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EXTENSIONS OF HOMEOMORPHISMS BETWEEN LIMBS OF THE MANDELBROT SET

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ABSTRACT. Using holomorphic surgery techniques, we construct a homeomorphism between a neighborhood of any limb without root point of the Mandelbrot set and a neighborhood of any other of equal denominator, in such a way that the limbs are mapped to each other. On the limbs, the homeomorphism coincides with that constructed in "Homeomorphisms between limbs of the Mandelbrot set" (*J. Geom. Anal.* **9** (1999), 327–390) which proves – without assuming local connectivity of the Mandelbrot set – that these maps are compatible with the embedding of the limbs in the plane. Outside the limbs, the constructed extension is quasi-conformal.

CONTENTS

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1. Introduction

Given the family of quadratic polynomials $Q_c(z) = z^2 + c$, we define the *filled Julia set* of Q_c as the set

$$K_c = \{z \in \mathbb{C} \mid \{Q_c^n(z)\}_{n\geq 0} \text{ is bounded}\},$$

and the *Julia set* J_c as the boundary of K_c. Both sets are bounded and completely invariant under Q_c. The complement of the filled Julia set is the basin of attraction of the superattracting point at infinity, which is always connected.

The polynomials Q_c have one single critical point in $\mathbb{C}$ which is $\omega = 0$. The behavior of this point plays a crucial role in determining the dynamics of Q_c and the topology of K_c. Indeed, the filled Julia set is connected if and only if it contains the critical point 0. If not, it is a Cantor set.

This dichotomy is reflected in the definition of the Mandelbrot set which is defined as follows (see Figure 1).

$$M = \{c \in \mathbb{C} \mid 0 \in K_c\}.$$

The Mandelbrot set is compact, full and connected and it is conjectured to be locally connected.

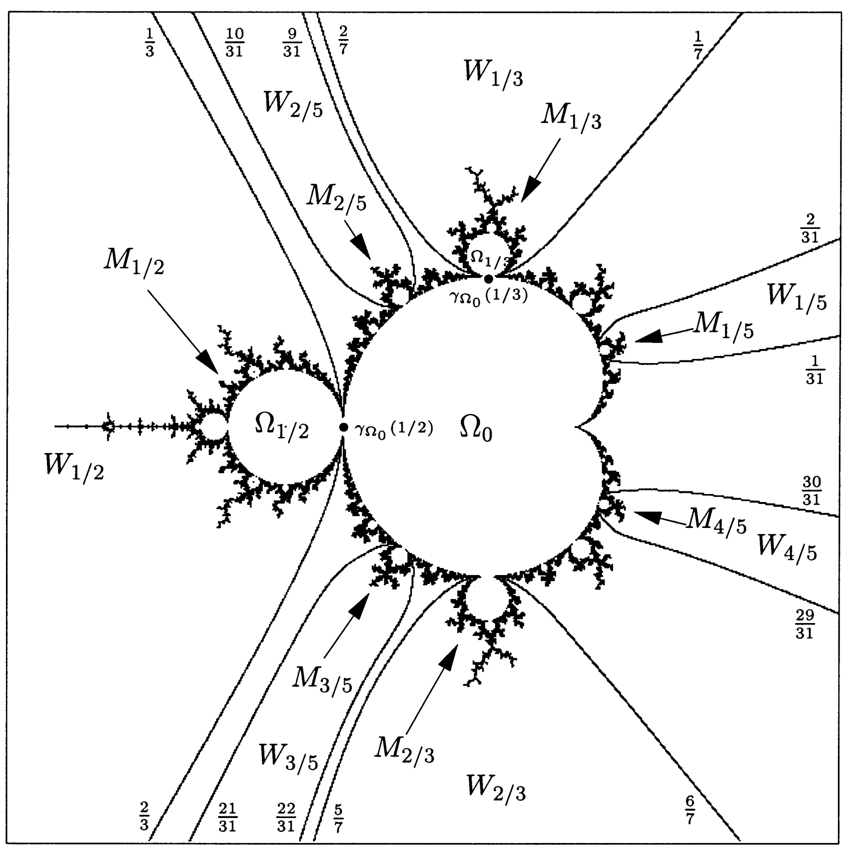

Figure 1. The boundary of the Mandelbrot set and certain wakes

The interior of M contains infinitely many connected components for which Q_c has an attracting periodic orbit. These are called *hyperbolic components* and it is conjectured that their union equals the interior of M. The boundary of each hyperbolic component Ω can be parametrized by a map $\gamma_\Omega : [0,1) \to \partial\Omega$ so that, at $c = \gamma_\Omega(t)$, the indifferent periodic orbit has multiplier $e^{2\pi i t}$. The point $c = \gamma_\Omega(0)$ is called the *root* of the hyperbolic component Ω.

The largest hyperbolic component consists of all parameter values c for which Q_c has an attracting fixed point, and we shall denote it by Ω_0. Its boundary is referred to as the main cardioid. At each boundary point $\gamma_{\Omega_0}(p/q)$, for any $p/q \in (0,1) \cap \mathbb{Q}$, there is attached a hyperbolic component $\Omega_{p/q}$ of period q.

We define the set $M^*_{p/q}$, to be the connected component of $M \setminus \overline{\Omega}_0$ attached to the main cardioid at the point $c = \gamma_{\Omega_0}(p/q)$. We then define the *$p/q$-limb* of M as $M_{p/q} = M^*_{p/q} \cup \gamma_{\Omega_0}(p/q)$ (see Figure 1). Hence the set $M^*_{p/q}$ is the limb without the root point.

In [BF], homeomorphisms between any two limbs of equal denominator were constructed. More precisely, the following theorem was proven.

Theorem ([BF])**.** *Given p/q and p'/q in $(0,1) \cap \mathbb{Q}$ and irreducible, there exists a homeomorphism*

$$\Phi^q_{pp'} : M_{p/q} \longrightarrow M_{p'/q}$$

which is holomorphic on the interior of $M_{p/q}$.

Moreover, it was shown that if the Mandelbrot set is assumed to be locally connected, then these homeomorphisms are compatible with the embeddings of the limbs in the plane, since a radial extension to the wake can be constructed. Our goal in this paper is to prove compatibility without assuming the extra hypothesis, in order to make this technique available for other parameter spaces in which local connectivity is proven to be false.

We shall prove that the embedding is preserved by extending the homeomorphisms of [BF] to a neighborhood of the limbs without root point. As it turns out, this extension will be quasi-conformal in the complement of the limb. It was pointed out to us by W. Jung that for this reason, even after MLC is proved, this new extension will be better than the radial one (see Remark 4.28). A more precise statement is as follows.

Main Theorem. *Given p/q and p'/q in $(0,1) \cap \mathbb{Q}$ and irreducible, there exist open sets $V_{p/q}$ and $V_{p'/q}$ intersecting M in $M^*_{p/q}$ and $M^*_{p'/q}$ respectively, and a homeomorphism*

$$\Lambda^q_{pp'} : V_{p/q} \longrightarrow V_{p'/q}$$

extending the homeomorphism $\Phi^q_{pp'} : M_{p/q} \to M_{p'/q}$, which is orientation preserving and quasi-conformal in $V_{p/q} \setminus M^*_{p/q}$ (see Figure 2).

Remarks 1.1.

a) Trivially, we can first restrict the domain of $\Lambda^q_{pp'}$ to obtain a map from $\widetilde{V}_{p/q} \subset W_{p/q}$ to $\widetilde{V}_{p'/q} \subset W_{p'/q}$, and then extend it quasi-conformally to the wake, thus obtaining a homeomorphism $\widetilde{\Lambda}^q_{pp'} : W_{p/q} \to W_{p'/q}$ which is quasi-conformal from $W_{p/q} \setminus M_{p/q}$ to $W_{p'/q} \setminus M_{p'/q}$.

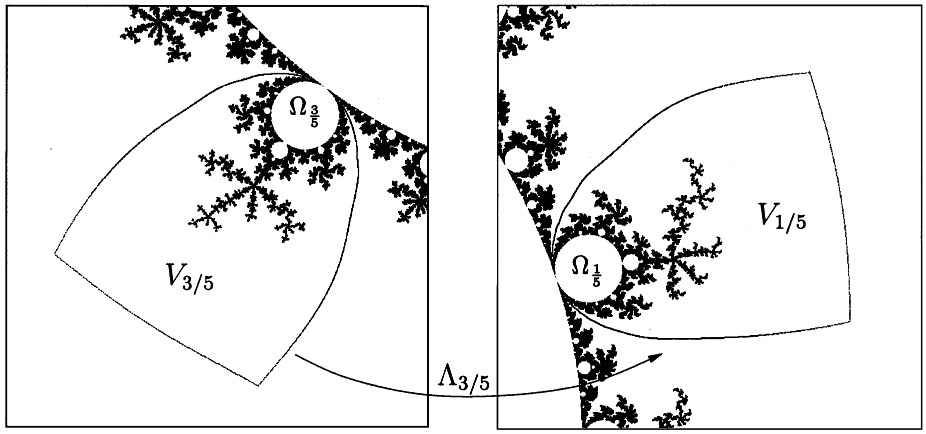

FIGURE 2. The map $\Lambda_{3/5}$

b) Moreover, Branner and Lyubich have recently announced that the homeomorphisms $\Phi^q_{pp'}$ in [BF] between limbs are quasi-conformal, after removing arbitrarily small neighborhoods of the root points. Hence the maps above are quasi-conformal from $W_{p/q}$ to $W_{p'/q}$, adjusting the domains likewise.
c) The proof of the Main Theorem immediately reduces to the case $p' = 1$. The rest of the homeomorphisms $\Lambda^q_{pp'}$ are obtained by composing the maps Λ^q_{p1} and their inverses for different values of p. To ease notation we will hereafter denote Λ^q_{p1} by $\Lambda_{p/q}$.

We will construct the extension using holomorphic surgery but, this time, we will have to deal also with polynomials with a disconnected Julia set.

An essential step in proving the bijectivity of $\Phi^q_{pp'}$ was that two polynomials that are hybrid equivalent and have a connected Julia set must also be affine conjugate. Since this is false if the Julia set is disconnected, the proof of injectivity will be completely different.

The paper is organized as follows. Section 2 contains some general preliminaries about dynamics of polynomials (which the expert reader may skip). In Section 3 we build up the necessary setup and notation to be able to give a precise statement of the Main Theorem. Section 4 is dedicated to the proof and divided into three main parts: the definition of the map in Section 4.2, the proof of continuity in Section 4.3, and the proof of injectivity in Section 4.4.

Notation. We shall denote the interior of a set A by $\text{int}(A)$ and uniform convergence on compact subsets by the symbol $\rightrightarrows$. The set $\mathbb{N}$ denotes the natural numbers $1, 2, \ldots$ without including 0.

2. PRELIMINARIES

2.1. **Dynamics of quadratic polynomials.** An essential tool to study the dynamics of complex polynomials is the map known as the Böttcher map or Böttcher parametrization. For any $c \in \mathbb{C}$ there exists a real number $\nu \geq 0$, a neighborhood

$\mathcal{U}_c$ of infinity and a unique holomorphic isomorphism tangent to the identity at infinity

$$\psi_c : \mathbb{C} \setminus \overline{\mathbb{D}}_{e^\nu} \to \mathcal{U}_c$$

which conjugates $Q_0(z) = z^2$ to the map Q_c. The map ψ_c is called the Böttcher parametrization of f around infinity. Its inverse is called the Böttcher coordinate.

If the critical point, $\omega = 0$, does not belong to the basin of infinity, and hence K_c is connected, the set $\mathcal{U}_c$ is in fact the complement of the filled Julia set and $\nu = 0$. In the case where K_c is disconnected, $\nu > 0$ can be chosen so that the critical point belongs to the boundary of $\mathcal{U}_c$ (see Figure 3). The Böttcher coordinates can be defined holomorphically past the set $\mathbb{C} \setminus \overline{\mathbb{D}}_{e^\nu}$ (see Proposition 3.2) but not globally.

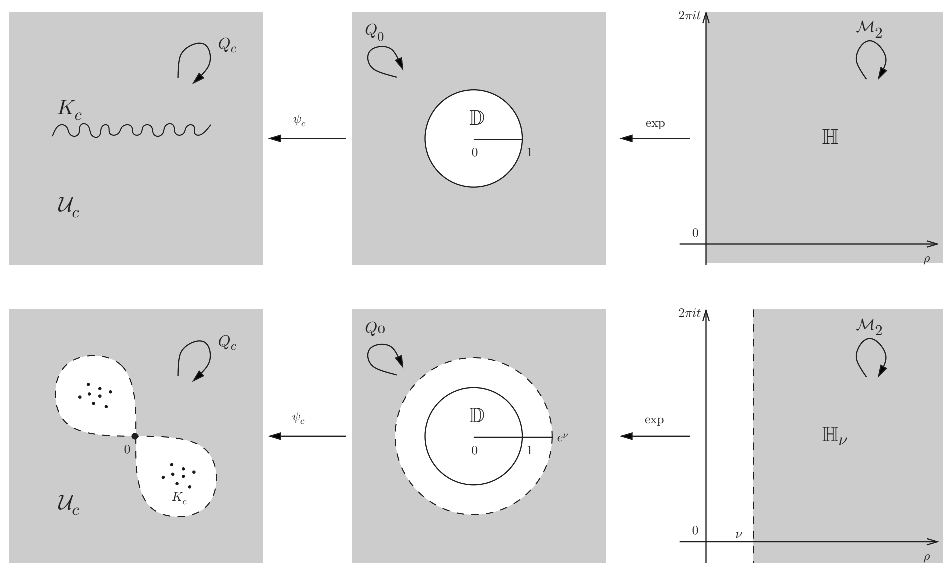

FIGURE 3. The Böttcher parametrization for both the connected and the disconnected case

We can also lift Q_0 to the map $\mathcal{M}_2(z) := 2z$ in the right half plane $\mathbb{H}$, the universal covering space.

In summary, the following diagram commutes

$$\begin{array}{ccc}
\mathbb{H}_\nu & \xrightarrow{\mathcal{M}_2} & \mathbb{H}_\nu \\
\text{exp}\downarrow & & \downarrow\text{exp} \\
\mathbb{C} \setminus \overline{\mathbb{D}}_{e^\nu} & \xrightarrow{Q_0} & \mathbb{C} \setminus \overline{\mathbb{D}}_{e^\nu} \\
\psi_c\downarrow & & \downarrow\psi_c \\
\mathbb{C} \setminus \mathcal{U}_c & \xrightarrow{Q_c} & \mathbb{C} \setminus \mathcal{U}_c
\end{array}$$

where $\mathbb{H}_\nu = \{\rho + 2\pi it \in \mathbb{H} \mid \rho > \nu\}$, and keeping in mind that $\nu = 0$ when K_c is connected.

We remark that in the case of K_c being connected and locally connected, ψ_c extends continuously to the boundary of $\mathbb{D}$, so that ψ_c is defined on $\mathbb{C} \setminus \mathbb{D}$. Even in the case when K_c is not locally connected, there is a set of points of full measure on $\partial\mathbb{D}$ where the radial extension of ψ_c is well defined. This set always includes the points with rational arguments.

The *potential* $G_c : \mathbb{C} \setminus K_c \to \mathbb{R}_+$ (Green's function) of K_c satisfies

$$\begin{cases} G_c(z) = \log\left(|\psi_c^{-1}(z)|\right) & \text{if } z \in \mathcal{U}_c, \\ G_c(z) = \frac{1}{2^n} G_c(Q_c^n(z)) & \text{if } Q_c^n(z) \in \mathcal{U}_c, \end{cases}$$

and hence $G_c(Q_c(z)) = 2G_c(z)$ for all $z \in \mathbb{C} \setminus K_c$. The potential measures the rate of escape of points under iteration of Q_c. The level sets of the potential function are called *equipotentials* (see Figure 4). Equipotentials in $\mathcal{U}_c$ are simple closed curves which correspond in the complement of $\overline{\mathbb{D}}_{e^\nu}$ to circles around the origin and on $\mathbb{H}_\nu$ to vertical lines. If K_c is connected, then all equipotentials are simple closed curves. If K_c is a Cantor set, then $\nu = G_c(0)$ and the equipotential of potential ν is a figure eight, the boundary of $\mathcal{U}_c$.

Given $t \in \mathbb{R}$ we denote by $R(t)$ the horizontal line in $\mathbb{H}$ with imaginary part equal to $2\pi t$, i.e.,

$$R(t) := \{\rho + 2\pi i t \in \mathbb{H} \mid \rho > 0\}.$$

If K_c is connected, we may transport $R(t)$ to the dynamical plane all the way. In that case, we define the *external ray of argument* t to be

$$R_c(t) = \psi_c\left(\exp(R(t))\right).$$

Note that $R_c(t)$ is an orthogonal trajectory to equipotentials.

If $R_c(t)$ has a limit when $\rho \to 0$, then it tends to a point of the Julia set which we denote by $R_c^*(t)$. We say that the ray *lands* at this point and we have

$$Q_c(R_c^*(t)) = R_c^*(2t).$$

All external rays with rational arguments land, and if K_c is locally connected, all external rays land.

If K_c is a Cantor set, we may transport $R(t)$ under $\psi_c \circ \exp$ on the part that intersects $\mathbb{H}_\nu$, obtaining a ray in $\mathcal{U}_c$. For a given $t \in \mathbb{R}$, the ray segment extends unbroken as an orthogonal trajectory to equipotentials of decreasing potential, either all the way to 0, or down to a level where it branches at the critical point 0 or an iterated preimage of 0.

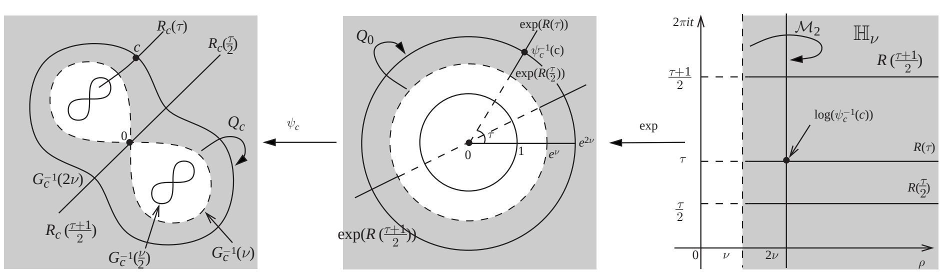

FIGURE 4. Equipotentials and external rays in a disconnected case. An argument t in the vertical axis must be interpreted as the actual value $2\pi i t$.

2.2. **The parameter plane of quadratic polynomials.** Let M denote the Mandelbrot set as defined in the introduction. The results in this section can be found in [DH1] or [Br].

The map $\phi_M : \mathbb{C} \setminus M \to \mathbb{C} \setminus \overline{\mathbb{D}}$ defined as $\psi_c^{-1}(c)$ is a conformal isomorphism. We define an *external ray of external argument* θ as

$$R_M(\theta) = \phi_M^{-1}\left(\exp(R(\theta))\right) = \phi_M^{-1}\left(\{e^{\rho+2\pi i\theta}\}_{0<\rho<\infty}\right).$$

If $R_M(\theta)$ has a limit $c \in \partial M$ when $\rho \to 0$, we say that $R_M(\theta)$ *lands* at c. It is known that all external rays with rational arguments land at either a root of a hyperbolic component or at a *Misiurewicz point*, i.e., a parameter value $c \in \partial M$ for which $\omega = 0$ is strictly preperiodic under Q_c.

There are exactly two external rays landing at each root point in M (except at $c = 1/4$). Given $p/q \in (0,1) \cap \mathbb{Q}$, we denote by $\theta^-_{p/q}$ and $\theta^+_{p/q}$ the arguments of the two external rays landing at the root point of $\Omega_{p/q}$, i.e., at $\gamma_{\Omega_0}(p/q) \in \partial\Omega_0$. Then, we define the *$p/q$-wake* of M, $W_{p/q}$, as the open subset of $\mathbb{C}$ that contains $M^*_{p/q}$ and is bounded by these two rays and $\gamma_{\Omega_0}(p/q)$ (see Figure 1).

The characterization of polynomials Q_c for which $c \in W_{p/q}$ is as follows. Consider the dynamical plane of Q_c. The polynomial has exactly two fixed points, both repelling, denoted by α_c and β_c. The fixed point β_c is the landing point of the ray $R_c(0)$. The fixed point α_c is the landing point of a periodic cycle of q rays, with combinatorial rotation number p/q. The arguments of these rays depend only on p/q and include $\theta^+_{p/q}$ and $\theta^-_{p/q}$. Moreover, these rays are unbranched, since neither the critical point nor any iterated preimage of it ever fall on them. It follows that all the preimages of these rays are also unbranched.

2.3. **Tools.** In the surgery construction we shall use the theory of quasi-conformal mappings, the Measurable Riemann Mapping Theorem, and what essentially is the theory of Polynomial-like mappings of Douady and Hubbard. For the main definitions and statements we refer to the Tools section in [BF], or to any of the original sources like [A, AB, DH2].

In this section, we point out a few important facts that we shall use when dealing with quadratic polynomials whose Julia set is disconnected.

Recall that two polynomials f and g are said to be *topologically equivalent* (or *locally topologically conjugate*) ($f \sim_{\text{top}} g$) if there exists a homeomorphism between a neighborhood of K_f and a neighborhood of K_g such that $g \circ h = h \circ f$. If the homeomorphism h can be chosen to be quasi-conformal, we say that f and g are *quasi-conformally equivalent* and denote it by $f \sim_{\text{qc}} g$. If h can be chosen so that moreover, $\overline{\partial} h = 0$ a.e. on K_f, then we say that f and g are *hybrid equivalent* and we denote it by $f \sim_{\text{hb}} g$. Finally, f and g are *holomorphically equivalent* if h is holomorphic. The strongest type of conjugacy is a *(global) holomorphic conjugacy* or *affine conjugacy* which is given by h being holomorphic and defined on all of $\mathbb{C}$ or, equivalently, affine.

Recall that the quadratic family is usually written in the form $Q_c(z) = z^2 + c$ because in this way, there is a unique representative of each affine conjugacy class. That is to say, if Q_c and $Q_{c'}$ are affine conjugate, then $c = c'$.

When dealing with polynomials Q_c with c in the Mandelbrot set, the same is true for the classes of hybrid equivalence because of the following fact.

Proposition 2.1 ([DH2]). *Let f and g be polynomials of degree $d > 1$ with K_f and K_g connected. If f and g are hybrid equivalent, then they are affine conjugate.*

But this is not true for polynomials with a disconnected Julia set. For quadratic polynomials Q_c with c outside of M we have the following.

Proposition 2.2. *All polynomials Q_c with $c \notin M$ are hybrid equivalent to each other.*

3. The Main Theorem

The goal of this section is to build up the necessary setup and notations to give a more precise statement of the Main Theorem. This setup will also be used in the proof. Throughout the section we fix $p/q \in (0,1) \cap \mathbb{Q}$ and consider polynomials Q_c with $c \in W_{p/q}$.

3.1. **In the dynamical plane.** Recall that for each $c \in W_{p/q}$, there are q rays landing at α_c.

The other preimage of α_c under Q_c is the point $\widetilde{\alpha}_c = -\alpha_c$. There are q additional rays landing at $\widetilde{\alpha}_c$, and their arguments are preimages under doubling of the arguments of the rays landing at α_c. Figure 5 shows an example of a Julia set in the 3/5-limb, together with the rays described above.

The rays landing at α_c and $\widetilde{\alpha}_c$ partition the dynamical plane into $2q-1$ closed subsets. We denote the subset containing the critical point by V_c^0, and the others by V_c^i or $\widetilde{V}_c^i = -V_c^i$ for $i = 1, 2, \dots q-1$ as shown in Figure 5. Note that these subsets have their counterparts in the right half plane, the same for all $c \in W_{p/q}$, hence we shall use the same notation but without the subscript c. For $1 \le i \le q$ we let $\theta^i \in (0,1)$ be the argument of the ray on the common boundary of V_c^{i-1} and V_c^i. In the same fashion, $\widetilde{\theta}^i$ denotes the argument of the ray $R_c(\widetilde{\theta}^i) = -R_c(\theta^i)$. Note that $R_c(\theta^i) = R_c(\widetilde{\theta}^i + 1/2)$.

Then, Q_c acts on these sets as follows:

$$\begin{array}{llll} V_c^0 & \xrightarrow{2-1} & V_c^p, & \\ V_c^i, \widetilde{V}_c^i & \xrightarrow{1-1} & V_c^{[i+p \pmod q]} & \text{for } 0 < i \le q-1, i \ne q-p, \\ V_c^{q-p}, \widetilde{V}_c^{q-p} & \xrightarrow{1-1} & V_c^0 \cup \bigcup_{i=1}^{q-1} \widetilde{V}^i. & \end{array} \tag{1}$$

We establish the following conventions: in the dynamical plane and in expressions with integer indices like $[i + p \pmod q]$ we will omit $\pmod q$, while in expressions with arguments, we will omit $\pmod 1$. In both cases, it should be understood that expressions should be taken $\pmod q$ and $\pmod 1$ respectively.

3.1.1. *Sectors.* For later purposes, we need to define some subsets which we call *sectors.* They should be viewed as neighborhoods of rays $R_c(\theta)$ that land.

Instead of viewing the sectors in the dynamical plane, it is better to think about them in the exterior of the unit disk or, even better, in the right half plane (see Figure 6).

Definition. For a fixed slope $s > 0$ we define the sector centered at $R(\theta)$ with slope s as

$$S(\theta) = S^s(\theta) = \{\rho + 2\pi i\, t \in \overline{\mathbb{H}} \mid |t - \theta| \le s\rho\}.$$

The boundary of the sector is the two half lines of slope $\pm 2\pi s$ which cross exactly at the root point of the sector $2\pi i\theta$ (see Figure 6). For any positive real $\lambda \in \mathbb{R}$, the map $\mathcal{M}_\lambda(z) = \lambda z$ maps the sector $S(\theta)$ homeomorphically and holomorphically

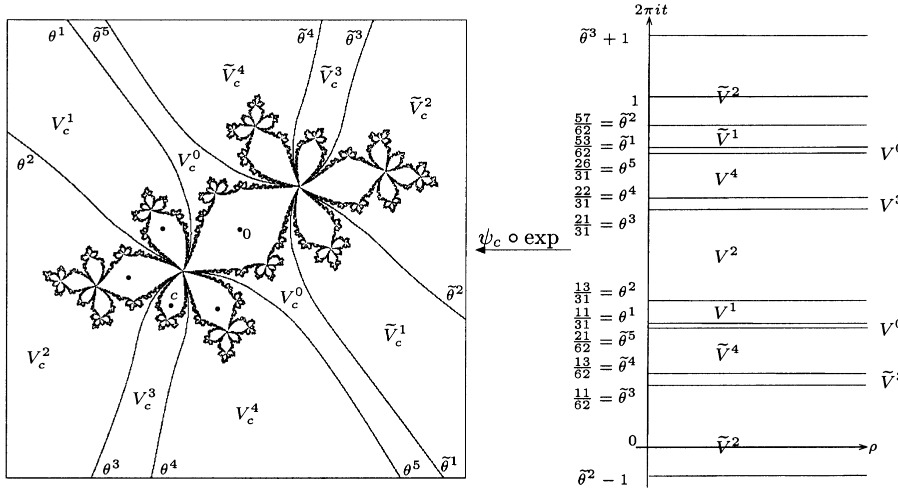

FIGURE 5. Left: the Julia set for the center of the main hyperbolic component $\Omega_{3/5}$ in $M_{3/5}$, the relevant rays and the nine subsets in the plane. Right: the partition in $\mathbb{H}$ for all $c \in W_{3/5}$, where we have marked the external arguments of the relevant rays, i.e., the relevant t-values.

onto the sector $S(\lambda\theta)$, sending a point of potential (i.e., real part) ρ to a point of potential $\lambda\rho$. Therefore, for all $\lambda \in \mathbb{R}$, the map

$$\mathcal{H}_\lambda(z) = \mathcal{H}_{\lambda,\theta}(z) = \lambda\, z - 2\pi i\theta(\lambda - 1)$$

is a homeomorphism from any sector $S(\theta)$ onto itself, mapping points of potential ρ to points of potential $\lambda\rho$. The map $\mathcal{H}_\lambda$ is multiplication by λ with respect to the root point of the sector.

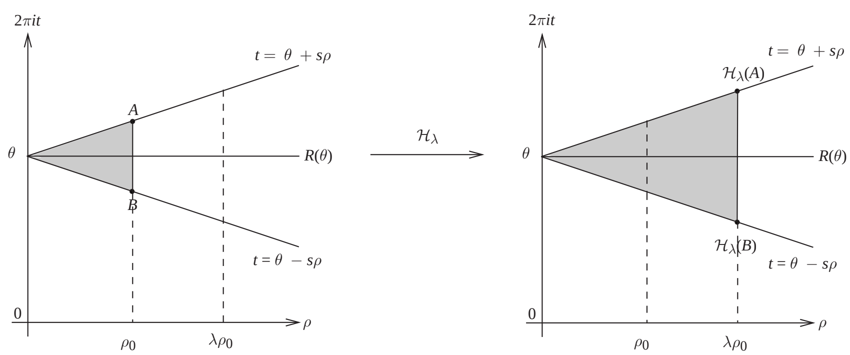

FIGURE 6. A sector and the homeomorphisms $\mathcal{H}_\lambda$

Note that, as they are defined, any two sectors in $\mathbb{H}$ overlap. To avoid overlapping of relevant sectors, we choose an arbitrary but fixed value $\eta > 0$ and set

$$S_n(\theta) = S_n^{\eta,s}(\theta) = \{\rho + 2\pi it \in S^s(\theta) \mid \rho \leq \frac{\eta}{2^n}\},$$

where $n \in \mathbb{N} \cup \{0\}$. We are interested in the sectors around the rays that land at the fixed point α_c and its symmetrical point $\widetilde{\alpha}_c$, and iterated preimages of these (see Figure 7 for an example). We set

$$S = S(\theta^1) \cup \cdots \cup S(\theta^q),$$

$$\widetilde{S} = S(\widetilde{\theta}^1) \cup \cdots \cup S(\widetilde{\theta}^q).$$

The following proposition assures that the restricted sectors do not overlap, if the slope s is chosen sufficiently small (see Figure 7). We refer to [BF] for the proof.

Proposition 3.1. *Fix* $\eta > 0$ *and* $0 < s < \frac{1}{2\eta(2^q-1)}$. *The sectors*

$$S_0^s(\theta^i),\ 1 \le i \le q \quad \textit{and} \quad S_n^s(\theta), \quad n \in \mathbb{N}$$

are all disjoint, where $2^n\theta = \theta^j$ *for some* $1 \le j \le q$.

A sector, as defined in the right half plane, can be transported to a sector in the dynamical plane *if the map* ψ_c *is well defined on* $\exp(S^s(\theta))$. In that case we define

$$S_c(\theta) = S_c^s(\theta) = \psi_c(\exp(S_0^s(\theta))),$$

which is a neighborhood of the ray $R_c(\theta)$ in the dynamical plane.

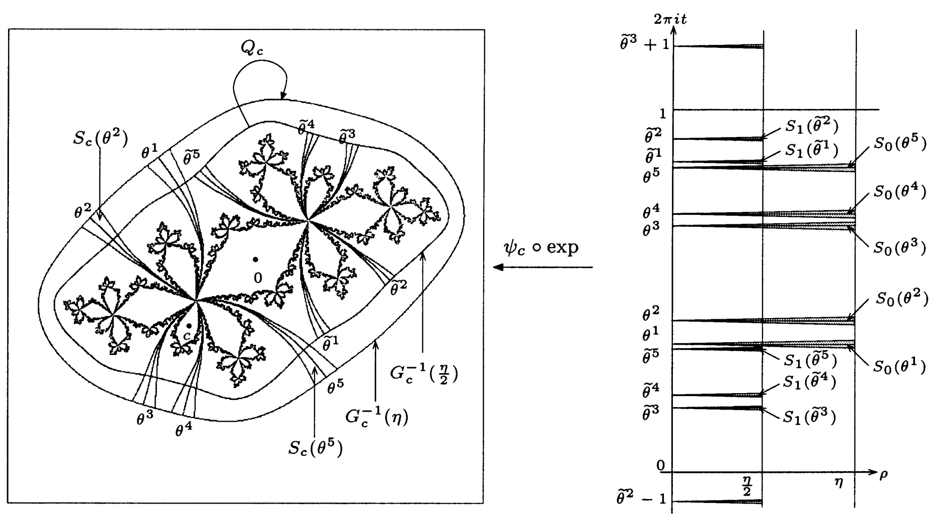

FIGURE 7. Examples of relevant sectors in the right half plane for $c \in M_{3/5}$ and their correspondents in the dynamical plane. 0-sectors and 1-sectors have been drawn, with slope $s < \frac{1}{2\eta(2^q-1)}$ (with $q = 5$).

In order to have sectors around the rays landing at α_c always well defined in dynamical plane for $c \in W_{p/q}$, no matter if the Julia set is connected or not, we shall restrict c-values to a neighborhood of the p/q-limb without root point. To find out what the appropriate restriction is, we need to study the Böttcher coordinates further.

3.1.2. *Slits.* In this section we want to make precise what the maximal domain of the Böttcher coordinates is.

Definition. Let τ and ν be such that $\nu = G_c(0)$ and $\log(\psi_c^{-1}(c)) = 2\nu + 2\pi i\tau$, where we have chosen the branch of the logarithm for which $0 \le \tau < 1$. A *critical slit in* $\mathbb{H}$ is any iterated preimage under doubling of the horizontal segments $\{\rho + 2\pi i(\tau + m) \mid 0 < \rho \le 2\nu, \quad m \in \mathbb{Z}\}$. More precisely, the critical slits are the horizontal segments (see Figure 8) of the form

$$\{\rho + 2\pi i(\frac{\tau + m}{2^n} + k) \quad | \quad 0 < \rho \le \frac{2\nu}{2^n}, \quad 0 \le m < 2^n, \quad n \in \mathbb{N}, k \in \mathbb{Z}\}.$$

The *critical slits* in $\mathbb{C} \setminus K_c$ are the union of the singular points of the vector-field $\operatorname{grad} G_c$ and their stable manifolds. Equivalently, these correspond to the preimages under the polynomial Q_c of the ray segment of argument τ and potential less than 2ν; if τ is periodic of period k under doubling, then take iterated preimages of the ray segment of argument τ and potential between $2\nu/2^k$ and 2ν. Critical slits in the dynamical plane correspond to critical slits in the right half plane.

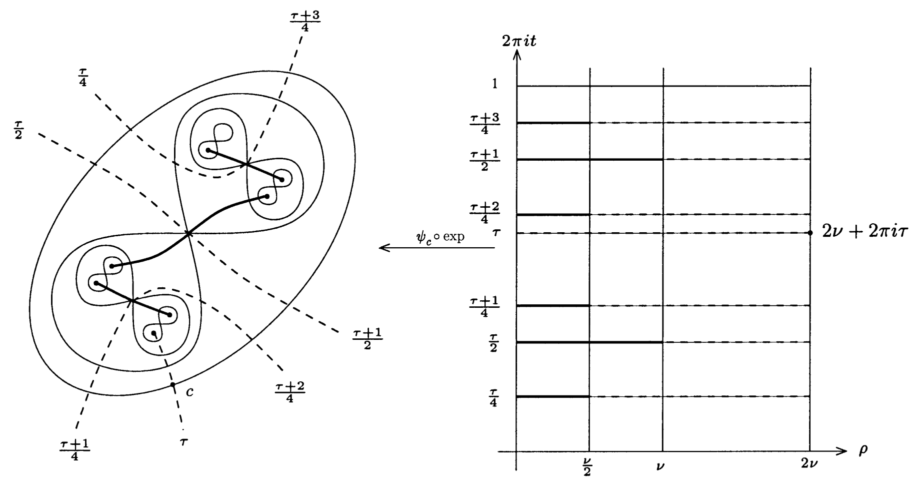

FIGURE 8. Critical slits in the dynamical plane and in the right half plane in a case where τ is not periodic under doubling

Proposition 3.2. *Let $\mathbb{C}_c^*$ denote the plane minus the closed unit disk after removing all the critical slits according to the chosen c-value. Likewise, let $\mathbb{H}_c^*$ (respectively $(\mathbb{C} \setminus K_c)^*$) be the right half plane $\mathbb{H}$ (resp. $(\mathbb{C} \setminus K_c)$) after removing all the critical slits and their translates by $2\pi i\mathbb{Z}$. Then, the map $\psi_c : \mathbb{C} \setminus \overline{\mathbb{D}}_{e^\nu} \to \mathcal{U}_c$ extends to a conformal isomorphism*

$$\psi_c : \mathbb{C}_c^* \longrightarrow (\mathbb{C} \setminus K_c)^*$$

conjugating Q_0 to Q_c. Hence, the map $\psi_c \circ \exp : \mathbb{H}_\nu \to \mathcal{U}_c$ extends to a conformal map

$$\psi_c \circ \exp : \mathbb{H}_c^* \longrightarrow (\mathbb{C} \setminus K_c)^*$$

conjugating the doubling map to Q_c.

Proof (Idea). The extension of ψ_c is obtained inductively through successive lifts. The construction is similar to the extension of the Böttcher map in a neighborhood of infinity to the set $\mathcal{U}_c$. Let $k \in \mathbb{N} \cup \{0\}$ be given and assume that

$$\psi_c : \mathbb{C}_c^* \cap (\mathbb{C} \setminus \overline{\mathbb{D}}_{e^{\nu/2^k}}) \to \{z \in (\mathbb{C} \setminus K_c)^* | G_c(z) > \nu/2^k\}$$

is a conformal isomorphism conjugating Q_0 to Q_c. Then we obtain the extension to

$$\psi_c : \mathbb{C}_c^* \cap (\mathbb{C} \setminus \overline{\mathbb{D}}_{e^{\nu/2^{k+1}}}) \to \{z \in (\mathbb{C} \setminus K_c)^* | G_c(z) > \nu/2^{k+1}\}$$

as the lift of $\psi_c \circ Q_0$ which extends ψ_c. □

3.2. **In the parameter plane.** Our goal in this section is to make sure that by restricting the c-values of $W_{p/q}$ appropriately, we can have the Böttcher coordinates always well defined on the relevant sectors. In this way, we shall be able to work with the sectors on the right half plane, independently of the value of c in the (restricted) domain.

We define

$$S_M(\theta) = S_M^s(\theta) = \phi_M^{-1}(\exp(S(\theta))),$$

which is a neighborhood of the ray $R_M(\theta)$ in the parameter plane. Let $\theta^{\pm}_{p/q}$ be the arguments of the two rays landing at the root point of the limb $M_{p/q}$ (observe that $\theta^-_{p/q} = \theta^p$ and $\theta^-_{p/q} = \theta^{p+1}$).

Definition. Given $\eta > 0$ and $s < \frac{1}{2\eta(2^q-1)}$ we define the set (see Figure 9)

$$W^{\eta,s}_{p/q} = \{c \in W_{p/q} \mid c \notin S^s_M(\theta^{\pm}_{p/q}) \text{ and } G_M(c) < \eta\}.$$

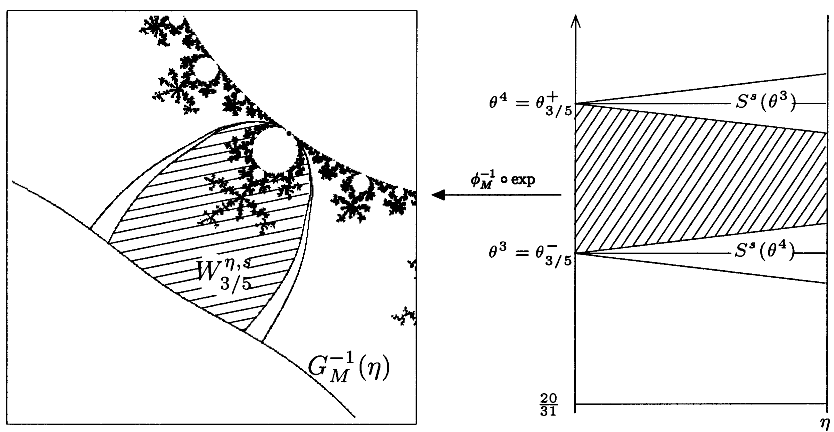

FIGURE 9. Sketch of the neighborhood $W^{\eta,s}_{3/5}$ of the limb $M_{3/5}$ and the correspondent of $W^{\eta,s}_{3/5} \setminus M_{3/5}$ in the right half plane

The main proposition is as follows.

Proposition 3.3. *If $c \in W^{\eta,s}_{p/q}$, then sectors in S and $\widetilde{S}$ are contained in $\mathbb{H}^*$. Hence, they project to sets S_c and $\widetilde{S}_c$ under $(\psi_c \circ \exp)$ so that sectors around the rays landing at α_c and $\widetilde{\alpha}_c$ are well defined.*

Proof. There is nothing to prove if $c \in M^*_{p/q}$. Hence, assume $c \in W^{\eta,s}_{p/q} \setminus M_{p/q}$. From the hypothesis $c \in W^{\eta,s}_{p/q}$, it follows directly that $\log(\phi_M(c)) \notin S^s(\theta^{\pm}_{p/q})$ and hence, $\log(\phi_M(c))$ cannot belong to any sector in S or $\widetilde{S}$. Therefore no preimage under doubling of $\log(\phi_M(c))$ can belong to any of these sets, since $\mathcal{M}_2$ maps $\widetilde{S}$ to S, and S to itself (up to vertical translation). It then follows that no critical slit can intersect S or $\widetilde{S}$. □

3.3. **Statement.** We are now ready to give a precise statement of the Main Theorem. For technical reasons (to be explained later in Section 4.2.1) we set $\eta' = 2^{(q-1)(1-1/p)}\eta$ and $s(\eta) = \frac{1}{2\eta'(2^q-1)}$. Since $\eta' > \eta$, we have $s(\eta) < \frac{1}{2\eta(2^q-1)}$.

Main Theorem. *Let $p/q \in (0,1) \cap \mathbb{Q}$. Then, for any $\eta > 0$ and any slope $s < s(\eta)$, there exists an injective map*

$$\Lambda_{p/q} : W^{\eta,s}_{p/q} \longrightarrow \mathbb{C}$$

such that,

- *it is a homeomorphism onto its image, which is an open set containing $M^*_{p/q}$;*
- $\Lambda_{p/q}|_{M_{p/q}} \equiv \Phi^q_{p1}$*; hence it is a homeomorphism between both limbs, holomorphic in the interior, and*
- *the map is quasi-conformal on $W^{\eta,s}_{p/q} \setminus M_{p/q}$.*

Therefore, this map is an extension of the homeomorphisms Φ^q_{p1} in [BF]. It follows without assuming local connectivity of M, that the homeomorphism $\Phi^q_{p1} : M_{p/q} \to M_{1/q}$ is compatible with the embedding of the limbs in the plane, and the same holds for $\Phi^q_{pp'} : M_{p/q} \to M_{p'/q}$.

4. Proof of the Main Theorem

4.1. **Idea of the proof.** We start with a quadratic polynomial Q_c with c in the p/q-wake. Without leaving the plane and using this polynomial, we define a new map g_c which presents the combinatorial properties of a quadratic polynomial in the $1/q$-wake. This new map is holomorphic everywhere except on the rays landing at α_c and $\widetilde{\alpha}_c$, where it is not even continuous.

To fix this problem, we choose some sectors around these rays (in the complement of the filled Julia set) and we define a new map f_c which is quasi-regular and equals g_c everywhere outside the sectors. This construction is done (up to where it is possible) on the right half plane (conveniently restricted), and brought back to the dynamical plane by means of the Böttcher parametrization. Hence, the necessary choices are made, once and for all, for all values of c. These choices are made in a very special way to obtain the following crucial fact: *although f_c is only quasi-regular, its q^{th} iterate f^q_c is holomorphic on the sectors.*

Up to this point, f_c is only defined on a topological disk X'_c which contains the Julia set. Moreover f_c maps X'_c to another topological disk X_c which contains X'_c compactly.

In other instances of surgery (for example in [BD] or [BF]), at this point one would construct an invariant almost-complex structure and integrate it to obtain a polynomial-like mapping conjugate to f_c. After that, the Straightening Theorem would be applied. In this proof, we shall do both steps at once.

Hence, as in the proof of the Straightening Theorem, the next step consists of extending f_c to a globally defined map F_c which is quasi-regular and conjugate to

$z \mapsto z^2$ on a neighborhood of infinity (precisely on $\mathbb{C} \setminus X_c$). We then construct an almost-complex structure σ_c on $\widehat{\mathbb{C}}$ that is invariant under F_c. It is at this point where a difficulty arises: we cannot apply the Shishikura Principle [S] which requires the map to be holomorphic everywhere except on regions where the orbits pass at most once. Indeed, orbits pass an unbounded number of times through the sectors where the map F_c is not holomorphic. Hence, it seems a priori that any invariant complex structure would have an unbounded dilatation ratio on these sectors. However, this problem is eliminated by using the crucial fact mentioned above: the q^{th} iterate F_c^q is holomorphic on the sectors. Therefore, the principle can be applied to F_c^q.

We finally apply the Measurable Riemann Mapping Theorem to integrate σ_c and obtain a quadratic polynomial $Q_{\Lambda(c)}$ conjugate to F_c.

This process provides the definition of the map $\Lambda_{p/q} : W^{\eta,s}_{p/q} \to \mathbb{C}$ as $\Lambda_{p/q}(c) = \Lambda(c)$. In Section 4.3, we prove that this map is continuous and that it is an extension of the map Φ^q_{p1} in [BF]. In Section 4.4, we show that it is injective and quasi-conformal outside the limb.

4.2. **Definition of $\Lambda_{p/q}$.**

4.2.1. *The combinatorial construction.* In this section, we start with a quadratic polynomial in the p/q-wake, and we construct a new map g_c which exhibits the combinatorial properties of a quadratic polynomial in the $1/q$-wake. This new map is holomorphic everywhere, except on the rays landing at α_c and $\widetilde{\alpha}_c$, where it has a shift discontinuity. We also define a topological disk, whose boundary is made of pieces of equipotential curves joined along these rays, such that g_c maps this disk outside itself (except for some pieces on these external rays).

Let p/q, θ^i, $\widetilde{\theta}^i$ for $i = 1, \dots, q$, V^i and $\widetilde{V}^i$ for $i = 0, 1, \dots, q-1$ be as in Section 3.1. We first establish some combinatorial facts and then proceed to define the new map.

Definition of $n[i]$. For $1 \le i \le q-1$, we define $n[i]$ to be the smallest positive integer such that

$$n[i]\, p \equiv i \pmod q.$$

We set also $n[0] = 0$ and $n[q] = q$.

Dynamically, $n[i]$ is the number of iterates of the quadratic polynomial Q_c that are necessary to map V_c^0 to V_c^i, for $1 \le i \le q-1$. Observe that $1 \le n[i] \le q-1$ and that $n[i]$ only depends on p/q. The set $\{n[0], n[1], \dots, n[q]\}$ is a permutation of the set $\{0, 1, \dots, q\}$.

Definition of $k[i]$. For $0 \le i \le q-1$, we define

$$k[i] = n[i+1] - n[i].$$

Note that $\sum_{i=0}^{q-1} k[i] = n[q] - n[0] = q$. Suppose $0 < i \le q-2$. Dynamically, if $k[i]$ is positive, it coincides with the number of iterates of Q_c needed to map V_c^i to V_c^{i+1} injectively. If $k[i]$ is negative, we need $|k[i]|$ iterates of Q_c to map V_c^{i+1} onto

V_c^i injectively. Hence, for $0 \le i \le q-2$ we have

$$\begin{array}{rcccll} Q_c^{k[0]} & : & V_c^0 & \xrightarrow{2-1} & V_c^1, & \\ Q_c^{k[i]} & : & V_c^i & \xrightarrow{1-1} & V_c^{i+1} & \text{if } 1 \le i \le q-2 \text{ and } k[i] > 0, \\ (Q_c^{-k[i]}|_{V_c^{i+1}})^{-1} & : & V_c^i & \xrightarrow{1-1} & V_c^{i+1} & \text{if } 1 \le i \le q-2 \text{ and } k[i] < 0, \\ Q_c^{k[q-1]} & : & V_c^{q-1} & \xrightarrow{1-1} & \bigcup_{i=1}^{q-1} \widetilde{V}_c^i \cup V_c^0. & \end{array}$$

Lemma 4.1. *For all* $0 \le i \le q-1$,

$$k[i] = \begin{cases} n[1] & \textit{if } k[i] > 0, \\ n[1]-q & \textit{if } k[i] < 0. \end{cases}$$

Proof. The set $\{n[0], n[1], \ldots, n[q]\}$ is a permutation of $\{0, 1, \ldots, q\}$, hence there is a unique element in $\{0, p, 2p, \ldots, (q-1)p\}$ which is congruent to each $0 \le i \le q-1$. The same is true for $\{-(q-1)p, \ldots, -2p, -p, 0\}$ since, for each $0 \le i \le q-1$, we have $-(q-n[i])p \equiv i \pmod q$.

We now subtract the equalities

$$\begin{aligned} n[i+1]p &= i+1+nq, \\ n[i]p &= i+mq, \end{aligned}$$

obtaining $(n[i+1]-n[i])p = 1+(n-m)q$. Hence, $k[i]p \equiv 1 \pmod q$. However, $-(q+1) \le k[i] \le q-1$. Therefore $k[i]$ equals $n[1]$ or $-(q-n[1]) = n[1]-q$. □

We observe that we have the symmetry $n[j]+n[q-j] = q$ for all $j = 0, \ldots, q$. Hence, $k[j-1] = k[q-j]$ and, in particular,

$$k[q-1] = k[0] = n[1].$$

It will be useful also to observe the following property.

Lemma 4.2. *For all* $0 \le i \le q$,

$$n[i]p \le i + (p-1)q.$$

Proof. We know that $n[i]p \equiv i \pmod q$. Hence $n[i]p = i + nq$ for some $n \in \mathbb{Z}$. Assume the lemma is false, i.e., $n > p-1$. Then, $n \ge p$ and $n[i]p \ge i + pq$, but this is a contradiction since $n[i]p \in \{0, p, 2p, \ldots, (q-1)p\}$. □

We now proceed to define the map g_c. Essentially, $g_c := Q_c^{k[i]}$ on V_c^i. More precisely,

Definition. On the complement of the set of rays that land at α_c and $\widetilde{\alpha}_c$ we define the map g_c to be

$$g_c(z) = \begin{cases} Q_c^{n[1]}(z) & \text{if } z \in V_c^i \text{ and } k[i] > 0,\ i = 0, \ldots, q-1, \\ (Q_c^{q-n[1]}|_{V_c^{i+1}})^{-1}(z) & \text{if } z \in V_c^i \text{ and } k[i] < 0,\ i = 1, \ldots, q-2, \\ g_c(-z) & \text{if } z \in \widetilde{V}_c^i,\ i = 1, \ldots, q-1. \end{cases}$$

By the remarks above, it follows that

$$\begin{array}{rclcll} g_c(V_c^i) & = & g_c(\widetilde{V}_c^i) & = & V_c^{i+1} & \text{for } 1 \le i \le q-2, \\ g_c(V_c^{q-1}) & = & g_c(\widetilde{V}_c^{q-1}) & = & \bigcup_{i=1}^{q-1} \widetilde{V}_c^i \cup V_c^0. & \end{array}$$

Hence we observe that, combinatorially, the dynamics of g_c are those of a quadratic polynomial in the $1/q$–wake. Moreover, g_c maps K_c to itself continuously, and is holomorphic in the interior of K_c.

Remark 4.3. Observe that points with a finite orbit (periodic or preperiodic) under Q_c are still points with a finite orbit under g_c. If Q_c has an attracting cycle, then g_c must also have an attracting cycle. Moreover, one can check that $g_c^q = Q_c^q$.

Clearly, this map needs to be modified since it is not continuous on the set of rays that land either at α_c or $\widetilde{\alpha}_c$ (although it is holomorphic everywhere else). We will now study these shift discontinuities in more detail.

Given any Jordan curve γ we denote by $B(\gamma)$ the bounded connected component of $\mathbb{C} \setminus \gamma$.

Keeping in mind that our goal is to obtain a polynomial–like mapping, we want to start by defining, for a given $\sigma > 0$, two simple closed curves $\widehat{\gamma}'_c = \widehat{\gamma}'_{\sigma,c}$ and $\widehat{\gamma}_c = \widehat{\gamma}_{\sigma,c}$, made out of pieces of equipotentials joined along rays, such that

(1) $g_c(\widehat{\gamma}'_c) = \widehat{\gamma}_c$, and
(2) $B(\widehat{\gamma}'_c) \subset B(\widehat{\gamma}_c)$.

We call $\sigma_0, \dots \sigma_{q-1}$ (resp. $\sigma'_0, \dots \sigma'_{q-1}$) the potential of $\widehat{\gamma}_c$ (resp. $\widehat{\gamma}'_c$) on $V_c^0, \dots, V_c^{q-1}$.

These potentials are not easy to find since the map g_c is a forward iterate of the polynomial on some regions while in others is a backward one. As a consequence, we cannot take $\widehat{\gamma}_c$ to be an equipotential curve and obtain that its preimage under g_c will be contained inside $B(\widehat{\gamma}_c)$. Neither is it possible to construct these curves out of pieces of equipotentials of potential $2^n\sigma$ for $n \in \mathbb{Z}$. In between two equipotential curves of potential σ and 2σ respectively, we will consider others of potential

$$2^{\frac{1}{p}}\sigma, 2^{\frac{2}{p}}\sigma, \dots, 2^{\frac{p-1}{p}}\sigma$$

and also these ones multiplied by 2, 2^2, etc., up to a maximum of $\sigma_0 = 2^{q-\frac{q-1}{p}}\sigma$. The idea for choosing the numbers σ_i and σ'_i is as follows. Set $\sigma'_0 = \sigma$. We know that, to map V_c^i to V_c^{i+1} we move $k[i]$ (whole) potential levels up or down, depending on $k[i]$ being positive or negative. This forces $\sigma_1 = 2^{k[0]}\sigma$. We take, by choice, $\sigma'_1 = 2^{-1/p}\sigma_1 = 2^{k[0]-1/p}\sigma$ and this again forces $\sigma_2 = 2^{k[0]+k[1]-1/p}\sigma$. As before we take by choice $\sigma'_2 = 2^{-1/p}\sigma_2$ and continue this procedure until we arrive at

$$\sigma'_{q-1} = 2^{k[0]+\cdots+k[q-2]-\frac{q-1}{p}}\sigma = 2^{n[q-1]-\frac{q-1}{p}}\sigma$$

and hence

$$\sigma_0 = 2^{k[q-1]}\sigma'_{q-1} = 2^{k[0]+\cdots+k[q-1]-\frac{q-1}{p}}\sigma = 2^{q-\frac{q-1}{p}}\sigma,$$

where we have used that $k[0] + \cdots + k[q-1] = q$.

We summarize this process in the following proposition (see Figure 10).

Proposition 4.4. *Given $\sigma > 0$, let $\widehat{\gamma}'_c$ be the curve made of pieces of equipotential curves (joined along rays) of potential*

$$\sigma'_i = 2^{n[i]-\frac{i}{p}}\sigma, \quad \text{on } V_c^i \cup \widetilde{V}_c^i, \text{ for } 0 \le i \le q-1.$$

Let $\widehat{\gamma}_c$ be the curve made of pieces of equipotential curves (joined along rays) of potential

$$\begin{aligned} \sigma_0 &= 2^{q-\frac{q-1}{p}}\sigma && \text{on } V_c^0 \cup \textstyle\bigcup_{i=1}^{q-1} \widetilde{V}_c^i, \\ \sigma_i &= 2^{k[i-1]}\sigma'_{i-1} && \\ &= 2^{n[i]-\frac{i-1}{p}}\sigma && \text{on } V_c^i, \text{ for } 1 \le i \le q-1. \end{aligned}$$

Then,

(1) *$g_c(\widehat{\gamma}'_c) = \widehat{\gamma}_c$, and*
(2) *$B(\widehat{\gamma}'_c) \subset B(\widehat{\gamma}_c)$.*

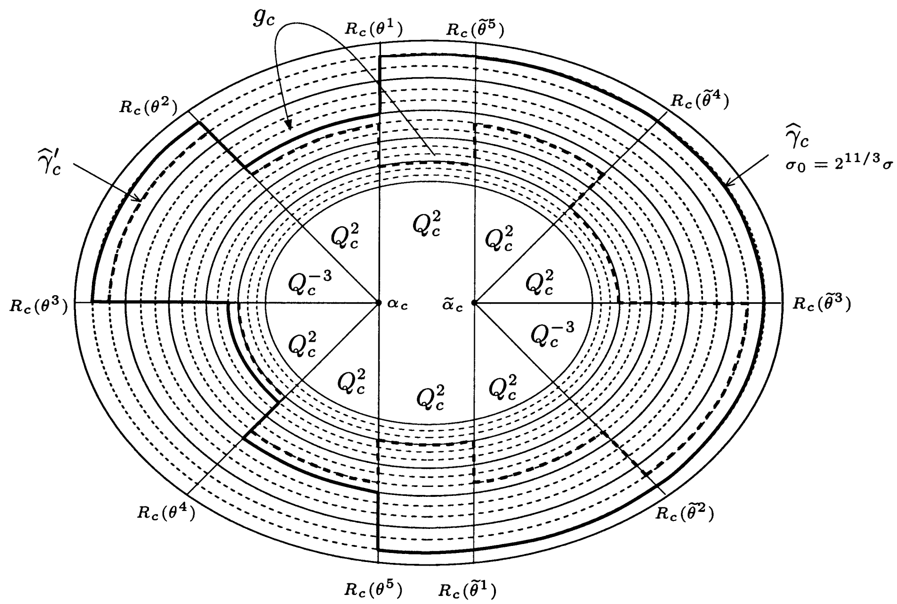

FIGURE 10. Sketch of the curves $\widehat{\gamma}_c$ (full-drawn) and $\widehat{\gamma}_c'$ (dotted) and the map g_c for a $c \in W_{3/5}$. The equipotentials drawn are of level $2^{k/3}\sigma$ where $-3 \le k \le 12$. For $i = 1, \dots, q-1$ the level $\sigma_i = 2^{1/3}\sigma_i'$.

Proof. Statement (1) is clear by construction.

For the sets $V_c^1, \dots, V_c^{q-1}$, statement (2) is clear from the definition. To prove it for $V_c^0 \cup \bigcup_{i=1}^{q-1} \widetilde{V}_c$, we need to show that $\sigma_i' < \sigma_0$ for any $0 \le i \le q-1$, i.e.,

$$\sigma_i' < 2^{q-\frac{q-1}{p}}\sigma, \quad \text{for all } 0 \le i \le q-1 .$$

We start with $i = 0$. Since $\sigma_0' = \sigma$, we only need to show that $q - \frac{q-1}{p} > 0$, or equivalently $\frac{1}{p}(q(p-1)+1) > 0$ which is clear since $p \ge 1$.

For $1 \le i \le q-1$ we must show

$$n[i] - \frac{i}{p} < q - \frac{q-1}{p},$$

or equivalently rearranging terms,

$$n[i]p - i - pq + q - 1 < 0.$$

From $n[i]p \le i + (p-1)q$ (Lemma 4.2) it follows that

$$n[i]p - i - pq + q - 1 \le i + (p-1)q - i - pq + q - 1 = -1 < 0$$

and we are done. □

In Proposition 4.4 we refer to an arbitrary $\sigma > 0$ and in Propositions 3.1 and 3.3 to an arbitrary $\eta > 0$ and a slope s bounded in terms of η. In order to have the equipotential of the critical point (the figure eight) completely contained in $B(\widehat{\gamma}_c')$ and, at the same time, ensure that a slope is chosen so that sectors do not overlap within $B(\widehat{\gamma}_c)$, we set $\eta' = \sigma_0 = 2^{q-\frac{q-1}{p}}\sigma$ and $\eta = 2\sigma$, i.e., $\eta' = 2^{(q-1)(1-\frac{1}{p})}\eta$, and choose a slope $s < \frac{1}{2\eta'(2^q-1)} = s(\eta)$.

4.2.2. *Smoothing on the right half plane.* In this section we modify the map g_c to construct a new map f_c which will be quasi-regular. The modification will be done only on the sectors around the rays where the discontinuities occur, i.e., in the sets S_c and $\widetilde{S}_c$ as defined in Section 3.1.1. Recall that, by Proposition 3.3, these are well defined sectors in the complement of the filled Julia set for all $c \in W^{\eta,s}_{p/q}$.

Since we want the entire process to vary continuously with the parameter c, we make the construction (up to where it is possible) once and for all on the right half plane $\mathbb{H}$, or rather on the cylinder $\mathbb{H}/2\pi i\mathbb{Z}$, unfolded as the infinite strip $(0,\infty)\times[0,2\pi i)$, and hence, once and for all for all values of c. Let us first redo or translate what we have done in the dynamical plane up to now, to the cylinder $\mathbb{H}/2\pi i\mathbb{Z}$ (see Figure 11).

Let V^0, V^i and $\widetilde{V}^i$ for $i=1,\dots,q-1$, denote the sets in $\mathbb{H}/2\pi i\mathbb{Z}$ corresponding to V^0_c, V^i_c and $\widetilde{V}^i_c$ respectively.

We define the map g to be as follows.

Definition. Let $(\rho, 2\pi\theta)\in\mathbb{H}/2\pi i\mathbb{Z}$ such that $0\le\theta<1$. Then,

$$g(\rho,2\pi\theta)=\begin{cases}(2^{k[i]}\rho, 2\pi(2^{k[i]}(\theta-\theta^i)+\theta^{i+1})) & \text{if } (\rho,2\pi\theta)\in V^i \text{ for } i=0,\dots,q-1,\\ g(\rho,2\pi(\theta+\frac{1}{2} \pmod 1)) & \text{if } (\rho,2\pi\theta)\in \widetilde{V}^i \text{ for } i=1,\dots,q-1.\end{cases}$$

It is easy to check that if K_c is connected, the following diagram commutes:

$$\begin{array}{ccc}\mathbb{H}/2\pi i\mathbb{Z} & \xrightarrow{g} & \mathbb{H}/2\pi i\mathbb{Z}\\ \downarrow{\scriptstyle \psi_c\circ\exp} & & \downarrow{\scriptstyle \psi_c\circ\exp}\\ \mathbb{C}_c\setminus K_c & \xrightarrow{g_c} & \mathbb{C}_c\setminus K_c\end{array}$$

If K_c is not connected, the same is true for at least all points with potential greater than the potential of $\omega=0$. Observe that g is independent of $c\in M_{p/q}$ and it is holomorphic everywhere except along those rays $R(\theta^i)\cup R(\widetilde{\theta}^i)$ for $i=1,\dots,q$ for which $k[i-1]\neq k[i]$.

In the dynamical plane, we constructed two curves $\widehat{\gamma}_c$ and $\widehat{\gamma}'_c$ made of pieces of equipotentials joined along rays, such that $B(\widehat{\gamma}'_c)\subset B(\widehat{\gamma}_c)$ and $g_c(\widehat{\gamma}'_c)=\widehat{\gamma}_c$. Following the usual notation, we denote by $\widehat{\gamma}$ and $\widehat{\gamma}'$ the corresponding curves in the cylinder. Then, $\widehat{\gamma}$ and $\widehat{\gamma}'$ are made of pieces of equipotential (vertical lines) of potentials σ_i and σ'_i as defined in Proposition 4.4. (See Figure 11 and compare with Figure 10.)

We now proceed to restrict the domain of definition of g. To that end, we shall consider sectors around the rays θ^i and $\widetilde{\theta}^i$ for $i=1,\dots,q$ and define two $\mathcal{C}^\infty$ curves γ and γ', which equal $\widehat{\gamma}$ and $\widehat{\gamma}'$ respectively, outside the sectors. That is, we will use the sectors to fix the jump discontinuities of the curves $\widehat{\gamma}$ and $\widehat{\gamma}'$. We first observe that these jump discontinuities can only be of three types. After a simple computation, one obtains the following lemma.

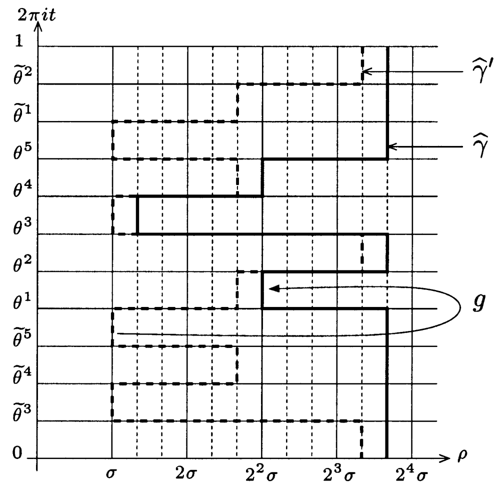

FIGURE 11. Sketch of the curves $\widehat{\gamma}$ (full-drawn) and $\widehat{\gamma}'$ (dotted) for all $c \in W_{3/5}$. The figure is drawn out of scale for clarity purposes.

Lemma 4.5. *Let σ_i and σ_i' be as in Proposition 4.4. Then, for $i = 1, \ldots, q-1$,*

$$\begin{aligned} \frac{\sigma_i'}{\sigma_{i-1}'} &= 2^{k[i-1]-\frac{1}{p}} &&= \begin{cases} 2^{n[1]-\frac{1}{p}} := 2^{J_1} & \text{if } k[i-1] > 0, \\ 2^{n[1]-q-\frac{1}{p}} := 2^{J_2} & \text{if } k[i-1] < 0, \end{cases} \\ \frac{\sigma_{i+1}}{\sigma_i} &= 2^{k[i]-\frac{1}{p}} &&= \begin{cases} 2^{n[1]-\frac{1}{p}} = 2^{J_1} & \text{if } k[i] > 0, \\ 2^{n[1]-q-\frac{1}{p}} = 2^{J_2} & \text{if } k[i] < 0, \end{cases} \\ \frac{\sigma_o'}{\sigma_{q-1}'} &= \tfrac{\sigma_1}{\sigma_0} &&= 2^{n[1]-q+\frac{q-1}{p}} := 2^{J_3}, \end{aligned}$$

where we have set $\sigma_q = \sigma_0$.

Therefore, to join the curve discontinuities we basically need three types of curves. To be more precise, let $\Sigma = \Sigma^s$ denote a *standard sector*, i.e., a sector of slope s centered at the real axis (see Figure 12). Let us choose a $\mathcal{C}^\infty$ curve, Γ_1, such that it connects the points $\sigma \cdot (1 - 2\pi i s)$ and $2^{J_1}\sigma \cdot (1 + 2\pi i s)$, and have vertical tangents at these two points. Likewise, choose Γ_2 (resp. Γ_3) joining the points $\sigma \cdot (1 - 2\pi i s)$ with $2^{J_2}\sigma \cdot (1 + 2\pi i s)$ (resp. $2^{J_3}\sigma \cdot (1 + 2\pi i s)$), and having vertical tangents at these points. Observe that for any $n \in \mathbb{Z}$, the homothecy $\mathcal{M}_{2^{n/p}}$ "translates" any of these curves to the right or to the left n/p potential levels in a holomorphic fashion. Likewise, the vertical translations $\mathcal{T}_\theta(\rho + 2\pi i t) = \rho + 2\pi i(t + \theta)$ move the curves to the sector $S(\theta)$.

Finally, we define

$$\gamma' = \begin{cases} \widehat{\gamma}' & \text{on } (\mathbb{H}/2\pi i\mathbb{Z}) \setminus (S \cup \widetilde{S}), \\ \mathcal{M}_{\frac{\sigma_{i-1}'}{\sigma}} \Gamma_1 + \mathcal{T}_\theta & \text{on } S(\theta),\ \theta \in \{\theta^i, \widetilde{\theta}^i\}, \text{ if } k[i] = n[1],\ i = 1, \ldots, q-1, \\ \mathcal{M}_{\frac{\sigma_{i-1}'}{\sigma}} \Gamma_2 + \mathcal{T}_\theta & \text{on } S(\theta),\ \theta \in \{\theta^i, \widetilde{\theta}^i\}, \text{ if } k[i] = n[1] - q,\ i = 1, \ldots, q-2, \\ \mathcal{M}_{\frac{\sigma_{q-1}'}{\sigma}} \Gamma_3 + \mathcal{T}_\theta & \text{on } S(\theta),\ \theta \in \{\theta^q, \widetilde{\theta}^q\}, \end{cases}$$

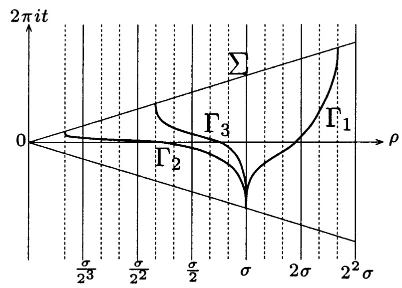

FIGURE 12. The standard sector and the curves Γ_1, Γ_2 and Γ_3, for $p/q = 3/5$. The potential lines are drawn out of scale for clarity purposes. In this case we have $J_1 = 2 - \frac{1}{3}$, $J_2 = -3 - \frac{1}{3}$ and $J_3 = -3 + \frac{4}{3} = -2 + \frac{1}{3}$.

and

$$\gamma = \begin{cases} \widehat{\gamma} & \text{on } (\mathbb{H}/2\pi i\mathbb{Z}) \setminus S, \\ \mathcal{H}_{2^{\frac{1}{p}},\theta^i}\gamma' & \text{on } S(\theta^i),\ i = 2, \ldots, q-1, \\ \mathcal{M}_{\frac{\sigma_0}{\sigma}}\Gamma_3 + \mathcal{T}_{\theta^1} & \text{on } S(\theta^1), \\ \mathcal{M}_{\frac{\sigma_{q-1}}{\sigma}}\Gamma_1 + \mathcal{T}_{\theta^q} & \text{on } S(\theta^q), \end{cases}$$

Let X and X' denote the subsets of the cylinder $\mathbb{H}/2\pi i\mathbb{Z}$ to the left of γ and γ' respectively. By construction, γ and γ' project under $\psi_c \circ \exp$ to $\mathcal{C}^\infty$ curves γ_c and γ'_c in dynamical plane such that $\overline{X'_c} \subset X_c$, where $X'_c := B(\gamma'_c)$ and $X_c := B(\gamma_c)$.

We shall modify the map g on the sectors around the rays of discontinuity and obtain a new $\mathcal{C}^1$ map $f : X' \to X$, which induces a quasi-regular map $f_c : X'_c \to X_c$. The procedure to define f on the sectors works as follows. Let us first define three types of quadrilaterals T_i, $i = 1, 2, 3$, inside a standard sector Σ^s, as the subsets of the sector bounded by the curves Γ_i, $2^{-\frac{1}{p}}\Gamma_i$ and the two line segments of the boundary of Σ^s (see Figure 13).

Set $T_i^{(0)} = T_i$ and $T_i^{(n)} = 2^{-\frac{n}{p}}T_i$. Choose a diffeomorphism from Γ_1 to Γ_2 and extend it to a diffeomorphism $\mathcal{D}^{(0)} : T_1 \to T_2$ such that $\mathcal{D}^{(0)}$ determines the same tangent map on the boundary of the sectors as the identity on the line segment with negative slope and $\mathcal{M}_{2^{J_2 - J_1}}$ on the line segment with positive slope. Moreover, we also require that

$$\mathcal{D}^{(0)} \circ \mathcal{M}_{2^{\frac{1}{p}}} = \mathcal{M}_{2^{\frac{1}{p}}} \circ \mathcal{D}^{(0)},$$

on $\mathcal{M}_{2^{\frac{-1}{p}}}(\Gamma_1)$.

Inductively, define

$$\mathcal{D}^{(n)} : T_1^{(n)} \longrightarrow T_2^{(n)}$$

such that the following diagram commutes.

$$\begin{array}{ccc} T_1^{(n-1)} & \xrightarrow{\mathcal{D}^{(n-1)}} & T_2^{(n-1)} \\ {\scriptstyle \mathcal{M}_{2^{\frac{-1}{p}}}}\downarrow & & \downarrow{\scriptstyle \mathcal{M}_{2^{\frac{-1}{p}}}} \\ T_1^{(n)} & \xrightarrow{\mathcal{D}^{(n)}} & T_2^{(n)} \end{array}$$

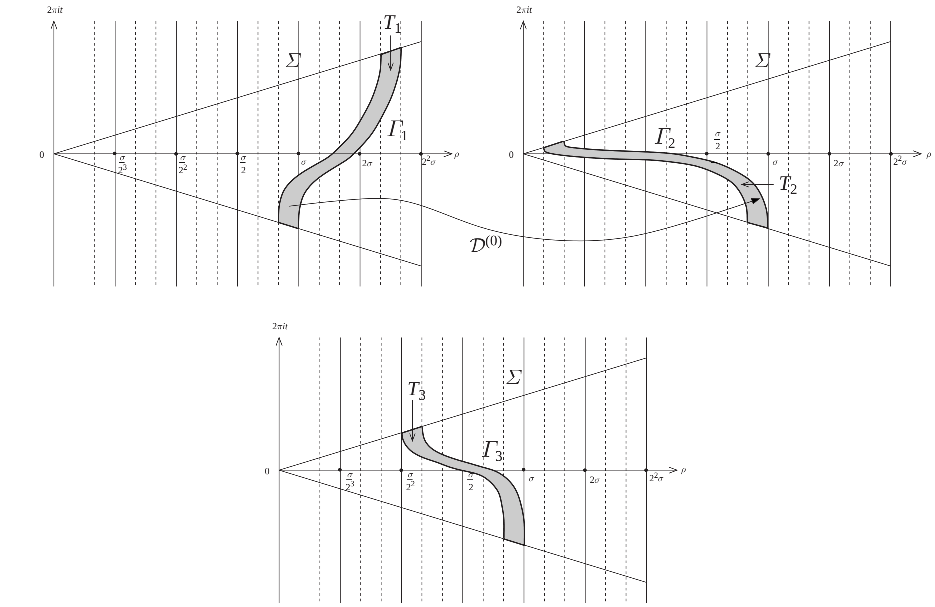

FIGURE 13. Three types of quadrilaterals and the map $\mathcal{D}^{(0)}$

Finally, set $\mathcal{D} : \Sigma^s \to \Sigma^s$ where $\mathcal{D}|_{T_1^{(n)}} = \mathcal{D}^{(n)}$.

The map $\mathcal{D} : \Sigma^s \to \Sigma^s$ is K-quasi-conformal for some constant $K > 1$. Indeed, $\mathcal{D}^{(0)}$ is a diffeomorphism on a compact set, and $\mathcal{D}$ consists of compositions of $\mathcal{D}^{(0)}$ with holomorphic maps.

Remark 4.6. Note that $\mathcal{D}$ could be defined as follows. Map the standard sector Σ by (the principal branch of) the logarithm onto a strip, symmetric around the real axis, with $|y| < \kappa$ where $\tan(\kappa) = 2\pi s$. We would choose a differentiable map $d : \log(\Gamma_1) \to \log(\Gamma_2)$ such that $d(x_1(y), y) = (x_2(y), y)$ where $(x_1(y), y) \in \log(\Gamma_1)$ and $(x_2(y), y) \in \log(\Gamma_2)$. Then, extend to the left by $d(x, y) = (x_2(y)+x-x_1(y), y)$, where (x, y) satisfies $x \leq x_1(y)$. This is a differentiable map which commutes with any horizontal translation, in particular translation by $\log(2)/p$. Set $\mathcal{D} = \exp \circ d \circ \log$, then $\mathcal{D}$ commutes with any multiplication by a real positive number, in particular multiplication by $2^{1/p}$ in Σ.

We proceed now to define f on the sectors. Abusing notation let $S(\theta)$ denote the restricted sector $S(\theta) \cap X$, and let $S'(\theta) = S(\theta) \cap X'$. We shall send each sector to the standard sector Σ by a conformal isomorphism, so that γ' is sent to Γ^1, Γ^2 or Γ^3 accordingly. We will apply $\mathcal{D}$ or $\mathcal{D}^{-1}$ and then bring the image back to fit correctly with the image sector. More precisely the procedure can be written as follows.

For $i = 1, \dots, q-1$, we define $f : S'(\theta^i) \to S(\theta^{i+1})$ as one of the following three compositions:

(a) If $k[i-1] = n[1]$ and $k[i] = n[1] - q$, then we let f be

$$S'(\theta^i) \xrightarrow{\mathcal{M}_{\frac{\sigma}{\sigma'_{i-1}}} \circ \mathcal{T}_{-\theta^i}} \Sigma \xrightarrow{\mathcal{D}} \Sigma \xrightarrow{\mathcal{T}_{\theta^{i+1}} \circ \mathcal{M}_{\frac{\sigma_i}{\sigma}}} S(\theta^{i+1}).$$

(b) If $k[i-1] = n[1] - q$ and $k[i] = n[1]$, then let f be

$$S'(\theta^i) \xrightarrow{\mathcal{M}_{\frac{\sigma}{\sigma'_{i-1}}} \circ \mathcal{T}_{-\theta^i}} \Sigma \xrightarrow{\mathcal{D}^{-1}} \Sigma \xrightarrow{\mathcal{T}_{\theta^{i+1}} \circ \mathcal{M}_{\frac{\sigma_i}{\sigma}}} S(\theta^{i+1}).$$

(c) Finally, if $k[i-1] = k[i]$, then we let f be

$$S'(\theta^i) \xrightarrow{g} S(\theta^{i+1}).$$

For $i = q$, we define $f : S'(\theta^q) \longrightarrow S(\theta^1)$ as $f \equiv g \equiv \mathcal{M}_{2^{n[1]}}$. For the sectors in $\widetilde{S}$ we define

$$f : S'(\widetilde{\theta}^i) \xrightarrow{\mathcal{T}_{\theta^i - \tilde{\theta}^i}} S'(\theta^i) \xrightarrow{f} S(\theta^{i+1}), \quad i = 1, \ldots, q-1$$

We end the definition of f by setting $f \equiv g$ everywhere outside the sectors.

The following proposition will be essential later.

Proposition 4.7. *The q-th iterate of the map f is holomorphic (wherever defined) on sectors of $S' \cup \widetilde{S}'$. In fact, $f^q = \mathcal{M}_{2^q}$ on these regions.*

Proof. For any $i = 1, \ldots, q$, the sector $S(\theta^i)$ is mapped onto itself after q iterations of f (wherever defined). At each step, the map is either holomorphic (if $k[i] = k[i-1]$), or it is basically $\mathcal{D}$ or $\mathcal{D}^{-1}$ (composed with holomorphic maps like translations or special homothecies) depending on $k[i]$ and $k[i-1]$. Since $\mathcal{D}$ commutes with $\mathcal{M}_{2^{1/p}}$, it only remains to prove that the number of times when $\mathcal{D}$ is applied equals the number of times when $\mathcal{D}^{-1}$ is applied and that the composition of the homothecies equal $\mathcal{M}_{2^q}$. If we set

$$\epsilon[i] = \begin{cases} 0 & \text{if } k[i-1] = k[i], \\ 1 & \text{if } k[i-1] > k[i], \\ -1 & \text{if } k[i-1] < k[i], \end{cases}$$

for $i = 1, \ldots, q$, this is equivalent to show that $\sum_{i=1}^{q-1} \epsilon[i] = 0$. To this end, consider the continuous piecewise linear map $k : [0, q-1] \to \mathbb{R}$ which results from joining the points $(i, k[i])$ for $i = 0, \ldots, q-1$ by a straight segment (see Figure 14). Since $k[i]$ can only take the values $n[1]$ or $n[1] - q$, every time the graph crosses the real axis with negative slope, it corresponds to a value $\epsilon[i] = 1$, while each time that it is crossed with positive slope, it corresponds to a value $\epsilon[i] = -1$. Since $k[0] = k[q-1] = n[1]$, it is clear that the graph of k has to cross the real axis the same number of times with positive slope as with negative slope. Hence, $\sum_{i=1}^{q-1} \epsilon[i] = 0$.

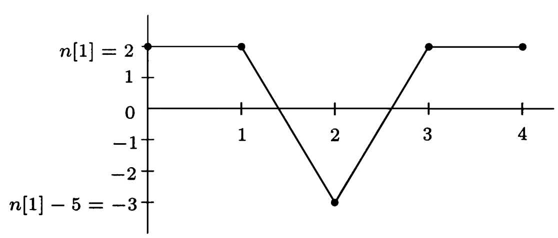

FIGURE 14. The graph of the piecewise-linear map k in the proof of Proposition 4.7 for $p/q = 3/5$

On any sector $S(\widetilde{\theta}^i)$, the map f only differs by a vertical translation from that on $S(\theta^i)$. Hence the qth iterate is also holomorphic.

To see that $f^q = \mathcal{M}_{2^q}$ on $S' \cup \widetilde{S}'$ we note that $\prod_{i=1}^{q} \mathcal{M}_{\frac{\sigma_i}{\sigma'_{i-1}}} = \mathcal{M}_{2^q}$. ∎

4.2.3. *Back to dynamical plane.* We have constructed a smooth map f on the cylinder which is a modification of g on the relevant sectors. Since we are considering values of $c \in W_{p/q}$ for which the filled Julia set might not be connected, we cannot apply the Böttcher map to simply project f to a map f_c on the complement of K_c. However, we showed in Proposition 3.3 that the Böttcher coordinates are well defined on the relevant sectors. Hence, we define $f_c : S'_c \to S_c$ as the map for which the following diagram commutes.

$$\begin{array}{ccc} S' & \xrightarrow{f} & S \\ {\scriptstyle \psi_c\circ\exp}\downarrow & & \downarrow{\scriptstyle \psi_c\circ\exp} \\ S'_c & \xrightarrow{f_c} & S_c \end{array}$$

We complete the definition of $f_c : X'_c \to X_c$ by setting

$$f_c(z) = \begin{cases} f_c(-z) & \text{if } z \in \widetilde{S}'_c, \\ g_c(z) & \text{if } z \in X'_c \setminus \operatorname{int}(S'_c \cup \widetilde{S}'_c). \end{cases}$$

Note that the two definitions coincide on the boundary of $S'_c \cup \widetilde{S}'_c$.

Remark 4.8. In fact, the diagram commutes as long as the Böttcher coordinates are well defined, in particular, down to the potential level of $\omega = 0$ (see Figure 15).

Proposition 4.9. *The map $f_c : X'_c \to X_c$ is quasi-regular.*

Proof. By construction, f_c is holomorphic on $X'_c \setminus (S'_c \cup \widetilde{S}'_c)$. On the sectors in S'_c, the map is defined as $f_c = (\psi \circ \exp) \circ f \circ (\psi_c \circ \exp)^{-1}$. Since f is K-quasi-conformal on sectors, so is f_c. This implies that f_c is also quasi-conformal on sectors in $\widetilde{S}'_c$ □

Remark 4.10. It follows from Proposition 4.7 and the definition of f_c that $f_c^q = Q_c^q$ wherever defined on $S'_c \cup \widetilde{S}'_c$.

4.2.4. *Extension of f_c to $\mathbb{C}$.* The following step is to extend f_c to a map $F_c : \mathbb{C} \to \mathbb{C}$ which is quasi-regular and conjugate to $z \mapsto z^2$ in a neighborhood of infinity (precisely in $\mathbb{C} \setminus X_c$).

For convenience, we shall from now on view the cylinder $\mathbb{H}/2\pi i\mathbb{Z}$ as the complement of the unit disk. Abusing notation, let γ, γ', X, X' and f denote the analogs to the objects with those names, now viewed on $\mathbb{C} \setminus \overline{\mathbb{D}}$ (see Figure 16). Note that X and X' are annuli with their outer boundaries included. Let A be the closed annulus bounded by γ and γ' or, equivalently, $A = X \setminus \operatorname{int}(X' \cup \overline{\mathbb{D}})$. In this model space we proceed now to extend $f : X' \to X$ to $F : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \overline{\mathbb{D}}$.

Choose $r > 1$ arbitrary and a Riemann mapping $\mathcal{R} : \widehat{\mathbb{C}} \setminus (X \cup \overline{\mathbb{D}}) \longrightarrow \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}_{r^2}$, mapping ∞ to ∞. Since X is locally connected, $\mathcal{R}$ extends continuously to a map on the closed sets $\mathcal{R} : \widehat{\mathbb{C}} \setminus \operatorname{int}(X \cup \overline{\mathbb{D}}) \longrightarrow \widehat{\mathbb{C}} \setminus \mathbb{D}_{r^2}$. We shall extend $\mathcal{R}$ to a quasi-conformal map $\mathcal{R} : \widehat{\mathbb{C}} \setminus \operatorname{int}(X' \cup \overline{\mathbb{D}}) \to \widehat{\mathbb{C}} \setminus \mathbb{D}_r$ in such a way that it conjugates f to $Q_0(z) = z^2$ on γ', the outer boundary of X'. Start by choosing $\mathcal{R}$ on γ' with this property, i.e., the following diagram commutes.

$$\begin{array}{ccc} \gamma' & \xrightarrow{f} & \gamma \\ {\scriptstyle \mathcal{R}}\downarrow & & \downarrow{\scriptstyle \mathcal{R}} \\ \partial\mathbb{D}_r & \xrightarrow{Q_0} & \partial\mathbb{D}_{r^2} \end{array}$$

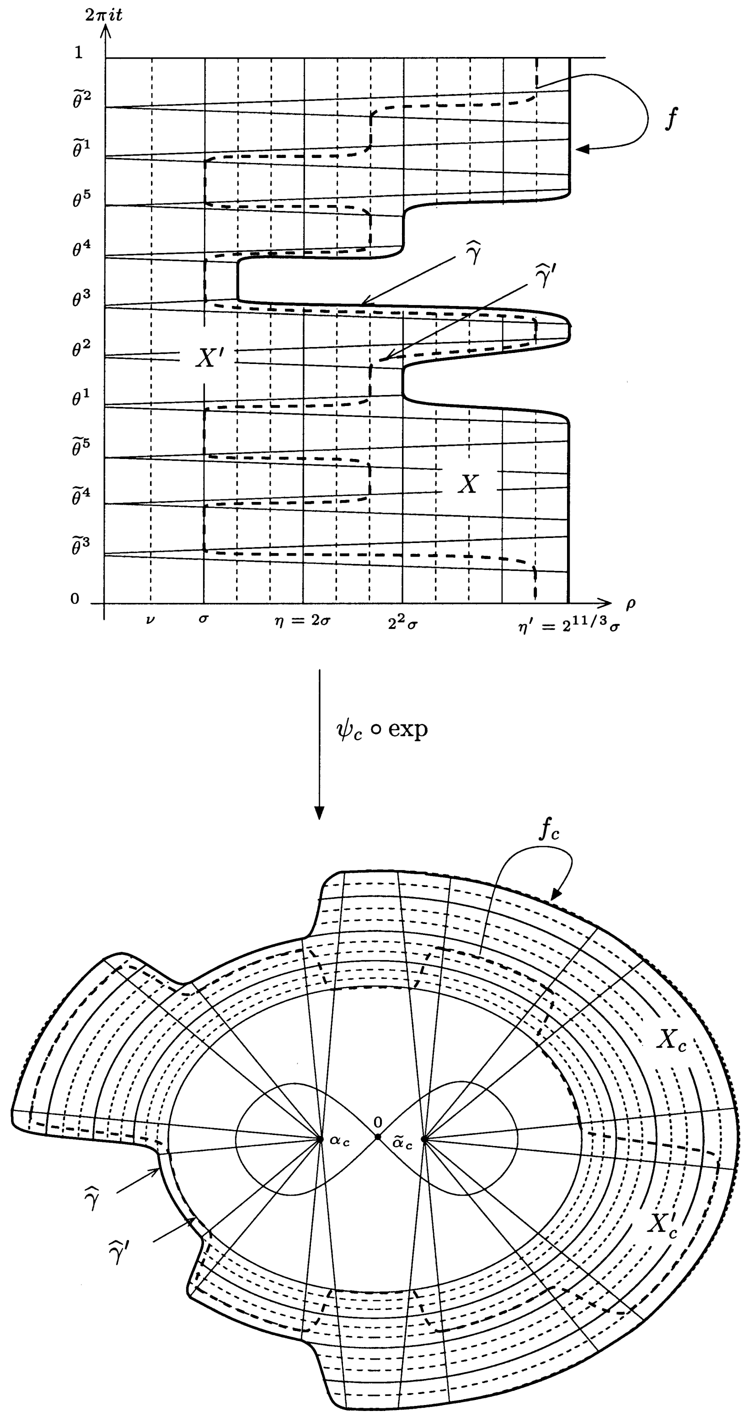

FIGURE 15. The maps $f : X' \to X$ and $f_c : X_c' \to X_c$ in a disconnected case. The map $(\psi_c \circ \exp)$ conjugates these two maps down to the potential level of 0.

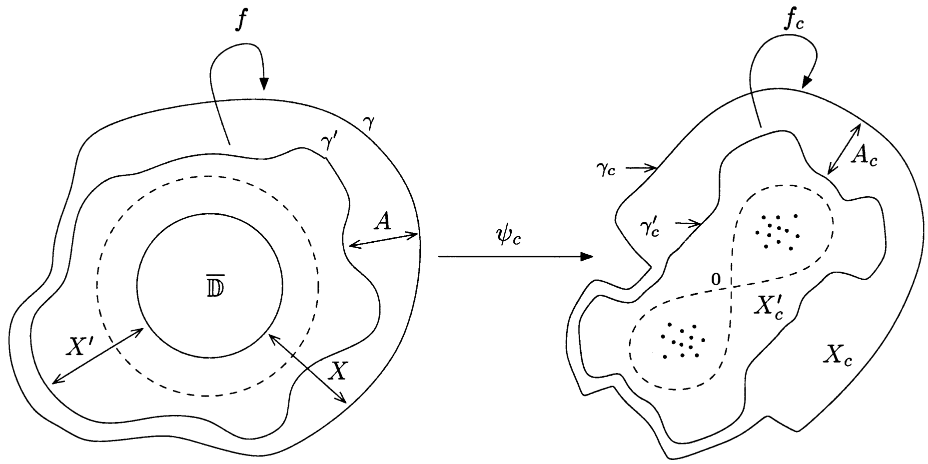

FIGURE 16. The setup in the complement of the unit disk

Since we have $\mathcal{R}$ defined on the boundaries of the annulus A, and quasi-symmetric, we can now extend it quasi-conformally to the interior of A. Therefore we have constructed a quasi-conformal map $\mathcal{R} : \widehat{\mathbb{C}} \setminus \operatorname{int}(X' \cup \overline{\mathbb{D}}) \to \widehat{\mathbb{C}} \setminus \mathbb{D}_r$ such that it conjugates f to Q_0 on γ'.

We may now define the extension of f as $F : \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ where,

$$F = \begin{cases} f & \text{on } X', \\ \mathcal{R}^{-1} \circ Q_0 \circ \mathcal{R} & \text{on } \widehat{\mathbb{C}} \setminus (X' \cup \overline{\mathbb{D}}). \end{cases}$$

Observe that, by construction, F is holomorphic everywhere except on $A \cup (S \cup \widetilde{S})$, where it is quasi-regular. Hence F is quasi-regular on all $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Back to dynamical plane, we define $F_c : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as

$$F_c = \begin{cases} f_c & \text{on } X'_c, \\ \psi_c \circ F \circ \psi_c^{-1} & \text{on } \widehat{\mathbb{C}} \setminus X'_c. \end{cases}$$

Remark 4.11. Observe that if K_c is connected, the equality $F_c = \psi_c \circ F \circ \psi_c^{-1}$ holds in all of $\widehat{\mathbb{C}} \setminus K_c$. If K_c is not connected, it is true on $(\widehat{\mathbb{C}} \setminus X'_c) \cup (S_c \cup \widetilde{S}_c)$ and, even more, down to wherever the Böttcher coordinates are well defined, in particular, down to the potential level of $\omega = 0$.

In any case, F_c is a quasi-regular map which is holomorphic everywhere except in $A_c \cup (S_c \cup \widetilde{S}_c)$, where $A_c = \psi_c(A)$ (see Figure 17). The dilatation ratio is bounded by a uniform constant since all choices were made once and for all on the complement of the unit disk.

4.2.5. *Holomorphic smoothing and definition of* $\Lambda_{p/q}$. We shall construct an almost complex structure σ_c on $\widehat{\mathbb{C}}$ which will be invariant under F_c. As usual, the construction starts in the model space, the complement of the disk. The dependence on the parameter c occurs mainly through the Böttcher coordinates.

Let σ_0 denote the standard complex structure which we put on $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}_r$. Define σ on $\widehat{\mathbb{C}} \setminus (X' \cup \overline{\mathbb{D}})$ as the pull back of σ_0 by the map $\mathcal{R}$, i.e., $\sigma = \mathcal{R}^* \sigma_0$. Observe that,

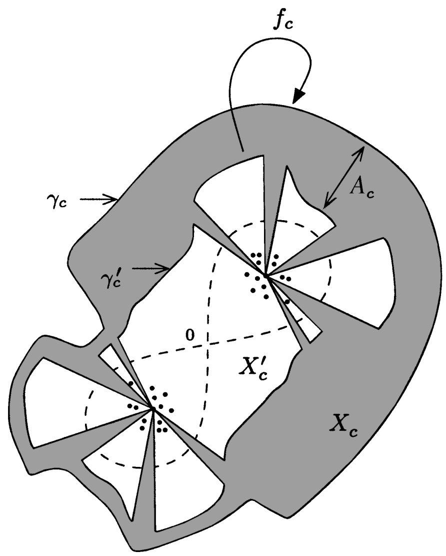

FIGURE 17. Shadowed, the region $A_c \cup (S_c \cup \widetilde{S}_c)$ where $F_c : \mathbb{C} \to \mathbb{C}$ is not holomorphic

since $\mathcal{R}$ is holomorphic on $\widehat{\mathbb{C}} \setminus (X \cup \overline{\mathbb{D}})$ we have that $\sigma = \sigma_0$ on this set. Likewise, σ has bounded distortion on the annulus A since $\mathcal{R}$ is quasi-conformal on A.

We now use the Böttcher coordinates to transport σ to the dynamical plane. To this end, define $\sigma_c = (\psi_c^{-1})^* \sigma$ on the set $\widehat{\mathbb{C}} \setminus X_c'$. Since ψ_c^{-1} is holomorphic, $\sigma_c = \sigma_0$ on $\widehat{\mathbb{C}} \setminus X_c$ and σ_c has bounded distortion on the annulus A_c. Next we use the map F_c to extend σ_c to X_c' by setting inductively

$$\sigma_c = (F_c^n)^* \sigma_c, \quad \text{on } F_c^{-n}(A_c),\ n > 0.$$

Notice that this is well defined since successive preimages of A_c form a nested sequence of sets with disjoint interiors (which are annuli as long as we are above the potential level of 0) (see Figure 18). Moreover, they cover all the complement of K_c since the orbit of any point in $X_c' \setminus K_c$ has one and only one point in the annulus A_c (after removing one of its boundaries). Finally, define $\sigma_c = \sigma_0$ on K_c.

Remark 4.12. These pull backs can be done in the complement of the unit disk defining an almost complex structure σ on this set. If K_c is connected, then $\sigma_c = (\psi_c^{-1})^* \sigma$ on $\widehat{\mathbb{C}} \setminus K_c$. If not, the equality is true at least down to the potential level of 0.

Proposition 4.13. *Let σ_c be the almost complex structure on $\widehat{\mathbb{C}}$ defined above. Then, σ_c is invariant under F_c (i.e., $F_c^* \sigma_c = \sigma_c$) by construction. Moreover, σ_c is quasi-conformally equivalent to the standard complex structure.*

Proof. By construction, it is clear that $F_c^* \sigma_c = \sigma_c$ on X_c'. We claim that $F_c^* \sigma_c = \sigma_c$ holds also on the annulus A_c. Since F_c maps A_c into $\widehat{\mathbb{C}} \setminus X_c$ where $\sigma_c = \sigma_0$, we

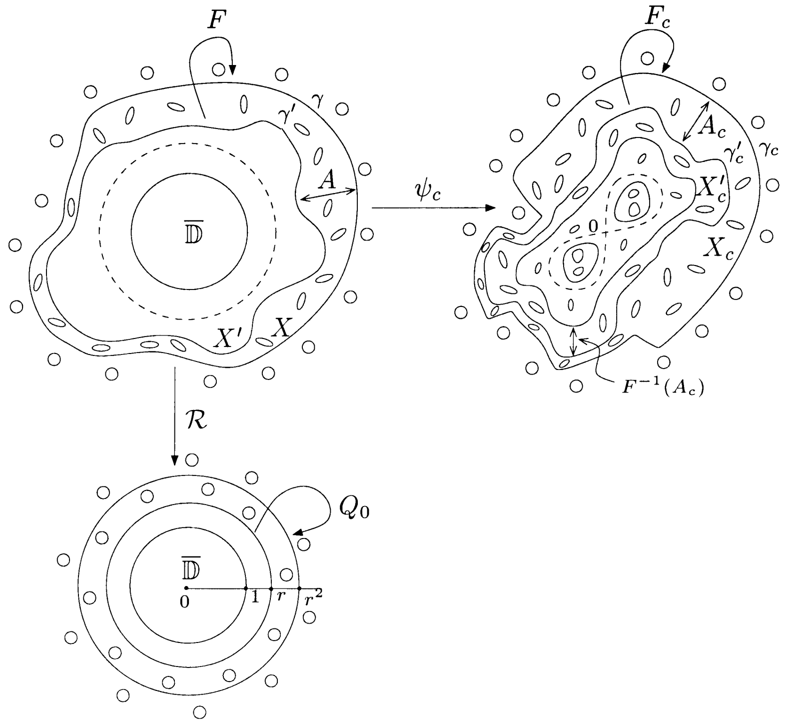

FIGURE 18. The complex structure σ_c on the successive preimages of A_c

must show that F_c transports σ_c on A_c to the standard structure σ_0. By definition, $F_c = \psi_c \circ \mathcal{R}^{-1} \circ Q_0 \circ \mathcal{R} \circ \psi_c^{-1}$. Hence,

$$\begin{aligned} F_c^* \sigma_0 = &\quad (\psi_c^{-1})^* \circ \mathcal{R}^* \circ Q_0^* \circ (\mathcal{R}^{-1})^* \circ \psi_c^* \sigma_0 \\ = &\quad (\psi_c^{-1})^* \circ \mathcal{R}^* \circ Q_0^* \circ (\mathcal{R}^{-1})^* \sigma_0 \\ = &\quad (\psi_c^{-1})^* \circ \mathcal{R}^* \circ Q_0^* \sigma_0 \\ = &\quad (\psi_c^{-1})^* \circ \mathcal{R}^* \sigma_0 \\ = &\quad (\psi_c^{-1})^* \sigma \\ = &\quad \sigma_c. \end{aligned}$$

It remains to be shown that σ_c has bounded distortion. We only need to prove it in $X_c \setminus K_c$ since $\sigma_c = \sigma_0$ everywhere else.

Let E_x be the infinitesimal ellipse defined at almost any point $x \in X_c \setminus K_c$ by σ_c. Clearly, if $x \in A_c$, the ratio of the axes is bounded by some constant K_1.

We first consider points on the sectors. Note that $F_c|_{S'_c}$ is an injective map and consider the compact set $T = T_c = \bigcup_{i=0}^{q-1} F_c^{-i}(A_c \cap S_c)$. On the set T, σ_c is obtained by a finite number of pull backs of the structure on A_c, and therefore the distortion is bounded by a constant K_2. Moreover, $T \setminus \gamma$ is a fundamental domain for $F_c^q : S_c \setminus T \to S_c$, i.e., if $x \in S_c \setminus T$, there exists a unique $n > 0$ such that $F_c^{nq}(x) \in T \setminus \gamma$. Hence,

$$E_x = (T_x F_c^{nq})^{-1}(E_{F_c^{nq}(x)}),$$

and then, the ratio of the axis is also bounded by K_2 since F_c^q is holomorphic on $S_c \setminus T$ (see Proposition 4.7 and Remark 4.10) (see Figure 19).

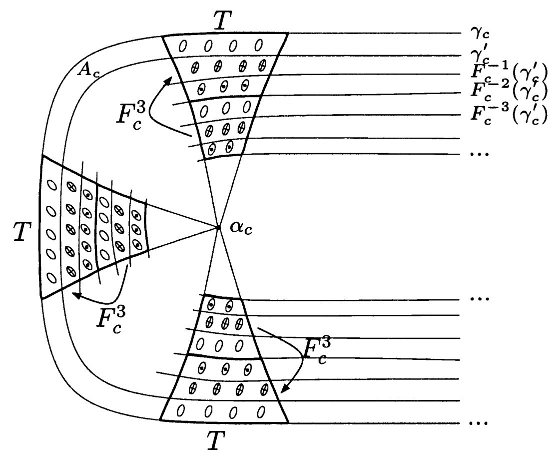

FIGURE 19. The complex structure σ_c on the sectors S_c. For simplification the sketch is drawn for $q = 3$. Moreover, the ellipse field is drawn in a symbolic way underlining that F_c^q is mapping each sector holomorphically into itself, so that the ellipse field in this sense repeats itself.

If $x \in \widetilde{S}_c'$, the bound on the ratio of the axis of E_x is also K_2, since $F_c(x) = F_c(-x)$.

If $x \notin (S_c' \cup \widetilde{S}_c')$, then either there exists n such that $F_c^n(x) \in (S_c' \cup \widetilde{S}_c')$ or the orbit of x never enters the sectors. In the first case, let n denote the smallest such number and then,

$$E_x = (T_x F_c^n)^{-1}(E_{F_c^n(x)}).$$

This ellipse has also bounded dilatation ratio (with K_2 as a bound) since F_c is analytic on all points $F_c^j(x)$ for $j = 0, \ldots, n-1$ (that is, outside of the sectors). In the second case, there exists a unique $n > 0$ such that $F_c^n(x) \in A_c$. By the same argument, the dilation ratio of E_x is bounded by K_1.

This concludes the proof of the proposition. □

We proceed now to integrate the almost complex structure. Applying the Measurable Riemann Mapping Theorem, we obtain a quasi-conformal homeomorphism $\varphi_c : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which integrates σ_c. That is, $\varphi_c^* \sigma_0 = \sigma_c$ and $\varphi_c \circ F_c \circ \varphi_c^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is holomorphic of degree two. If we choose φ_c so that it fixes 0 and ∞ and is of the form $\mathcal{R}(z) + \mathcal{O}(1)$, then it is unique and the composition map is a centered quadratic polynomial. It is also monic, since at infinity the map takes the form

$$\varphi_c \circ F_c \circ \varphi_c^{-1}(z) = \varphi_c \circ \psi_c \circ \mathcal{R}^{-1} \circ Q_0 \circ \mathcal{R} \circ \psi_c^{-1} \circ \varphi_c^{-1}(z) = z + \mathcal{O}(z).$$

Hence, it can be written as

$$Q_{\Lambda(c)} = z^2 + \Lambda(c),$$

which gives the definition of $\Lambda_{p/q} : W_{p/q}^{\eta,s} \to \mathbb{C}$ as $\Lambda_{p/q}(c) = \Lambda(c)$. We will write $\Lambda(c)$ whenever the dependence on p/q is understood.

We observe that Λ is well defined once we have chosen the slope s, the bound η, the boundaries of X and X', the smoothing f of g, the real number $r > 0$, and the map $\mathcal{R}$. However, recall that all polynomials outside M are hybrid equivalent. Hence, the resulting $\Lambda(c)$ may depend on these choices in the case when the Julia set is disconnected. This is the reason why we have made all the choices once and for all in the right half plane (or the complement of $\overline{\mathbb{D}}$).

4.2.6. *The Böttcher map of $Q_{\Lambda(c)}$.* A useful consequence of this construction is the fact that one can obtain an expression for the Böttcher map of the new polynomial in terms of the integrating map. More precisely, we have the following proposition.

Proposition 4.14. *Given $c \in W^{\eta,s}_{p/q}$, let φ_c, $\Lambda(c)$, etc., be as above. Then, the Böttcher map of $Q_{\Lambda(c)}$ can be written as*

$$\psi_{\Lambda(c)} = \varphi_c \circ \psi_c \circ \mathcal{R}^{-1} \quad \text{on } \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}_r.$$

Proof. By construction, the following diagram commutes.

$$\begin{array}{ccc}
\widehat{\mathbb{C}} \setminus \mathbb{D}_r & \xrightarrow{Q_0} & \widehat{\mathbb{C}} \setminus \mathbb{D}_{r^2} \\
\mathcal{R} \big\uparrow & & \big\uparrow \mathcal{R} \\
\widehat{\mathbb{C}} \setminus (X' \cup \overline{\mathbb{D}}) & \xrightarrow{F} & \widehat{\mathbb{C}} \setminus (X \cup \overline{\mathbb{D}}) \\
\psi_c \big\downarrow & & \big\downarrow \psi_c \\
\widehat{\mathbb{C}} \setminus X'_c & \xrightarrow{F_c} & \widehat{\mathbb{C}} \setminus X_c \\
\varphi_c \big\downarrow & & \big\downarrow \varphi_c \\
\widehat{\mathbb{C}} \setminus \varphi_c(X'_c) & \xrightarrow{Q_{\Lambda(c)}} & \widehat{\mathbb{C}} \setminus \varphi_c(X_c)
\end{array}$$

Observe that the map $\varphi_c \circ \psi_c \circ \mathcal{R}^{-1} : \widehat{\mathbb{C}} \setminus \mathbb{D}_r \to \widehat{\mathbb{C}} \setminus \varphi_c(X_c)$ transports the standard complex structure to itself and therefore it is holomorphic. Moreover, it maps ∞ to ∞, and it conjugates $Q_{\Lambda(c)}$ to Q_0. It follows that it is the Böttcher map of $Q_{\Lambda(c)}$. □

Corollary 4.15. *The boundaries of the sets $\varphi_c(X_c)$ and $\varphi_c(X'_c)$ are equipotential curves of the polynomial $Q_{\Lambda(c)}$ of potential $2\log(r)$ and $\log(r)$ respectively.*

4.3. **Continuity of $\Lambda_{p/q}$ and other properties.** The goal of this section is to prove that the map Λ is continuous and that it coincides with the homeomorphisms Φ^q_{p1} in [BF] on the limbs. Prior to that, we state some lemmas and observe some important properties of the map.

The following rigidity lemma is crucial for the construction to work.

Lemma 4.16 ([DH2, p. 304])**.** *Let $c_1 \in \partial M$ and $c_2 \in \mathbb{C}$. Suppose that the polynomials Q_{c_1} and Q_{c_2} are quasi-conformally conjugate. Then, $c_1 = c_2$.*

The following lemma is the analog to that in p. 313 of [DH2].

Lemma 4.17. *Let $\{c_n\}_{n>0}$, $c_n \in M_{p/q}$, be a sequence converging to $c_0 \in M_{p/q}$. Let $\lambda_n = \Lambda(c_n)$ for $n \geq 0$. Assume λ_* is an accumulation point of the sequence $\{\lambda_n\}_{n>0}$. Then, the polynomials Q_{λ_0} and Q_{λ_*} are quasi-conformally conjugate.*

Proof. Let $\varphi_n = \varphi_{c_n}$ be the integrating maps which are all quasi-conformal maps of the sphere with dilatation ratio bounded by a uniform constant K, and normalized so that $\varphi_n(0) = 0$, $\varphi_n(\infty) = \infty$ and φ_n is tangent to $\mathcal{R}(z)$ at infinity. Also, $\overline{\partial}\varphi_n$ have support in a fixed compact set. Since the space of such maps is compact with respect to uniform convergence, there exists a subsequence $\{\varphi_{n_k}\}$ which converges uniformly on compact sets to a $K-$quasi-conformal map φ_*. Abusing notation, we denote this subsequence by $\{\varphi_n\}$.

The quasi-regular maps F_{c_n} constructed by surgery depend continuously on the parameter c, since the sectors involved in the construction do so. Then, $F_{c_n} \rightrightarrows F_{c_0}$ and

$$Q_{\lambda_n} = \varphi_n \circ F_{c_n} \circ \varphi_n^{-1} \rightrightarrows \varphi_* \circ F_{c_0} \circ \varphi_*^{-1} =: Q_*.$$

Observe that Q_* must be a holomorphic map of $\widehat{\mathbb{C}}$ of degree two since it is the uniform limit of holomorphic maps of $\widehat{\mathbb{C}}$ of degree two. Moreover, Q_* is centered since the critical point is $\varphi_*(0) = 0$ and monic because the Böttcher map $\varphi_* \circ \psi_{c_0} \circ \mathcal{R}^{-1}$ is tangent to the identity at infinity. Hence Q_* is of the form $z^2 + \lambda$ and in fact, $Q_*(z) = Q_{\lambda_*}(z) = z^2 + \lambda_*$ since $Q_{\lambda_n} \rightrightarrows Q_*$ and $\lambda_n \to \lambda_*$ by hypothesis. We conclude then that

$$Q_{\lambda_*} \sim_{qc} F_{c_0} \sim_{qc} Q_{\lambda_0}$$

and the lemma follows. □

Proposition 4.18. *The map $\Lambda : W^{\eta,s}_{p/q} \to \mathbb{C}$ sends the interior of the limb $M_{p/q}$ to the interior of the limb $M_{1/q}$; the boundary of $M_{p/q}$ to the boundary of $M_{1/q}$, and the rest of points in $W^{\eta,s}_{p/q} \setminus M_{p/q}$ to points in $\mathbb{C} \setminus M$.*

Proof. If c belongs to a hyperbolic component of $M_{p/q}$ and hence has an attracting cycle, then $Q_{\Lambda(c)}$ also has an attracting cycle (see Remark 4.3) and therefore $\Lambda(c)$ belongs to a hyperbolic component of $M_{1/q}$.

If c belongs to a non-hyperbolic component of the interior of $M_{p/q}$, then the Julia set J_c has positive measure and it carries an invariant line field. Following the surgery construction in detail, as in Section 5.4 in [BF], one can check that $J_{\Lambda(c)}$ must also have positive measure and carry an invariant line field. Hence $\Lambda(c)$ belongs to a non-hyperbolic component of the interior of $M_{1/q}$.

Suppose $c \in \partial M_{p/q}$. Let $\{c_n\}_{n\geq 0}$, $c_n \in \partial M_{p/q}$ be a sequence of Misiurewicz points (i.e., $\omega = 0$ is strictly preperiodic under Q_{c_n}) converging to c. Recall that this sequence exists since Misiurewicz points are dense in the boundary of the Mandelbrot set. Let $\lambda = \Lambda(c)$ and $\lambda_n = \Lambda(c_n)$. The critical point of Q_{λ_n} must still be strictly preperiodic, and hence λ_n is Misiurewicz and belongs to the boundary of $M_{1/q}$. Now, let $\lambda_* \in \partial M_{1/q}$ be any accumulation point of the sequence $\{\lambda_n\}$ which must exist since $\partial M_{1/q}$ is a compact set. By Lemma 4.17, the polynomials Q_λ and Q_{λ_*} are quasi-conformally conjugate, but we also know that $\lambda_* \in \partial M_{1/q}$. Hence, it follows from Lemma 4.16 that $\lambda = \lambda_* \in \partial M_{1/q}$.

Finally, let $c \in W^{\eta,s}_{p/q} \setminus M_{p/q}$. Then, the critical orbit under Q_c is unbounded. It is also clear from the surgery construction that the critical orbit under $Q_{\Lambda(c)}$ is unbounded and therefore $\Lambda(c) \in \mathbb{C} \setminus M$. □

We are now ready to prove the continuity of the map Λ. First observe that, since the integrating map φ_c conjugates F_c to the polynomial $Q_{\Lambda(c)}$, the critical point and the critical value of F_c (i.e., 0 and $F_c(0)$) must be mapped to the critical point and the critical value of $Q_{\Lambda(c)}$ respectively (i.e., 0 and $\Lambda(c)$). Hence,

$$\Lambda(c) = \varphi_c(F_c(0)) = \varphi_c(Q_c^{n[1]}(0)), \tag{2}$$

since $0 \in V_c^0 \setminus S_c$ and $F_c = Q_c^{n[1]}$ on this set.

Theorem 4.19. *The map Λ is continuous.*

Proof. We consider two separate cases. Suppose $c_0 \notin \partial M_{p/q}$. Let V be a neighborhood of c_0 in $W_{p/q}^{\eta,s}$ such that $V \cap \partial M_{p/q} = \emptyset$. We claim that the almost complex structure σ_c we constructed, varies continuously with $c \in V$. To see this, observe that $\psi_c(z)$ is a holomorphic map in both variables, $c \in V$ and $z \in \mathcal{U}_c$. Now, σ is defined once and for all outside $\mathbb{D}$ and then transported to $\mathcal{U}_c$ by the tangent map of $\psi_c(z)$. Finally, $\mathcal{U}_c$ is the complement of K_c if this set is connected or the complement of a figure eight that contains K_c if it is disconnected. In both cases, the boundary of $\mathcal{U}_c$ also moves continuously with $c \in V$ and the claim follows.

Hence, we conclude from the Measurable Riemann Mapping Theorem including dependence on parameters, that the map $(c, z) \mapsto (c, \varphi_c(z))$ is jointly continuous where, as above, φ_c is the integrating map. Thus the map $c \mapsto \varphi_c(Q_c^{n[1]}(0))$ is continuous and this equals $\Lambda(c)$ by Equation (2).

Now suppose $c_0 \in \partial M_{p/q}$. The same argument cannot be applied since there is a discontinuity of the almost complex structure at all parabolic points. Let $\{c_n\}_{n>0}$ be an arbitrary sequence of parameter values $c_n \in W_{p/q}^{\eta,s}$ such that $c_n \to c_0$. Let $\lambda_n = \Lambda(c_n)$ for $n \geq 0$. For any accumulation point λ_* of $\{\lambda_n\}$ we must show that $\lambda_* = \lambda_0$.

From Lemma 4.17 it follows that Q_{λ_*} and Q_{λ_0} are quasi-conformally conjugate. From Proposition 4.18 we know that $\lambda_0 \in \partial M_{1/q}$. Hence we conclude from Lemma 4.16 that $\lambda_* = \lambda_0$. □

The following proposition states that the map Λ coincides with the homeomorphism Φ_{p1}^q constructed in [BF].

Proposition 4.20. *If $c \in M_{p/q}$, then $\Lambda_{p/q}(c) = \Phi_{p1}^q(c)$. Hence, $\Lambda_{p/q}$ is a homeomorphism on the limb $M_{p/q}$ which is holomorphic in the interior.*

Proof. In [BF] we constructed for each $p/q \in (0,1) \cap \mathbb{Q}$ a homeomorphism $\phi_{p/q} : M_{p/q} \to L_{q,0}$, where $L_{q,0}$ denotes the 0-limb of the connectedness locus L_q of the family of polynomials $P_\lambda(z) = \lambda z(1 + \frac{z}{q})^q$. The homeomorphism $\Phi_{p1}^q : M_{p/q} \to M_{1/q}$ equals the composition $\phi_{1/q}^{-1} \circ \phi_{p/q} : M_{p/q} \to M_{1/q}$. In order to prove that $\Lambda_{p/q} = \Phi_{p1}^q$ on $M_{p/q}$ we shall prove that $\phi_{1/q} \circ \Lambda_{p/q} = \phi_{p/q}$ on $M_{p/q}$. We shall briefly recall the surgery construction in [BF] leading to the definition of $\phi_{p/q}$, leaving out technical details.

Let $c \in M_{p/q}$ be chosen arbitrarily. We truncate the plane by cutting away the wedges $V_c^1, \ldots, V_c^{q-1}$ and identify points equipotentially on the two bounding rays $R_c(\theta^1)$ and $R_c(\theta^q)$. We denote this truncated plane by $\mathbb{C}_c^T = (V_c^0 \cup \bigcup_{i=1}^{q-1} \tilde{V}_c^i)/\sim$. Then we construct the first return map of Q_c on the truncated plane, that is on V_c^0 and each $\tilde{V}_c^j$ we apply the smallest number of iterates of Q_c that maps the sets into the allowed space. The first return map of Q_c is then Q_c^q on $\mathrm{int}(V_c^0)$ and $z \mapsto Q_c^{q-n[j]}(-z)$ on $\mathrm{int}(\tilde{V}_c^j)$, $j = 1, \ldots, q-1$. To obtain the polynomial P_λ with $\phi_{p/q}(c) = \lambda$, we restrict the first return map, and smoothen it on sectors around the lines of discontinuity, ray segments of $R_c(\tilde{\theta}^j)$, $j = 1, \ldots, q-1$, such that the resulting map, say p_c, is quasi-regular. This map is hybrid equivalent to the polynomial P_λ. In [BF] we argued that other choices in the construction result in maps that are hybrid equivalent to p_c, hence also to P_λ. Starting from $Q_{\Lambda(c)}$ we construct in a similar manner a map $p_{\Lambda(c)}$. By a rigidity argument analog to Proposition 2.1, to finish the proof we only need to show that $p_c \sim_{hb} p_{\Lambda(c)}$.

Observe that if we form the composition g_c^{q-j} on V_c^j, $j = 0, 1, \ldots, q-1$, then we obtain $g_c^{q-j} = Q_c^{q-n[j]}$ since

$$V_c^j \xrightarrow{Q_c^{k[j]}} V_c^{j+1} \xrightarrow{Q_c^{k[j+1]}} \ldots \xrightarrow{Q_c^{k[q-2]}} V_c^{q-1} \xrightarrow{Q_c^{k[q-1]}} V_c^0 \cup \bigcup_{i=1}^{q-1} \tilde{V}_c^i,$$

and $k[j] + k[j+1] + \cdots + k[q-2] + k[q-1] = q - n[j]$. It follows that the first return map of g_c equals Q_c^q on $\mathrm{int}(V_c^0)$ and $z \mapsto Q_C^{q-n[j]}(-z)$ on $\mathrm{int}(\tilde{V}_c^j)$. Hence the first return map of g_c coincides with the first return map of Q_c.

We note that $f_c \sim_{hb} Q_{\Lambda(c)}$, and if we carry through the surgery construction starting from f_c, we obtain a quasi-regular map, say $\tilde{p}_c$, that is hybrid equivalent to $p_{\Lambda(c)}$. Since we can use the choices made when starting from f_c as choices when starting from Q_c we have $p_c \sim_{hb} \tilde{p}_c$ and all together

$$p_{\Lambda(c)} \sim_{hb} \tilde{p}_c \sim_{hb} p_c \sim_{hb} P_\lambda.$$

□

Remark 4.21. Note that since $\varphi_c(\gamma_c)$ is an equipotential of level $2\log(r)$, where r is the arbitrary number chosen in connection with the Riemann mapping

$$\mathcal{R} : \widehat{\mathbb{C}} \setminus \mathrm{int}(X \cup \overline{\mathbb{D}}) \to \widehat{\mathbb{C}} \setminus \mathbb{D}_{r^2},$$

it follows that

$$G_M(\Lambda(c)) = 2\log(r),$$

if $c \in W_{p/q}$ and $G_M(c) = \eta$, while

$$G_M(\Lambda(c)) < 2\log(r)$$

if $c \in W_{p/q}^{\eta,s}$.

Note that the image $\Lambda(W_{p/q}^{\eta,s})$ may not be contained entirely in $W_{1/q}$. The Riemann mapping $\mathcal{R}$ is uniquely determined up to post-composition by a rotation. Hence when η, s and γ have been chosen, then the angle spanned by the arc $\mathcal{R}(\gamma \cap (V^1 \setminus (S(\theta^1) \cup S(\theta^2))))$ on $\partial\mathbb{D}_{r^2}$ is determined. If this angle is larger than $\frac{1}{2^q-1}$, the span of $W_{1/q}$, then the image of $W_{p/q}^{\eta,s}$ cannot fit into $W_{1/q}$.

Note however that the map Λ does depend on the different choices. Let us choose for instance an arbitrary $c' \in W_{p/q}^{\eta,s} \setminus M_{p/q}$ and let $r' > 0$ be such that $2\log(r') = G_M(\Lambda(c'))$. Let $\mathcal{R}' : \widehat{\mathbb{C}} \setminus \mathrm{int}(X \cup \overline{\mathbb{D}}) \to \widehat{\mathbb{C}} \setminus \mathbb{D}_{(r')^2}$ be a Riemann mapping satisfying $\mathcal{R}'(\infty) = \infty$. If we continue the construction from here on to obtain a map $\Lambda' = \Lambda'_{p/q} : W_{p/q}^{\eta,s} \to \mathbb{C}$, then we can be sure that $\Lambda'(c') \neq \Lambda(c')$, since $G_M(\Lambda(c')) < 2\log(r')$.

4.4. **Injectivity and quasi-conformality of $\Lambda_{p/q}$ outside the limb.** To show that the map $\Lambda_{p/q} : W_{p/q}^{\eta,s} \to \mathbb{C}$ is a homeomorphism in all of its domain onto its image it remains to solve the problem of injectivity on the complement of the p/q-limb, $W_{p/q}^{\eta,s} \setminus M_{p/q}$.

We shall do so by giving an alternative expression for the integrating map φ_c in the cases when K_c is not connected. This will lead to a new expression of $\Lambda(c)$ for which injectivity will be simpler to check. We remark that this argument cannot be used for points in the limb but only those in the complement.

To this end, let c_0 denote the center of $\Omega_{p/q}$, the main hyperbolic component of $M_{p/q}$ (that of period q) and let $c \in W^{\eta,s}_{p/q} \setminus M_{p/q}$. Let ψ_{c_0} and ψ_c be the two respective Böttcher maps. Recall that the set $\mathcal{U}_c$ was defined as the set of those points in dynamical plane that lie in the complement of the filled figure eight that corresponds to the potential level of $\omega = 0$. Define the map

$$\begin{array}{rccc} h_c: & \mathcal{U}_c & \longrightarrow & \mathbb{C} \setminus K_{c_0} \\ & z & \longmapsto & (\psi_{c_0} \circ \psi_c^{-1})(z). \end{array}$$

(See Figure 20.) Note that h_c is injective and holomorphic for any $c \in W^{\eta,s}_{p/q} \setminus M_{p/q}$.

Remark 4.22. In fact, the set $\{(c,z) \mid c \in W^{\eta,s}_{p/q} \setminus M_{p/q}, z \in \mathcal{U}_c\}$ is open in $(W^{\eta,s}_{p/q} \setminus M_{p/q}) \times \mathbb{C}$. In other words, for any given $\widetilde{c} \in W^{\eta,s}_{p/q} \setminus M_{p/q}$ and any $\widetilde{z} \in \mathcal{U}_{\widetilde{c}}$, there exists a neighborhood $U_{\widetilde{c}} \in W^{\eta,s}_{p/q} \setminus M_{p/q}$ of $\widetilde{c}$, and a neighborhood $V_{\widetilde{z}}$ of $\widetilde{z}$, such that $V_{\widetilde{z}} \subset \mathcal{U}_c$ for all $c \in U_{\widetilde{c}}$. Moreover, the map $(c,z) \longmapsto (c, h_c(z))$ is well defined in $U_{\widetilde{c}} \times V_{\widetilde{z}}$ and it is holomorphic in both its variables, c and z.

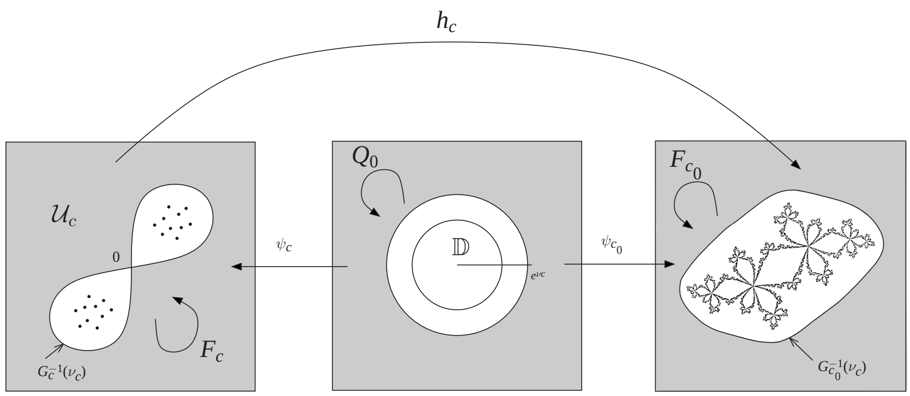

FIGURE 20. The set $\mathcal{U}_c$ and the map h_c

An important property of this map is that it provides a conjugacy between F_c and F_{c_0}.

Lemma 4.23. *The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{U}_c & \xrightarrow{F_c} & \mathcal{U}_c \\ {\scriptstyle h_c}\downarrow & & \downarrow{\scriptstyle h_c} \\ \mathbb{C} \setminus K_{c_0} & \xrightarrow{F_{c_0}} & \mathbb{C} \setminus K_{c_0} \end{array}$$

Proof. Recall (from Remark 4.11) that

$$F_c = \psi_c \circ F \circ \psi_c^{-1},$$

on $\mathcal{U}_c$, where F is a map defined on the complement of the unit disk independently of c. Then,

$$h_c \circ F_c = (\psi_{c_0} \circ \psi_c^{-1}) \circ (\psi_c \circ F \circ \psi_c^{-1}) = \psi_{c_0} \circ F \circ \psi_c^{-1}.$$

On the other hand,

$$F_{c_0} \circ h_c = (\psi_{c_0} \circ F \circ \psi_{c_0}^{-1}) \circ (\psi_{c_0} \circ \psi_c^{-1}) = \psi_{c_0} \circ F \circ \psi_c^{-1}.$$

□

We shall now make a parallel construction for polynomials in $W_{1/q} \setminus M_{1/q}$. Set $\lambda_0 = \Lambda_{p/q}(c_0)$, this is the center of $\Omega_{1/q}$, the main hyperbolic component of the $1/q$-limb. Let $\mathcal{U}_\lambda$ be as above in the dynamical plane of Q_λ.

Similarly as before, we define a map

$$\begin{array}{rccc} H_\lambda : & \mathcal{U}_\lambda & \longrightarrow & \mathbb{C} \setminus K_{\lambda_0} \\ & z & \longmapsto & (\psi_{\lambda_0} \circ \psi_\lambda^{-1})(z) \end{array}$$

which is holomorphic and injective.

The analog to Lemma 4.23 is also true and it is proven in the same way.

Lemma 4.24. *The following diagram commutes.*

$$\begin{array}{ccc} \mathcal{U}_\lambda & \xrightarrow{Q_\lambda} & \mathcal{U}_\lambda \\ {\scriptstyle H_\lambda}\downarrow & & \downarrow{\scriptstyle H_\lambda} \\ \mathbb{C} \setminus K_{\lambda_0} & \xrightarrow{Q_{\lambda_0}} & \mathbb{C} \setminus K_{\lambda_0} \end{array}$$

The two maps h_c and $H_{\Lambda(c)}$, together with the integrating map for the center point c_0, give the following key expression for the integrating map for any c not inside the limb.

Proposition 4.25. *The integrating map φ_c can be written as*

$$\varphi_c = H_{\Lambda(c)}^{-1} \circ \varphi_{c_0} \circ h_c,$$

on $\mathcal{U}_c \cap \varphi_c^{-1}(\mathcal{U}_{\Lambda(c)})$.

Proof. On the smaller set $\mathbb{C} \setminus X'_c$, Proposition 4.25 can also be stated by saying that the following diagram commutes:

$$\begin{array}{ccccc} \mathbb{C} \setminus X'_{c_0} & \xleftarrow{\psi_{c_0}} & \mathbb{C} \setminus (X' \cup \overline{\mathbb{D}}) & \xrightarrow{\psi_c} & \mathbb{C} \setminus X'_c \\ {\scriptstyle \varphi_{c_0}}\downarrow & & {\scriptstyle \mathcal{R}}\downarrow & & \downarrow{\scriptstyle \varphi_c} \\ \mathbb{C} \setminus \varphi_{c_0}(X'_{c_0}) & \xleftarrow{\psi_{\lambda_0}} & \mathbb{C} \setminus \mathbb{D}_r & \xrightarrow{\psi_{\Lambda(c)}} & \mathbb{C} \setminus \varphi_c(X'_c), \end{array}$$

which we have proven in Proposition 4.14.

Observe that this argument cannot be applied deeper since the expression for the Böttcher maps in terms of $\mathcal{R}$ applies only to $\mathbb{C} \setminus \mathbb{D}_r$. However, we shall use Lemmas 4.23 and 4.24 to pull back the equality.

For any $z \in \mathcal{U}_c \cap \varphi_c^{-1}(\mathcal{U}_{\Lambda(c)})$, there exists $n \geq 0$ such that $F_c^n(z) \in \mathbb{C} \setminus X'_c$. Hence the proposition applies to $F_c^n(z)$ and we have

$$H_{\Lambda(c)} \left(\varphi_c \left(F_c^n(z)\right)\right) = \varphi_{c_0} \left(h_c \left(F_c^n(z)\right)\right).$$

Since z and $F_c^n(z)$ are in $\mathcal{U}_c$ and the Böttcher maps are defined in this set we have that

$$h_c\left(F_c^n(z)\right) = F_{c_0}^n\left(h_c(z)\right). \tag{3}$$

Since $z \in \varphi_c^{-1}(\mathcal{U}_{\Lambda(c)})$, it follows that $\varphi_c(z) \in \mathcal{U}_{\Lambda(c)}$ and hence $H_{\Lambda(c)}(\varphi_c(z))$ is well defined. Moreover,

$$Q_{\lambda_0}^n\left(H_{\Lambda(c)}\left(\varphi_c(z)\right)\right) = H_{\Lambda(c)}\left(Q_{\Lambda(c)}^n\left(\varphi_c(z)\right)\right). \tag{4}$$

Now, by construction we know that $\varphi_c\left(F_c^n(z)\right) = Q_{\Lambda(c)}^n\left(\varphi_c(z)\right)$. Hence, equation (3) can be written as

$$H_{\Lambda(c)}\left(Q_{\Lambda(c)}^n\left(\varphi_c(z)\right)\right) = \varphi_{c_0}\left(F_{c_0}^n\left(h_c(z)\right)\right)$$

or, using (4), as

$$Q_{\lambda_0}^n\left(H_{\Lambda(c)}\left(\varphi_c(z)\right)\right) = Q_{\lambda_0}^n\left(\varphi_{c_0}\left(h_c(z)\right)\right).$$

By taking the appropriate branches of the inverse of Q_{λ_0}, we obtain that

$$H_{\Lambda(c)}\left(\varphi_c(z)\right) = \varphi_{c_0}\left(h_c(z)\right)$$

and the proposition follows. □

If $c \in W_{p/q}^{\eta,s} \setminus M_{p/q}$, then $Q_c^{n[1]}(0)$ belongs to the set $\mathcal{U}_c$ since $n[1] \geq 1$ and $\Lambda(c) = \varphi_c(F_c(0))$ belongs to the set $\mathcal{U}_{\Lambda(c)}$. Hence the prospective critical value $F_c(0) = Q_c^{n[1]}(0)$ belongs to the set $\mathcal{U}_c \cap \varphi_c^{-1}(\mathcal{U}_{\Lambda(c)})$ (see Figure 21), and Proposition 4.25 holds for this point. We have then proved

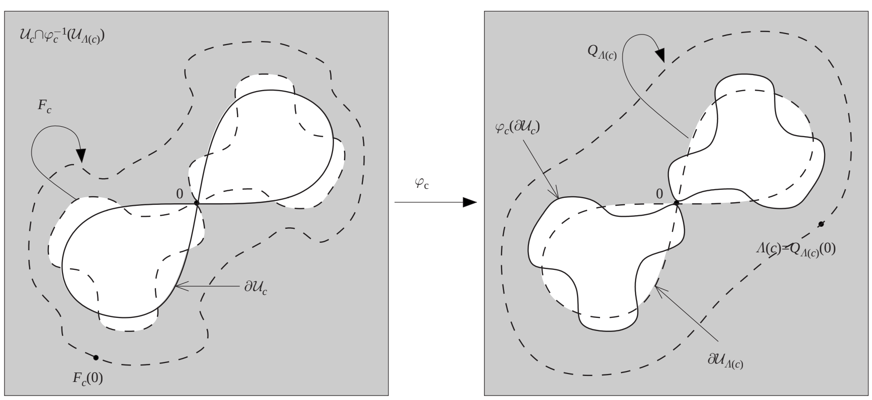

FIGURE 21. The set $\mathcal{U}_c \cap \varphi_c^{-1}(\mathcal{U}_{\Lambda(c)})$ and the location of $F_c(0) = Q_c^{n[1]}(0)$ and of $Q_{\Lambda(c)}(0)$

Proposition 4.26. *Let $c \in W_{p/q}^{\eta,s} \setminus M_{p/q}$. Then,*

$$\Lambda(c) = \varphi_c\left(Q_c^{n[1]}(0)\right) = H_{\Lambda(c)}^{-1}\left(\varphi_{c_0}\left(h_c\left(Q_c^{n[1]}(0)\right)\right)\right).$$

It remains to be shown,

Proposition 4.27. *The map $c \mapsto \Lambda(c)$ is a quasi-conformal injection.*

Proof. Let $\hbar : W_{p/q} \setminus M_{p/q} \longrightarrow \operatorname{int}\left(V^1_{c_0} \setminus K_{c_0}\right)$ be defined by

$$\hbar(c) = h_c(Q_c^{n[1]}(0)) = h_c(Q_c^{n[1]-1}(c)).$$

We shall first show that this map is well defined and it is a holomorphic isomorphism.

Clearly, any $c \in W_{p/q} \setminus M_{p/q}$ can be viewed as well in $V_c^p \setminus K_c$. Then, the map $Q_c^{n[1]-1}$ sends c to a point in $V_c^1 \setminus K_c$, which will be mapped, by the Böttcher coordinates $\psi_{c_0} \circ \psi_c^{-1}$, into $V^1_{c_0} \setminus K_{c_0}$. The composition is clearly holomorphic since all maps are holomorphic with respect to c and z. Moreover, it is proper, onto and of degree one. To see this, observe that $\hbar$ maps the rays $R_M(\theta^{\pm}_{p/q})$ bounding $W_{p/q}$ bijectively onto the rays $R_{c_0}(\theta^1)$ and $R_{c_0}(\theta^2)$ bounding $V^1_{c_0}$. Indeed, let $c \in R_M(\theta^-_{p/q})$ and be of potential ρ. Then, in dynamical plane, $c \in R_c(\theta^p)$ and is of potential ρ (recall that $\theta^p = \theta^-_{p/q}$); the image $Q_c^{n[1]-1}(c) \in R_c(\theta^1)$ and is of potential $2^{n[1]-1}\rho$. It follows that $h_c(Q_c^{n[1]-1}(c)) \in R_{c_0}(\theta^1)$ and is of potential $2^{n[1]-1}\rho$. Hence $\hbar$ maps $R_M(\theta^-_{p/q})$ bijectively onto $R_{c_0}(\theta^1)$ and, similarly, it maps $R_M(\theta^+_{p/q})$ bijectively onto $R_{c_0}(\theta^2)$. To finish the argument we observe that when c tends to $\partial M_{p/q}$, then $\hbar(c)$ tends to ∂K_{c_0}.

Next, consider the following map:

$$\begin{array}{rccc} \mathcal{H}: & \mathbb{C} \setminus M & \longrightarrow & \mathbb{C} \setminus K_{\lambda_0}, \\ & \lambda & \longmapsto & H_\lambda(\lambda). \end{array}$$

We observe that $\mathcal{H}$ is also a holomorphic isomorphism since we may write

$$\mathcal{H}(\lambda) = \psi_{\lambda_0}\left(\psi_\lambda^{-1}(\lambda)\right) = \psi_{\lambda_0}\left(\phi_M(\lambda)\right),$$

which is a composition of the two holomorphic isomorphisms $\phi_M : \mathbb{C} \setminus M \longrightarrow \mathbb{C} \setminus \overline{\mathbb{D}}$ and $\psi_{\lambda_0} : \mathbb{C} \setminus \overline{\mathbb{D}} \longrightarrow \mathbb{C} \setminus K_{\lambda_0}$.

Finally, observe (see Figure 22) that the map $\Lambda = \mathcal{H}^{-1} \circ \varphi_{c_0} \circ \hbar$ is a quasi-conformal homeomorphism onto its image, being a composition of two holomorphic isomorphisms and a quasi-conformal map. Indeed, note that

$$\hbar(W^{\eta,s}_{p/q} \setminus M_{p/q}) = (V^1_{c_0})^{2^{n[1]-1}\eta,s} \setminus K_{c_0}$$

where $(V^1_{c_0})^{2^{n[1]-1}\eta,s}$ is the dynamical wake restricted by part of the equipotential of potential η and slope lines of slope s of sectors around $R_{c_0}(\theta^1)$ and $R_{c_0}(\theta^2)$. Thus the set $\varphi_{c_0}\left((V^1_{c_0})^{2^{n[1]-1}\eta,s} \setminus K_{c_0}\right)$ is a quasi-conformal image within $\mathbb{C} \setminus K_{\lambda_0}$ which is finally mapped by a holomorphic isomorphism onto a subset of $\mathbb{C} \setminus M$. □

This ends the proof of the Main Theorem.

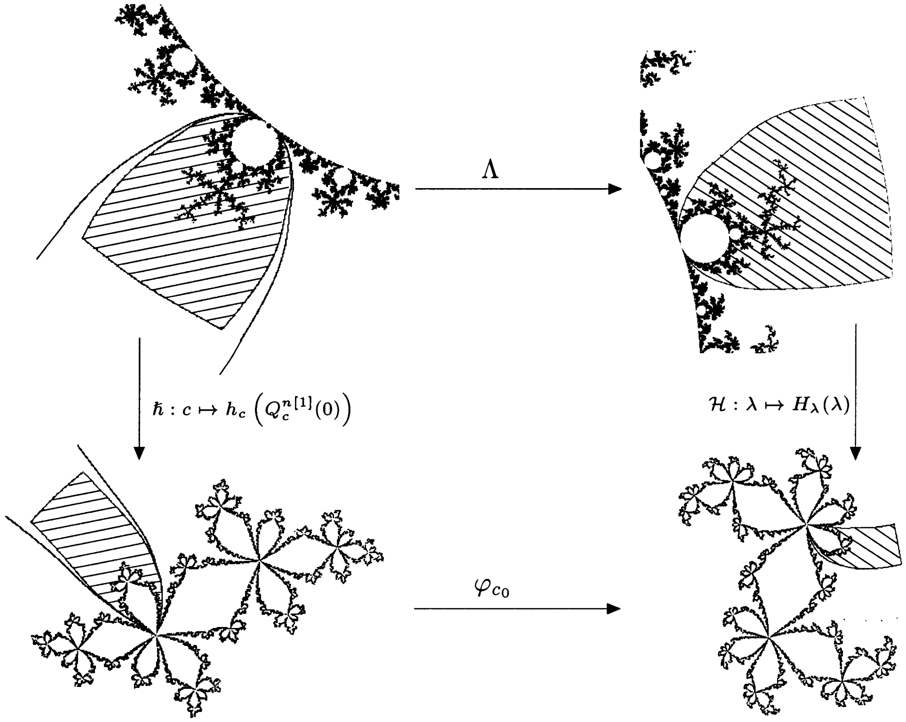

FIGURE 22. Commutative diagram relating parameter spaces and dynamical planes, as in Proposition 4.26

Remark 4.28. In this paper we have constructed an extension $\Lambda_{p/q}$ of the homeomorphism $\Phi^q_{p1} : M_{p/q} \to M_{1/q}$ and proved that $\Lambda_{p/q}$ is quasi-conformal outside $M_{p/q}$. As noted in the introduction we can deduce that $\Lambda_{p/q}$ (after a restriction – if necessary – followed by an extension) gives rise to a homeomorphism from $W_{p/q}$ onto $W_{1/q}$ which is quasi-conformal outside $M_{p/q}$. As further mentioned in the introduction, the combinatorial extension of Φ^q_{p1} described in [BF] assuming local connectivity of the Mandelbrot is not quasi-conformal outside $M_{p/q}$. We end this paper by describing why this is so. The combinatorial extension is defined for each $c \in W_{p/q} \setminus M_{p/q}$ with $\Phi_M(c) = e^{\rho+2\pi i\theta}$ as

$$(\rho, \theta) \mapsto (\rho, \Theta(\theta))$$

where $\Theta : [\theta^-_{p/q}, \theta^+_{p/q}] \to [\theta^-_{1/q}, \theta^+_{1/q}]$ is obtained through combinatorial surgery as described in section 7.1.2 in [BF]. The map is of the form $(\rho, \theta) \mapsto (\rho, h(\theta))$ with $h : I_1 \to I_2$ a homeomorphism between intervals. Indeed, such a map is quasi-conformal if and only if h is bi-Lipschitz. In our case, Θ is not Lipschitz, thus Φ^q_{p1} is not quasi-conformal. To see that Θ is not Lipschitz we compare para-patterns in $W_{p/q}$ and $W_{1/q}$.

We call a parameter value c an α-Misiurewicz point if the critical point eventually falls on the fixed point α_c. As usual fix q and consider an arbitrary p/q. Each α-Misiurewicz point in $M_{p/q}$ is the landing point of q rays of external arguments, say $\nu_1 < \nu_2 < \cdots < \nu_q$. The lengths of the intervals $[\nu_j, \nu_{j+1}]$ for $j = 1, 2, \ldots, q-1$ are of the form $2^{\sigma^p(j)}/D$ where D is a common denominator depending on c and

$\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(q-1)$ is a permutation of $0, 1, \ldots, q-2$. Note that for $p = 1$ the permutation is trivial, i.e. $(\sigma^p(1), \sigma^p(2), \ldots, \sigma^p(q-1)) = (0, 1, \ldots, q-2)$, and for any p we have $\sigma^p(p) = 0$. We consider in the limb $M_{p/q}$ the tree of – what we shall call – *dominating* α-Misiurewicz points together with the tree of external arguments associated to those. Let c^p denote the first dominating α-Misiurewicz point in $M_{p/q}$, i.e., the one of lowest pre-period, and let $\nu_1^p, \ldots, \nu_q^p$ denote the external arguments in increasing order of the q rays landing at c^p. Let W_j^p denote the sub-wake within $W_{p/q}$ bounded by $\mathcal{R}_M(\nu_j^p), \mathcal{R}_M(\nu_{j+1}^p)$ and c^p. Inductively, let $c^p_{j_1,\ldots,j_k}$ denote the dominating α-Misiurewicz point in the sub-wake $W^p_{j_1,\ldots,j_k}$ i.e., the one of lowest pre-period, and let $\nu^p_{j_1,\ldots,j_k,1}, \ldots, \nu^p_{j_1,\ldots,j_k,q}$ denote the external arguments in increasing order of the q rays landing at $c^p_{j_1,\ldots,j_k}$; here $W^p_{j_1,\ldots,j_k}$ denotes the sub-wake within $W^p_{j_1,\ldots,j_{k-1}}$ bounded by $\mathcal{R}_M(\nu^p_{j_1,\ldots,j_{k-1}})$, $\mathcal{R}_M(\nu^p_{j_1,\ldots,j_{k-1}+1})$ and $c^p_{j_1,\ldots,j_{k-1}}$. The surgery map Φ^q_{p1} respects the tree of dominating α-Misiurewicz points, and the combinatorial surgery map Θ respects the tree of associated external arguments, especially

$$\Theta(\nu^p_{j_1,\ldots,j_k}) = \nu^1_{j_1,\ldots,j_k}.$$

Consider in particular the two arguments in the kth generation: $\nu^p_{p,\ldots,p,p}$ and $\nu^p_{p,\ldots,p,p+1}$. A simple computation shows that

$$\frac{\Theta(\nu^p_{p,\ldots,p,p+1}) - \Theta(\nu^p_{p,\ldots,p,p})}{\nu^p_{p,\ldots p,p+1} - \nu^p_{p,\ldots,p}} = 2^{k(p-1)}. \tag{5}$$

Since $2^{k(p-1)}$ is unbounded when k tends to infinity, the map Θ is not Lipschitz (see Figures 23 and 24).

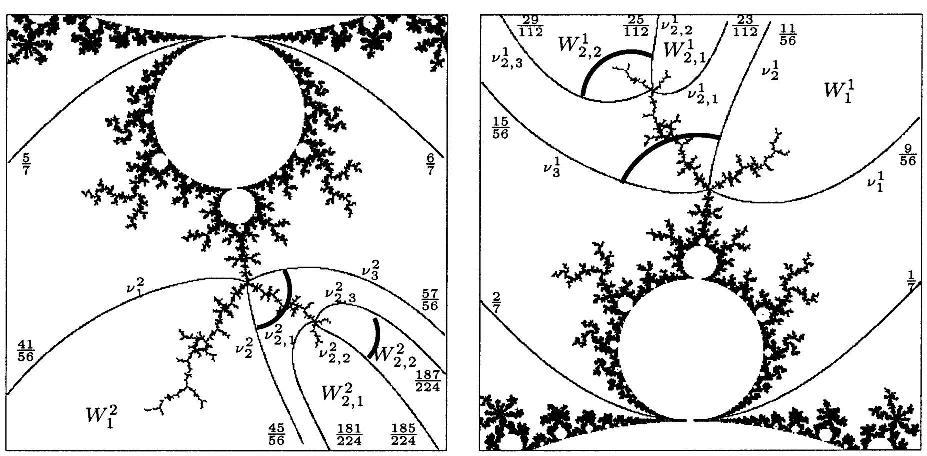

FIGURE 23. Some external arguments of the tree of dominating α-Misiurewicz points in the 2/3 and 1/3 limbs, corresponding to levels $k = 1$ and $k = 2$. Highlighted, we find the intervals in equation (5) for these two levels. The ratios are 2^1 and 2^2 respectively.

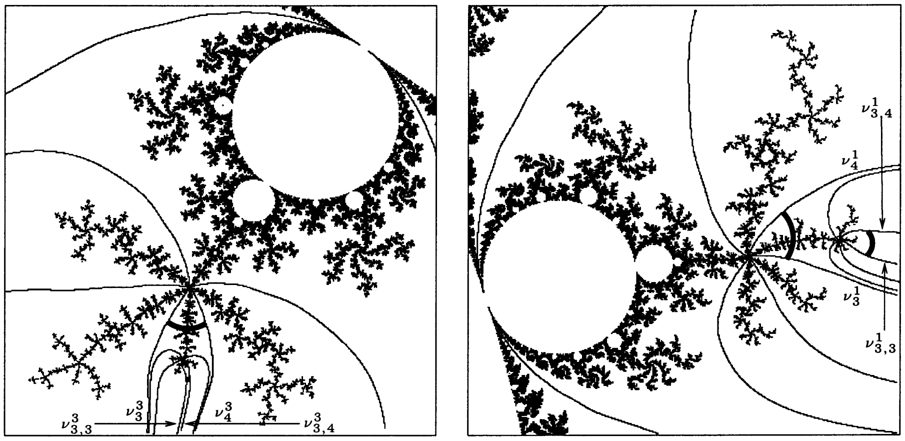

FIGURE 24. Some external arguments of the tree of dominating α-Misiurewicz points in the 3/5 and 1/5 limbs, corresponding to levels $k = 1$ and $k = 2$. Highlighted, we find the intervals in equation (5) for these two levels. The ratio for $k = 1$ is $\frac{47/992-39/992}{695/992-693/992} = 2^2$ and $\frac{171/3968-163/3968}{11111/15872-11109/15872} = 2^4$ for $k = 2$.

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