\rho(\pi_1(M)))/\rho(\pi_1(M))$.*

3. The critical exponents and Nayatani's metrics

In this section, we briefly review the critical exponent of Kleinian groups and Nayatani's metric.

Definition 3.1. The *critical exponent* $\delta(\Gamma)$ of a Kleinian group Γ acting on $S^n \cup \mathbb{H}^{n+1}$ is defined by

$$\delta(\Gamma) = \inf\{s > 0 \ ; \ G(x, y, s) = \sum_{\gamma \in \Gamma} \exp(-sd(x, \gamma y)) < \infty\},$$

where x, y are points in $\mathbb{H}^{n+1}$ and $d(x, y)$ is the hyperbolic distance between x and y. The value $\delta(\Gamma)$ does not depend on the choices of x, y.

The infinite series $G(x, y, s)$ is called the *Poincaré series* of Γ.

If Γ is convex cocompact, the critical exponent $\delta(\Gamma)$ is equal to the Hausdorff dimension $\dim_H \Lambda(\Gamma)$ of the limit set $\Lambda(\Gamma)$. The so-called Patterson-Sullivan density ([20], [26]) μ_o which has its base point at the origin o of $\mathbb{H}^{n+1}$ (regarded as a unit ball in $\mathbb{R}^{n+1}$) coincides with $\delta(\Gamma)$-dimensional Hausdorff measure up to a constant multiple. See also [18, Chapter 4].

In the rest of this section, we assume $n \geq 3$. In [17], Nayatani constructed a Γ-invariant Riemannian metric g on $\Omega(\Gamma)$, which is compatible to the conformal structure naturally defined on $\Omega(\Gamma)$. The construction is as follows:

Let Γ be a Kleinian group acting on S^n with $\delta = \delta(\Gamma) > 0$ and let $d\mu$ denote the Patterson-Sullivan density on $\Lambda(\Gamma)$ with the base point at o. For convex cocompact Γ, we can take the $\delta(\Gamma)$-dimensional Hausdorff measure on $\Lambda(\Gamma)$ as we have mentioned above. Define a Riemannian metric g on $\Omega(\Gamma)$ by

$$g(\xi) = \left(\int_{\Lambda(\Gamma)} \left(\frac{1}{2}|\xi - \eta|\right)^{-\delta} d\mu(\eta) \right)^{2/\delta} g_0(\xi), \quad \xi \in \Omega(\Gamma),$$

where $|\xi - \eta|$ is the Euclidean distance between $\xi, \eta \in S^n \subset \mathbb{R}^{n+1}$. Then this metric is Γ-invariant (see [17, §2]). We denote the metric on $\Omega(\Gamma)/\Gamma$ induced from this metric by g_Γ. Now we consider the case $\delta(\Gamma) = 0$. This occurs if and only if Γ is an elementary group, a group whose limit set consists of at most two points. Since we concentrate our attention to convex cocompact Kleinian groups, $\delta(\Gamma) = 0$ implies that the limit set consists of exactly two points and $\Omega(\Gamma)/\Gamma$ is conformally covered by $\mathbb{R} \times S^{n-1}$. Such manifolds admit compatible metrics which come from the standard product Riemannian metric on $\mathbb{R} \times S^{n-1}$. In what follows, in the case of $\delta(\Gamma) = 0$, we mean by a manifold with Nayatani's metric a quotient of $\mathbb{R} \times S^{n-1}$ with the standard product metric. We note that this metric has positive scalar curvature under the assumption $n \geq 3$.

This metric g_Γ has some nice properties and should be considered as a distinguished representative of a Kleinian structure (see [17] for the detail). What we need in this paper is the following.

Proposition 3.2 ([17, Corollary 3.4]). *Let $\Omega^n(\Gamma)/\Gamma$ be a compact Kleinian manifold with $\delta(\Gamma) > 0$. If $\Omega^n(\Gamma)/\Gamma$ is not covered by a torus T^n, then g_Γ has positive (resp. zero, resp. negative) scalar curvature if and only if $\delta(\Gamma) < (n-2)/2$ (resp. $\delta(\Gamma) = (n-2)/2$, resp. $\delta(\Gamma) > (n-2)/2$).*

We remark here that if Γ is convex cocompact, $\Omega(\Gamma)/\Gamma$ is not covered by T^n.

4. Proof of theorems

This section is devoted to the proof of Theorems 1, 2, 3, and 4. The basic ingredient of the proofs are explained in the preceding sections and [7, §§2–4]. For the (virtual) cohomological dimension of Kleinian groups, see [7, §4]. See also references cited there. After assuming them, the proofs are quite simple.

Proof of Theorem 1. Since $\Gamma = \rho(G)$ is convex cocompact, we can extend the action of Γ to S^N through a series of totally geodesic embeddings $S^n \hookrightarrow S^{n+1} \hookrightarrow \cdots \hookrightarrow S^N$ so that $\Omega^N(\Gamma)/\Gamma$ is compact ($N \geq n$) as explained in section 2 of [7]. We denote the new Kleinian group (this is also convex cocompact) by the same symbol Γ and the corresponding representation by $\rho^N : G \longrightarrow \mathrm{Conf}(S^n) \subset \mathrm{Conf}(S^N)$. Take N so that $\delta(\Gamma) < (N-2)/2$ where $\delta(\Gamma)$ is the critical exponent of Γ. Since the critical exponent does not change under this extension of the action of Γ, this is possible. Clearly we may assume $N \geq 3$ and $\Omega^N(\Gamma)$ is simply connected. Since Γ is torsion-free, $\Omega^N(\Gamma)/\Gamma$ is a manifold and Nayatani's metric associated to Γ has positive scalar curvature by Proposition 3.2. Let Φ be a developing map of $(M, C) = \Omega^N(\Gamma)/\Gamma$. By a theorem of Schoen and Yau (Theorem 2.5 in the present paper), Φ is injective and (M, C) is conformal to $\Phi(\tilde{M})/(\mathrm{hol}(\Phi))(G)$ (note that $G = \pi_1(M)$ by the injectivity of Φ). Since $\Phi(\tilde{M})$ is connected and conformal to $\Omega^N(\Gamma)$, there is $\psi \in \mathrm{Conf}(S^N)$ such that $\psi(\Phi(\tilde{M})) = \Omega^N(\Gamma)$. Thus we may assume $\Phi(\tilde{M}) = \Omega^N(\Gamma)$, in particular, $(\mathrm{hol}(\Phi))(G) = \Gamma$. Define an automorphism α of G

by $\alpha = (\mathrm{hol}(\Phi))^{-1} \circ \rho^N$. It is clear that the map $\rho \mapsto \rho \circ \alpha$ defines a homeomorphism on $\mathrm{Hom}(G, \mathrm{Conf}(S^n))$. Therefore we may assume $\rho^N = \mathrm{hol}(\Phi)$ in what follows.

Take a small neighborhood W of C in $\mathcal{FC}(M)$ which can be regarded as the space of all flat conformal structures as in Remark 1 following Definition 2.1. Thus we may assume every element in W contains positive scalar curvature metric. Let $\tilde{W}$ be the inverse image of W under the quotient map $\widetilde{\mathcal{FC}}(M) \longrightarrow \mathcal{FC}(M)$. Obviously $\Phi \in \tilde{W}$. Take a neighborhood V of $\Phi \in \tilde{W}$ as in Theorem 2.3 and let $V_0 = V \cap \tilde{W}$. Since every element in $\tilde{W}$ is a developing map of a flat conformal structure containing a positive scalar curvature metric, it is injective by Theorem 2.5. In particular, the associated holonomy representation is faithful and discrete, and the domain of discontinuity of its image is exactly the image of the developing map.

Now we can take $\mathrm{hol}(V_0)$ as U in the statement of Theorem 1. In fact, as we have seen above, for any $\rho' \in \mathrm{hol}(V_0) \subset \mathrm{hol}(\tilde{W})$, ρ' is injective, and the corresponding conformally flat manifold is recovered as $\Omega^N(\rho'(G))/\rho'(G)$. Since $\Omega^N(\Gamma)/\Gamma$ is compact and diffeomorphic to $\Omega^N(\rho'(G))/\rho'(G)$, $\Omega^N(\rho'(G))/\rho'(G)$ is also compact. Recalling $\mathrm{vcd}\, G \leq n < N$, we see that $\rho'(G)$ is convex cocompact by [7, Proposition 4.9]. Also since both Φ and Φ' are injective, we have a diffeomorphism $\Phi' \circ \Phi^{-1} : \Omega^N(\Gamma) \longrightarrow \Omega^N(\rho'(\Gamma))$. Since both groups are convex cocompact, this map can be extended to a quasiconformal mapping defined on the whole S^N by [27]. Let φ be $\Phi' \circ \Phi^{-1}$. Then, for any $\gamma \in G$,

$$\begin{aligned} \Phi' \circ \gamma &= \varphi \circ \Phi \circ \gamma = \varphi \circ \rho(\gamma) \circ \Phi \\ &= \varphi \circ \rho(\gamma) \circ (\varphi^{-1} \circ \varphi) \circ \Phi \\ &= \varphi \circ \rho(\gamma) \circ \varphi^{-1} \circ \Phi'. \end{aligned}$$

Thus we have $\rho'(G) = \varphi\rho(G)\varphi^{-1}$.

Since we have been looking at Γ acting on S^N, to complete the proof, we have to show φ descends to S^n. Since $\mathrm{Hom}(G, \mathrm{Conf}(S^n)) \longrightarrow \mathrm{Hom}(G, \mathrm{Conf}(S^N))$ induced from $S^n \longrightarrow S^N$ is easily seen to be an embedding, in particular a homeomorphism onto its image, $U \cap \mathrm{Hom}(G, \mathrm{Conf}(S^n))$ is open in $\mathrm{Hom}(G, \mathrm{Conf}(S^n))$. Let this open set be U in our statement. Let $\varphi = D(\mathrm{hol}(\Phi')) \circ \Phi^{-1}$, by taking $D(\mathrm{hol}(\Phi'))$ instead of Φ' in the construction above. By Lemma 2.2 (2), we see this map preserves $S^n \subset S^N$. The extension of this map φ to the whole S^n given as above is a desired conjugation.

Note that the restriction $\varphi : \Omega^n(\rho(G)) \longrightarrow \Omega^n(\rho'(G))$ is a smooth map by our construction. Recall the extension of the action of Γ has been done by means of a series of totally geodesic embeddings $S^n \hookrightarrow S^{n+1} \hookrightarrow \cdots \hookrightarrow S^N$. By our proof of Lemma 2.2 (2), we see that φ preserves each $S^m, n \leq m \leq N$. The action of Γ on $\mathbb{H}^{n+1}$ is conjugate to that on a hemisphere of S^{n+1}; φ actually gives the conjugation not only on S^n but on $\mathbb{H}^{n+1}$. This completes the proof. ∎

Remark 3. In the proof above, we have used the torsion-free assumption to apply the holonomy theorem (Theorem 2.3), for the assumption implies $\Omega^N(\Gamma)/\Gamma$ is a manifold.

Proof of Theorem 2. First note that if a developing map Φ of a flat conformal structure C on a compact manifold M is injective, then the image $\Phi(\tilde{M})$ is a connected component of the domain of discontinuity of the Kleinian group $\rho(\pi_1(M))$, where $\rho = \mathrm{hol}(\Phi)$. Therefore, under the assumption (2) in Theorem 2, $\rho(\pi_1(M))$ is convex cocompact and its domain of discontinuity is connected and coincides with $\Phi(\tilde{M})$ by [7, Proposition 4.9]. Consequently (M, C) is a convex cocompact Kleinian manifold. Thus what we have to prove in what follows is the openness and closedness of the set of flat conformal structures with injective developing map under our assumptions. The closedness is clear; since if a sequence of injective maps converges to a local diffeomorphism with respect to compact-C^∞ topology, the limit is clearly an injective map. Now we turn to the openness of the set. First assume $\pi_1(M)$ is torsion-free. Let N be as in the proof of Theorem 1. We denote by M^N the smooth manifold diffeomorphic to $\Omega^N(\rho^N(\pi_1(M)))/\rho^N(\pi_1(M))$. Let $[\Phi] \in \tilde{\mathcal{T}}_C(M)$ be the element represented by Φ. Since hol is a local homeomorphism, there is a neighborhood U of $[\Phi]$ on which hol' is homeomorphic. Since an embedding $S^n \longrightarrow S^N$ whose image is a round sphere induces an embedding $\mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^n)) \longrightarrow \mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^N))$, by taking a composition with $\mathrm{hol}' : \tilde{\mathcal{T}}_C(M) \longrightarrow \mathrm{Hom}(\pi_1(M), \mathrm{Conf}(S^n))$ and taking U sufficiently small if necessary, we have an embedding of U into $\tilde{\mathcal{T}}_C(M^N)$. We can define this embedding so that $[\Phi]$ corresponds to the element in $\tilde{\mathcal{T}}_C(M^N)$ represented by $\Omega^N(\rho^N(\pi_1(M)))/\rho^N(\pi_1(M))$. Then the proof of Theorem 1 shows every element in U is represented by a developing map of the form $\varphi \circ \Phi$. Since Φ has been assumed to be injective and φ restricted to S^n is a homeomorphism, it is injective. Now suppose $\pi_1(M)$ has a torsion element. Since $\rho = \mathrm{hol}(\Phi)$ is injective, $\rho(\pi_1(M))$ is isomorphic to $\pi_1(M)$ and finitely generated. Thus, by Selberg's lemma, there is a torsion-free finite index subgroup $G_0 \in \pi_1(M)$. Let $(M_0, C_0) = \Phi(\tilde{M})/\rho(G_0)$. Clearly M_0 is a compact manifold and the discussion above can be applied. On the other hand there is an obvious injective continuous map $\widetilde{\mathcal{F}C}(M) \longrightarrow \widetilde{\mathcal{F}C}(M_0)$ and it is easy to see this induces a continuous map $f : \tilde{\mathcal{T}}_C(M) \longrightarrow \tilde{\mathcal{T}}_C(M_0)$. Let $U' \subset \tilde{\mathcal{T}}_C(M_0)$ be U in the torsion-free case above. Then $f^{-1}(U)$ is open and a neigborhood of $[\Phi]$. Since every element in U is represented by an injective developing map, so is true for $f^{-1}(U)$. Thus in both cases we have shown the openness of the set in $\tilde{\mathcal{T}}_C(M)$. Since $\mathrm{Conf}(S^n)$ and $\mathrm{Diff}(M)/\mathrm{Diff}_C(M)$ act homeomorphically on $\tilde{\mathcal{T}}_C(M)$ and the action preserves the property that represented by injective developing maps, recalling $\mathcal{M}(M)$ is equipped with the quotient topology, we see the same is true for $\mathcal{M}(M)$. This completes the proof. □

Proof of Theorem 3. From Theorem 2 and Lemma 2.4, we know there is a continuous open map hol'' from $\mathcal{T}_0$ into $R(\pi_1(M), \mathrm{Conf}(S^n))$. Thus the remaining is to prove the injectivity of this map. Suppose $\mathrm{hol}''([C]) = \mathrm{hol}''([C'])$ for $[C], [C'] \in \mathcal{T}_0$. By taking suitable representatives $\Phi,\ \Phi' \in \widetilde{\mathcal{F}C}(M)$, we may assume $\mathrm{hol}(\Phi) = \mathrm{hol}(\Phi')$. Now Φ and Φ' are both injective by Theorem 2 and our choice of $\mathcal{T}_0$. Thus we have a diffeomorphism $(\Phi')^{-1} \circ \Phi$ of $\tilde{M}$, which must commute with the deck transformation group. Therefore Φ and Φ' lie in the same orbit of $\widetilde{\mathrm{Diff}_C}(M)$ in $\widetilde{\mathcal{F}C}(M)$. This completes the proof. □

On the space $\mathcal{T}_C(M)$, there is a complete distance defined in the same way as the Teichmüller distance on the Teichmüller space (see [8] for the detail). Let $\mathrm{Homeo}_C(M)$ be the group of homeomorphisms on M which has a lift to $\tilde{M}$ commuting all the deck transformations. Denote by $K(\varphi; C_1, C_2)$ the maximal dilatation of a quasiconformal mapping $\varphi : (M, C_1) \longrightarrow (M, C_2)$.

Definition 4.1. The *Teichmüller pseudodistance* d on $\mathcal{T}_C(M)$ is defined by

$$\mathrm{dist}([C_1], [C_2]) = \log \inf K(\varphi; C_1, C_2)$$

for $[C_1], [C_2] \in \mathcal{T}_C(M)$ represented by C_1 and C_2 respectively. The infimum above is taken over all qc mappings in $\mathrm{Homeo}_C(M)$. If this becomes a distance, we call it the *Teichmüller distance* on $\mathcal{T}_C(M)$. If M is compact, this turns out to be a complete distance on $\mathcal{T}_C(M)$ and the topology defined by dist coincides with the quotient topology coming from $\widetilde{\mathcal{F}C}(M)$.

Remark 4. If M is compact, the infimum can be replaced by the minimum. In other words, there exists a quasiconformal mapping satisfying $\log K(\varphi; C_1, C_2) = \mathrm{dist}(C_1, C_2)$. See [8, §7]. This also follows from a result in [11].

First note that the critical exponent is invariant under the conjugate action of $\mathrm{Conf}(S^n)$. Therefore we have a function δ' on a subset of $R(G, \mathrm{Conf}(S^n))$ consisting of equivalence classes of faithful discrete representation which is induced from the function $\Gamma \mapsto \delta(\Gamma)$. Let M, $\mathcal{T}_0$ be as in Theorem 2. Then through the embedding given by Theorem 3, we have the function $\delta'' : \mathcal{T}_0 \longrightarrow \mathbb{R}$ defined by $\delta''([C]) = \delta'(\mathrm{hol}''([C]))$.

Theorem 4.2. *Let M, $\mathcal{T}_0$ be as in Theorem 2. Then the function δ'' on $\mathcal{T}_0$ is uniformly continuous with respect to the Teichmüller distance.*

Proof. The proof is basically the same as [19, Theorem 5]. Recall that for convex cocompact Kleinian groups the critical exponent and the Hausdorff dimension of the limit set agrees. Let $[C_j] \to [C]$ with respect to dist. Then there is a sequence $\{\varphi_j\}$ of quasiconformal mappings such that $\log K(\varphi; C_j, C) = \mathrm{dist}([C_j], [C]) \to 0$. For each C_j, $(M, C_j) \cong \Omega(\Gamma_j)/\Gamma_j$ for some convex cocompact Γ_j. Now φ_j is lifted to a map $\Omega(\Gamma_j) \longrightarrow \Omega(\Gamma)$ and extended to a quasiconformal mapping defined on the whole S^n with the same maximal dilatation by [27]. By taking a suitable conjugate of Γ_j, we may assume each φ_j fixes three common points. Then the proof is straightforward from the uniform Hölder estimate for such quasiconformal mappings in [5] and the definition of the Hausdorff dimension. □

Proof of Theorem 4. (1) This is clear from Theorem 1.

(2) We have only to prove the continuity of the critical exponent around a given representation $\rho \in \mathrm{Hom}(G, \mathrm{Conf}(S^n))$ whose image $\Gamma = \rho(G)$ is convex cocompact Kleinian group. Take N as in the proof of Theorem 1. Then $\Omega^N(\Gamma)/\Gamma$ satisfies the assumption of Theorem 2. Therefore around $\rho^N \in \mathrm{Hom}(G, \mathrm{Conf}(S^N))$ the critical exponent is continuous by Theorem 3 and Theorem 4.2 above. On the other hand, $\mathrm{Hom}(G, S^n) \longrightarrow \mathrm{Hom}(G, S^N)$ is an embedding as we have mentioned in the proof of Theorem 1, we see the critical exponent is also continuous around $\rho \in \mathrm{Hom}(G, \mathrm{Conf}(S^n))$. □

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