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A UNIQUENESS THEOREM FOR HARMONIC FUNCTIONS ON THE UPPER-HALF PLANE

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ABSTRACT. Consider harmonic functions on the upper-half plane $R^2_+ = \{(x,y)|\ y > 0\}$ satisfying the boundary condition $u_y = -\exp(u)$ and the constraint $\int_{R^2_+} \exp(2u) < \infty$. We prove that all such functions are of form (1.2) below.

1. INTRODUCTION

Let R^2_+ be the upper-half plane $\{(x,y)\,|\,y>0\}$. Consider the problem

$$\left\{\begin{array}{c} \Delta u = 0 \ \text{ in } \ y>0, \\ u_y = -\exp(u) \ \text{ on } \ y=0, \\ \int_{R^2_+} \exp(2u) < \infty. \end{array}\right. \tag{1.1}$$

Here we assume that $u(x,y)$ is a C^2 function on the upper-half plane down to the boundary. One easily verifies that functions of form

$$u(x,y) = \ln\Big(\frac{2y_1}{(x-x_1)^2+(y+y_1)^2}\Big), \tag{1.2}$$

where x_1 is any real number and y_1 is any positive number, are solutions of (1.1). These solutions are in fact fundamental solutions of the Laplacian equation with a singularity at $(x_1, -y_1)$ on the lower-half plane.

Problem (1.1) comes out of a geometric context. Suppose u is a solution of (1.1). Then

$$g_{ij} = \exp(2u)\delta_{ij} \quad \text{for } \ i,j = 1,2$$

defines a Riemannian metric on the upper-half plane that is conformal to the Euclidean metric. In this metric, the Gaussian curvature is still zero everywhere (cf. [A]); however, the geodesic curvature of the boundary $y=0$, with x as parameter, is one at every point. The integral

$$\int_{R^2_+} \exp(2u)$$

is the area of the upper-half plane. Essentially, we prove that the metric is in fact induced by a stereographic projection from the upper-half plane to a unit circle.

There is a counterpart of (1.1) in higher dimensions. There we consider positive harmonic functions on the upper-half space satisfying $u_{x_n} = -u^{n/(n-2)} (n \geq 3)$

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on the boundary $x_n = 0$. These functions also serve as the ones that attain a best constant in the Sobolev trace embedding theorem, (cf. [E]). The uniqueness theorem in higher dimensions has been established by Chipot et al [CSF], by Li and Zhu [LZ], and by the author [O] using different methods. The method used by the author in [O] does not extend to the plane. Fortunately, here we show that the classical complex analysis can be used to make up what was left out.

2. An Analytic Function

Let $z = x+iy$. We will identify z with (x, y). Let u be a solution of (1.1) and let v be a conjugate of u satisfying the Cauchy-Riemann equations $u_x = v_y, \quad u_y = -v_x$. Define an analytic function by

$$f(z) = \int_0^z \exp(u + iv)dz. \tag{2.1}$$

We have

Lemma 2.1. *The function $f(z)$ satisfies*

(1) $|f'(z)| = \exp(u(z)) \neq 0$ *for all z on the upper-half plane including the boundary;*

(2) $f(z)$ *maps the boundary $y = 0$ into a unit circle and the curvature of the planar curve $\{f(x, 0) \mid -\infty < x < \infty\}$, with x as parameter, is one for each x; and*

(3) $\int_{R^2_+} |f'(z)|^2 = \int_{R^2_+} \exp(2u) < \infty.$

Remark. The function $f(z)$ arises from the following geometric consideration. Let $I = \exp(2u)(dx^2 + dy^2)$ and $II = 0$. Then I and II are the first and second fundamental forms of a parametric surface in R^3 because the Gauss-Codazzi equations are satisfied (cf. any textbook on differential geometry). In addition, by $II = 0$ the surface is planar. Thus I and II define a conformal mapping from R^2_+ to a plane in R^3 and indeed $f(z)$ in (2.1) is essentially such a mapping if we understand the complex plane as a subset of the three dimensional space.

Proof. We only need to verify the second statement by calculating the curvature of the planar curve $\{f(x, 0) \mid -\infty < x < \infty\}$ with x as parameter. The curvature equals

$$\begin{aligned}
& \frac{\mathrm{Im}(\overline{f'(z)}f''(z))}{|f'(z)|^3} \quad (z = x = (x, 0)) \\
= \ & \mathrm{Im}\,\frac{\exp(u(z) - iv(z))\exp(u(z) + iv(z))(u_x + iv_x)}{\exp(3u(z))} \\
= \ & \mathrm{Im}\,\frac{1}{\exp(u(z))}(u_x + iv_x) \\
= \ & \frac{1}{\exp(u(z))}(v_x) = \frac{1}{\exp(u(z))}(-u_y) = 1.
\end{aligned}$$

□

3. The Uniqueness

We will determine u from the following uniqueness theorem on an analytic function f satisfying the three conditions in Lemma 2.1. In the proof of Theorem 3.1

below, we use the well-known Picard's Little Theorem and Picard's Great Theorem on several occasions. We refer to [K] particularly for a proof of Picard's theorems using a geometric analysis technique.

Let C be the unit circle that contains $\{f(x,0) \mid -\infty < x < \infty\}$, let D be the disk with C as the boundary, and let w_0 be the center of D.

Theorem 3.1. *The function $f(z)$ in Lemma 2.1 is of the form*

$$f(z) = w_0 + e^{i\theta}\frac{z - z_1}{z - \overline{z_1}}, \tag{3.1}$$

where θ is a real number and $z_1 = x_1 + iy_1$ is a point on the upper-half plane.

Proof. Let $\Omega = f(R^2_+)$, the range set of f on R^2_+. For every $w \in \Omega$, let $\chi(w)$ be the number of the z in R^2_+ such that $f(z) = w$; that is, $\chi(w)$ is the number of times f on R^2_+ takes on the value w. At the end we will see that Ω is nothing but D and $\chi(w)$ is identically equal to one for every w in Ω. At present, however, it is only apparent that Ω is an open set because $f'(z) \neq 0$ everywhere and thus f is locally one-to-one. Next, we have that

$$\int_\Omega \chi(w) = \int_{R^2_+} |f'(z)|^2. \tag{3.2}$$

The derivation of the above equality is as follows. Let $\chi_R(w)$ be the number of times f on $R^2_+ \cap B_R$ takes on the value w. Then $\chi_R(w)$ is piecewise constant and finite-valued. The equality

$$\int_\Omega \chi_R(w) = \int_{R^2_+ \cap B_R} |f'(z)|^2$$

is simply the result of a change of variables in the integration. Noting that $\chi_R(w) \to \chi(w)$ from below and monotonically as $R \to \infty$, we have (3.2).

By $\int_{R^2_+} |f'(z)|^2 < \infty$ and (3.2), we know that $\chi(w)$ is finite almost everywhere on Ω.

There are three possible cases and we examine them one by one: 1) $w_0 \notin D$, 2) $w_0 \in D$ and $\chi(w_0)$ is finite, and 3) $w_0 \in D$ and $\chi(w_0)$ is infinite.

Case I. w_0, the center of D, is not in Ω.

We prove that this case in fact cannot happen. Suppose there is such an f. Define the analytic extension

$$g(z) = \begin{cases} f(z) - w_0 & \text{if } z \in \overline{R^2_+}, \\ \frac{1}{\overline{f(\bar{z})} - \overline{w_0}} & \text{if } z \in R^2_-. \end{cases} \tag{3.3}$$

Clearly, $g(z)$ is an entire function. Moreover, because $\chi(w) < \infty$ for almost every w in Ω and thus $f(z)$ takes on almost every value finite times, $g(z)$ also takes on almost every value finite times. The Picard Little Theorem tells us that $g(z)$ must have a pole at infinity, for otherwise $g(z)$ takes on every value, with one possible exception, infinite times. That is, $g(z)$ must be a polynomial. But no polynomial maps the real line to a circle. We have come to the desired contradiction.

Case II. $w_0 \in \Omega$ and $\chi(w_0) < \infty$.

Let $z_1, \ldots, z_n$ be all the z in R^2_+ satisfying $f(z) = w_0$. Again define $g(z)$ as in (3.3). We note that this time $g(z)$ has simple roots at $z_1, \ldots, z_n$ and simple poles at

$\overline{z_1}, \ldots, \overline{z_n}$. Again, $g(z)$ takes on almost every value finite times. The Picard Great Theorem tells us that $g(z)$ cannot have ∞ as an essential singular point. Thus $g(z)$ must be a rational function and it follows that

$$g(z) = e^{i\theta} \frac{z - z_1}{z - \overline{z_1}} \cdots \frac{z - z_n}{z - \overline{z_n}}$$

for a real number θ. Furthermore, we show that n must be one.

On the one side, we observe that $g'(z) = f'(z)$ on the upper-half plane and

$$g'(z) = -\overline{f'(\bar{z})}/(\overline{f(\bar{z})} - \overline{w_0})^2$$

on the lower-half plane. Thus $g'(z)$ has no roots and has only $\overline{z_1}, \ldots, \overline{z_n}$ as poles. On the other side, we show that $g'(z)$ must have a root if $n \geq 2$. For this purpose, let $P_1(z) = (z - z_1) \ldots (z - z_n) = z^n + a_{n-1} z^{n-1} + \ldots + a_0$ and let $P_2(z) = \overline{P_1(\bar{z})} = (z - \overline{z_1}) \ldots (z - \overline{z_n}) = z^n + \overline{a_{n-1}} z^{n-1} + \ldots + \overline{a_0}$. Then

$$\begin{aligned} g(z) &= e^{i\theta} \frac{z^n + a_{n-1} z^{n-1} + \cdots + a_0}{z^n + \overline{a_{n-1}} z^{n-1} + \cdots + \overline{a_0}} \\ &= e^{i\theta} (1 + \frac{c_m z^m + \cdots + c_0}{z^n + \overline{a_{n-1}} z^{n-1} + \cdots + \overline{a_0}}), \end{aligned}$$

where $0 \leq m < n$ and $c_m \neq 0$. It's simple to have that

$$g'(z) = e^{i\theta} \frac{c_m (m - n) z^{m+n-1} + \cdots}{(z^n + \overline{a_{n-1}} z^{n-1} + \cdots + \overline{a_0})^2}$$

where the terms omitted in the numerator have powers less than $m + n - 1$. Should n be greater than or equal to two, $m + n - 1 \geq 1$. Necessarily, the numerator would have at least a root. Furthermore, because this numerator also equals $P_1'(z) P_2(z) - P_2'(z) P_1(z)$, which does not vanish at any of the $z_1, \ldots, z_n, \overline{z_1}, \ldots, \overline{z_n}$, we conclude that $g'(z)$ would vanish at a point should $n \geq 2$. Therefore $n = 1$ and

$$g(z) = e^{i\theta} \frac{z - z_1}{z - \overline{z_1}}, \tag{3.4}$$

where θ is a real number and z_1 is a point on the upper-half plane.

Case III. $w_0 \in \Omega$ and $\chi(w_0) = \infty$.

We show that this case also cannot happen. Let w_1 be another point inside D other than the center w_0 and consider

$$h(z) = w_0 + \frac{(f(z) - w_0) - (w_1 - w_0)}{1 - (\overline{w_1 - w_0})(f(z) - w_0)}.$$

Apparently, $h(z)$ also maps the real line into C. We choose w_1 so that 1) $f(z)$ takes on w_1 finite times, and 2) $f(z)$ takes on $w_0 + 1/(\overline{w_1 - w_0})$ finite times–if $f(z)$ takes on $w_0 + 1/(\overline{w_1 - w_0})$ at all. We choose w_1 this way to ensure that the function $h(z) - w_0$ have a finite number of simple zeros and poles. Such a w_1 always exists. We include here a few lines as a proof.

Consider $\chi(w) = 0$ if $w \notin \Omega$. Then $\chi(w) < \infty$ for almost every w on the whole complex plane. It follows that $\chi(w_0 + 1/(\overline{w - w_0})) < \infty$ for almost every w on the whole complex plane as well. Hence

$$\chi(w) + \chi(w_0 + \frac{1}{(\overline{w - w_0})}) < \infty$$

almost everywhere. We can always choose a w_1 such that both $\chi(w_1)$ and $\chi(w_0 + 1/(\overline{w_1 - w_0}))$ are finite.

We proceed to define the analytic extension of $h(z) - w_0$:

$$g(z) = \begin{cases} h(z) - w_0 & \text{if } z \in \overline{R^2_+}, \\ \frac{1}{\overline{h(\bar{z})} - \overline{w_0}} & \text{if } z \in R^2_- . \end{cases}$$

The function $g(z)$ also has a finite number of simple zeros and poles on the whole complex plane. Again, $g(z)$ takes on almost every value finite times. The Picard Great Theorem says that $g(z)$ cannot have ∞ as an essential singular point. Thus $g(z)$ must be a rational function, but then $f(z)$ must be a rational function too, which led to $\chi(w_0) < \infty$, a contradiction to the assumption for the case.

In summary, we conclude that only Case II with $n = 1$ may happen and from (3.4),

$$f(z) = w_0 + e^{i\theta} \frac{z - z_1}{z - \overline{z_1}}.$$

□

It follows from Theorem 3.1 that a solution of (1.1) satisfies

$$\begin{aligned} u(x, y) &= u(z) = \ln |f'(z)| = \ln |\frac{2y_1}{(z - \overline{z_1})^2}| \\ &= \ln \frac{2y_1}{(x - x_1)^2 + (y + y_1)^2}, \end{aligned}$$

where $z_1 = x_1 + iy_1$ and $y_1 > 0$.

4. A Remark

If we drop the restriction $\int_{R^2_+} \exp(2u) < \infty$ in (1.1), there are infinitely many more solutions. In fact, let $h(z)$ be any entire analytic function such that $h(z)$ takes on real values for z on the real line and $h'(z) \neq 0$ for z everywhere (for examples, $\exp(z)$, $\exp(\exp(z))$). Then $u(z) = \ln |f'(z)|$ with

$$f(z) = \frac{h(z) - i}{h(z) + i}$$

is such a solution of (1.1). Particularly, the analytic function

$$f(z) = \frac{e^z - i}{e^z + i}$$

maps the real line to a half circle. The harmonic function $u(x, y)$ from this specific $f(z)$ is not of form (1.2) but satisfies the boundary condition $u_y = -\exp(u)$ and

$$\int_{-\infty}^{\infty} \exp(u(x, 0))dx = \int_{-\infty}^{\infty} |f'(x, 0)|dx = \pi < \infty,$$

and of course the integral $\int_{R^2_+} \exp(2u)$ is infinite. This example also shows that the following assumption

$$\int_{-\infty}^{\infty} \exp(u(x, 0))dx < \infty$$

cannot be used to replace the finiteness of the integral of $\exp(2u)$ in (1.1).

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