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MATING KLEINIAN GROUPS ISOMORPHIC TO $C_2 * C_5$ WITH QUADRATIC POLYNOMIALS

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Abstract. Given a quadratic polynomial $q : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ and a representation $G : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of $C_2 * C_5$ in $PSL(2, \mathbb{C})$ satisfying certain conditions, we will construct a 4 : 4 holomorphic correspondence on the sphere (given by a polynomial relation $p(z, w)$) that *mates* the two actions: The sphere will be partitioned into two completely invariant sets Ω and Λ. The set Λ consists of the disjoint union of two sets, Λ_+ and Λ_-, each of which is conformally homeomorphic to the filled Julia set of a degree 4 polynomial P. This filled Julia set contains infinitely many copies of the filled Julia set of q. Suitable restrictions of the correspondence are conformally conjugate to P on each of Λ_+ and Λ_-. The set Λ will not be connected, but it can be joined up using a family $\mathcal{C}$ of completely invariant curves. The action of the correspondence on the complement of $\Lambda \cup \mathcal{C}$ will then be conformally conjugate to the action of G on a simply connected subset of its regular set.

1. Background and motivation

The theories of iterated rational maps [3], [7] and Kleinian groups [2], [10], both acting on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ exhibit a number of striking similarities, which arise from the fact that in both cases $\hat{\mathbb{C}}$ is partitioned into two completely invariant sets, namely the regular set Ω and the limit Λ in the case of a Kleinian group, and the Fatou set F and the Julia set J in the case of a rational map. Orbits of points under the group or under backward iteration of the rational map accumulate on the limit or Julia set respectively, whereas the action of the group or rational map on the regular or Fatou set is discontinuous and equicontinuous.

One can *mate* two abstractly isomorphic Fuchsian groups G_1 and G_2 which are topologically conjugate on the upper half plane by *gluing* them together at their limit sets. This is realised by a third quasi-Fuchsian group G whose regular set consists of two simply connected components. On each of these components the action of G is conformally conjugate to one of the G_i. Similarly, one can mate two hyperbolic quadratic polynomials q_1 and q_2 (which both lie in the main cardiod of the Mandelbrot set) via a third rational map R by gluing them together at their Julia sets. The Fatou set of R will consist of two completely invariant components,

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on each of these the action of R is conjugate to the action of one of the q_i on its filled Julia set. If these Fatou components have interiors, then the conjugacies to the filled Julia sets will be conformal on these interiors.

The aim of this article is to mix the two notions above and to construct matings between Kleinian groups and polynomials. In other words we seek to construct a multivalued map on the Riemann sphere $\hat{\mathbb{C}}$ which behaves like the group on one part of the sphere and like the polynomial on its complement, so that the action matches on the boundary. This boundary should contain both the Julia set of the polynomial and the limit set of the group. The first examples of such matings are due to S. Bullett and C. Penrose [6], who mated representations of $C_2 * C_3$ in $PSL(2,\mathbb{C})$ (in particular the modular group) with quadratic polynomials.

The multivalued maps realising matings will be neither groups nor rational maps, but *holomorphic correspondences*:

Definition 1.1. A holomorphic correspondence of bidegree $m : n$ is a multivalued map $z \to w$ of the Riemann sphere defined by $p(z,w) = 0$ where p is a polynomial of degree m in z and n in w.

Rational maps and Kleinian groups themselves are examples of holomorphic correspondences: Let R be a rational map of degree m. Then it can be expressed as the $m : 1$ correspondence $z \to w$ defined by $R(z) - w = 0$. Similarly, if G is a finitely generated Kleinian group with generators $x_1, \ldots, x_n$, then the orbit of a point z under G can be expressed as the orbit of z under the $n : n$ correspondence $z \to w$ defined by

$$(x_1(z) - w)(x_2(z) - w) \ldots (x_n(z) - w) = 0.$$

1.1. **Hecke groups and shift actions.** All matings that have been constructed up to now involve *Hecke groups* or *non-contact Hecke groups*, as defined below. In order to clarify why these groups lend themselves particularly well to be mated (a fact that was noticed by S. Bullett and C. Penrose) and to give an intuitive idea of how a mating might look, we give a brief review of the properties of such groups.

Definition 1.2. For $n \geq 3$, the *Hecke group* G_n is the faithful discrete representation of $C_2 * C_n$ in $PSL(2,\mathbb{R})$ generated by the matrices:

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho = \begin{pmatrix} 0 & 1 \\ -1 & -2\cos(\pi/n) \end{pmatrix}.$$

The Hecke group G_n acts on the complex upper half-plane with limit set $\hat{\mathbb{R}} = \mathbb{R} \cup \infty$. The Hecke group G_3 is the *modular group* $PSL(2,\mathbb{Z})$.

Note that for a Hecke group G_n there are fundamental domains D_σ and D_ρ for the generators σ and ρ respectively, whose boundaries touch in two 'contact points' p and ∞, both in the limit set. The interiors of D_σ and D_ρ together cover the whole of $\hat{\mathbb{C}} - \{p, \infty\}$ (see Figure 1).

Definition 1.3. A Kleinian group G is called a *non-contact Hecke group of degree* n, for $n \geq 3$, if it is a discrete faithful representation of $C_2 * C_n$ in $PSL(2,\mathbb{C})$, and if there exist fundamental domains D_σ and D_ρ for the order 2 generator σ and the order n generator ρ respectively, whose interiors together cover the whole of the Riemann sphere.

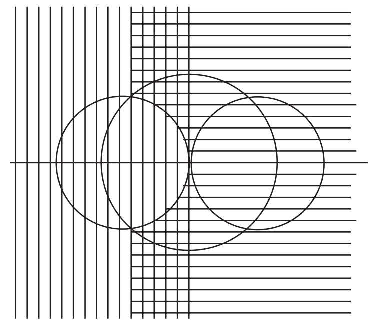

FIGURE 1. The vertically shaded region is the left half-plane, a fundamental domain for σ. The horizontally shaded region is a fundamental domain for ρ. The central circle is the unit circle, meeting the other two circles in the points $-cos(\pi/n) \pm isin(\pi/n)$, which are the fixed points of ρ, and $cos(\pi/n) \pm isin(\pi/n)$. The boundaries of the two domains meet each other in the points 0 and ∞.

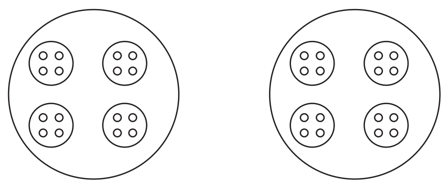

FIGURE 2. The outside of the disc on the left is D_σ and the outside of the disk on the right is D_ρ. Within $\hat{\mathbb{C}} - D_\sigma$ and $\hat{\mathbb{C}} - D_\rho$ we see the beginning of the nested sequences of disks, consisting of images of D_ρ under words of the form $\sigma\rho^j\sigma\rho^i$, for $0 \leq i, j \leq 4$ (these lie within $\hat{\mathbb{C}} - D_\sigma$) and of images of D_σ under words of the form $\rho^j\sigma\rho^i\sigma$, $0 \leq i, j \leq 4$ (these lie within $\hat{\mathbb{C}} - D_\rho$).

If G is a non-contact Hecke group, then, by the Klein combination theorem (see [10]) its limit set is a Cantor set which is contained in a quasi-circle. The limit set in this case is the disjoint union of two sets L_+ and L_- given by

$$L_+ = \bigcup \{ \bigcap_{n=0}^{\infty} \sigma\rho^{i_1}\sigma\rho^{i_2} \ldots \sigma\rho^{i_n}(D_\rho) \}$$

and

$$L_- = \bigcup \{ \bigcap_{n=0}^{\infty} \rho^{i_1}\sigma\rho^{i_2}\sigma \ldots \rho^{i_n}\sigma(D_\sigma) \},$$

where the unions are taken over all infinite words of the form $\sigma\rho^{i_1}\sigma\rho^{i_2}\ldots$ and $\rho^{i_1}\sigma\rho^{i_2}\sigma\ldots$ respectively, with $i_j \in \{1, 2, \ldots, n-1\}$ (see Figure 2).

Every point component of L_+ is given by a unique infinite word $\sigma\rho^{i_1}\sigma\rho^{i_2}\ldots$; hence we can parameterise L_+ by infinite sequences $\{i_n - 1\}_{n\geq 1}$ in $0, 1, 2, \ldots, n-2$. The generators $\sigma\rho, \sigma\rho^2, \ldots, \sigma\rho^{n-1}$ act as 'shift to the right and insert one of

$0, 1, 2, \ldots, n-2$ at the beginning' on this parameterisation. The same is true for L_- and the maps $\rho\sigma, \ldots, \rho^{n-1}\sigma$.

If $G = G_n$ is a Hecke group, then it is also true that each point in $\mathbb{R}^+ = \{x \in \mathbb{R} | 0 \leq x \leq \infty\}$ is given by an infinite word in $0, 1, 2, \ldots, n-2$ as above. Although in this case the word might not be unique, the ambiguity turns out to be a familiar one: consider the unit interval I with points given by their base $n-1$ expansion $.x_1 x_2 \ldots$. Then the map $h : \mathbb{R}^+ \to I$ which sends a point given by an infinite word $i_1 i_2 \ldots$ to the point with base $n-1$ expansion $.i_1 i_2 \ldots$ is a homeomorphism which conjugates each of $\sigma\rho, \ldots, \sigma\rho^{n-1}$ to the shift to the right with insertion of $0, 1, \ldots, n-2$ respectively. The same parameterisation exists for $\mathbb{R}^- = \{x \in \mathbb{R} | -\infty \leq x \leq 0\}$; in this case the maps $\rho^i\sigma$ carry to branches of the inverse shift (see [11]).

It is the fact that these groups carry an action of the shift on their limit sets which enables us to mate them with polynomials: let P be a polynomial of degree $n-1$. The filled Julia set K_P is defined to be $K_P = \bigcap_{n=0}^{\infty} P^{-1}(U)$, for some sufficiently large topological disc U in $\mathbb{C}$. In a neighbourhood of infinity we have a system of external rays, arising from the conjugacy between P and $z \to z^{n-1}$ (the "Böttcher Coordinate") which exists on this neighbourhood. These external rays can be continued to all of the complement of K_P, using P to "pull back" existing rays. The rays branch if they hit an escaping critical point. If all of the rays land on ∂K_P (which happens, for example, when all connected components of K_P are locally connected), then we can assign to each point in ∂K_P the angle or angles of the rays landing at it. Thus, if we write the angles in base $n-1$ notation, the boundary of K_P is parameterised by infinite sequences is $0, 1, \ldots, n-2$, with the action of P being carried to the left shift.

Given a (non-contact) Hecke group G of degree n, we get a semi-conjugacy $\phi_+ : L_+ \to \partial K_P$ conjugating the generators $\sigma\rho, \ldots, \sigma\rho^{n-1}$ to the inverse branches of P. Similarly, we get a semi-conjugacy $\phi_- : L_- \to \partial K_P$ conjugating the generators $\rho\sigma, \ldots, \rho^{n-1}\sigma$ to the inverse branches of P.

A mating between P and G can now be imagined as follows: the limit set of G is contained in a completely invariant quasi-circle (or a circle if $G = G_n$), which divides the sphere into two completely invariant sets H_1 and H_2. If we shrink geodesics in H_1 that connect points in L_+ and L_- which have the same image under the semi-conjugacies ϕ_+ and ϕ_- respectively, we obtain two sets Λ_+ and Λ_- homeomorphic to K_P. If G is a non-contact Hecke group, then, in the process of shrinking geodesics, we glue together or fold the parts of the quasi-circle which are the gaps in the Cantor sets L_+ and L_-. The new object we obtain is a sphere partitioned into the sets Λ_+ and Λ_-, H_2 and a family $\mathcal{C}$ of curves representing what used to be the gaps in the Cantor sets. An $n-1 : n-1$ correspondence F can be considered to be a mating of G and P, if:

- $\Lambda = \Lambda_+ \sqcup \Lambda_-$, H_2 and $\mathcal{C}$ are each completely invariant under F.
- The restriction of F to Λ_+ is a $1 : n-1$ correspondence and conjugate to the action of P^{-1} on K_P. This conjugacy is conformal on the interior of Λ_+, if it has any.
- The restriction of F^{-1} to Λ_- is a $1 : n-1$ correspondence and conjugate to the action of P^{-1} on K_P. This conjugacy is conformal on the interior of Λ_-, if it has any.

- The remaining $n-2$ branches of F on Λ_- send Λ_- homeomorphically onto Λ_+.
- On H_2, the correspondence F is conformally conjugate to the group G, acting as the $n-1:n-1$ correspondence $z \to w$ defined by

$$(\sigma\rho z - w)(\sigma\rho^2 z - w)\dots(\sigma\rho^{n-1} z - w) = 0.$$

In [4], S. Bullett and W. Harvey used quasi-conformal surgery to construct $2:2$ correspondences with this behaviour, which mate (non-contact) Hecke groups of degree 3 with quadratic polynomials. The aim of this article is to illustrate how this construction can be generalised to matings involving groups G of any degree n. Because of the nature of the construction, the polynomials P of degree $n-1$ which are mated with a group G, are of a particular type: they have $(n-1)/2$ quadratic-like restrictions if n is odd, and $(n-2)/2$ quadratic-like restrictions and one polynomial-like restriction of degree 1, if n is even. Moreover, all these quadratic-like restrictions have the same critical value. Although these polynomials are of a certain type, we find some flexibility within the type of matings constructed here: given any quadratic polynomial q with connected Julia set and any integer $m \geq 1$, one can construct a degree $n-1$ polynomial P of the type described above, such that a restriction of the iterate P^m of P is quadratic-like and hybrid-equivalent to q. Hence K_P contains infinitely many copies of K_q. This polynomial can then be mated with G. It is for this reason that we consider the correspondences constructed in this paper to be matings between *quadratics* and groups, rather than between degree $n-1$ polynomials and groups. Varying the integer m gives infinitely many different degree $n-1$ polynomials which have the property that a restriction of an iterate is hybrid-equivalent to q. Moreover, each such degree $n-1$ polynomial can be fitted into our mating construction in various ways, each giving rise to a *different* mating. Thus, given any q and G, there are infinitely many correspondences that realise a mating between the two.

It is in this point that the matings involving groups of degree $n > 3$ significantly differ from the examples involving groups of degree 3 constructed in [4].

In this paper, we shall generalise the construction of matings in [4] to the case $n = 5$. For all other n the construction works in an analogous way and one will not encounter any difficulties that have not been dealt with here.

To be precise, we shall prove that given any non-contact Hecke group of degree 5 and any quadratic polynomial in the Mandelbrot set, there exists a $4:4$ correspondence from the 2-parameter family $z \to w$ defined by

$$\left[\left(\frac{aw-1}{w-1}\right)^2 + \lambda\left(\frac{aw-1}{w-1}\right)\left(\frac{az+1}{z+1}\right) + \left(\frac{az+1}{z+1}\right)^2 - k\right]$$
$$\left[\left(\frac{aw-1}{w-1}\right)^2 + (1-\lambda)\left(\frac{aw-1}{w-1}\right)\left(\frac{az+1}{z+1}\right) + \left(\frac{az+1}{z+1}\right)^2 - \lambda^2 k\right] = 0$$

where $\lambda = -2cos(2\pi/5)$ and $a, k \in \mathbb{C}$, which realises a mating between the two.

As mentioned before, there are in fact infinitely many such matings. We shall explicitly describe the simplest ones in the main body of the paper. In section 11 we shall give an idea of how more complicated ones are constructed.

The matings between a quadratic q and the Hecke group G_5 arise from letting one of the parameters in the 2-parameter family above tend to a certain limit (this corresponds to shrinking the gaps in the Cantor limit sets of non-contact Hecke groups until the limit set is joined up). We shall not discuss such matings here. They are described in detail in [9] and [11].

2. Tools

We shall make use of the Measurable Riemann Mapping Theorem and Douady and Hubbard's Straightening Theorem.

Theorem 2.1 (Measurable Riemann Mapping Theorem [1])**.** *Any almost complex structure on a Riemann surface U with bounded dilatation ratio is integrable.*

Definition 2.2. Let U and V be subsets of $\hat{\mathbb{C}}$, homeomorphic to open disks so that the closure of U is properly contained in V. Suppose $f : U \to V$ is a proper holomorphic map of degree d. Then f is called *polynomial-like* of degree d.

The *filled Julia set* of f is defined to be

$$K_f = \bigcap_{n=0}^{\infty} f^{-n}(U).$$

A polynomial-like map f and a polynomial g of the same degree are said to be *hybrid equivalent* if there exists a quasi-conformal homeomorphism ϕ conjugating f to g, such that the dilatation of ϕ is zero on K_f.

Theorem 2.3 (Straightening Theorem [8])**.** *Every polynomial-like map f is hybrid equivalent to a polynomial g of the same degree. If K_f is connected, then g is unique up to affine conjugation.*

3. The definition of a mating

Definition 3.1. We say that a holomorphic $4:4$ correspondence $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a mating between a quadratic polynomial $q : z \to z^2 + c$ and a non-contact Hecke Group G via a degree 4 polynomial P if $\hat{\mathbb{C}}$ is partitioned into two sets Ω and Λ with the following properties:

- Λ and Ω are both completely invariant under F.
- Λ is the disjoint union of two sets (themselves not connected) Λ_+ and Λ_-, where Λ_+ is forward invariant under F and mapped 1:4 onto itself by F, Λ_- is backward invariant under F and mapped 4:1 onto itself by F. The remaining branches of F map Λ_- 3:3 onto Λ_+.
- Λ_+ and Λ_- have neighbourhoods N_+ and N_- respectively, on which the restrictions $F^{-1}|_{N_+}$ and $F|_{N_-}$ are hybrid equivalent to P. Thus Λ_+ and Λ_- are homeomorphic to K_P and the conjugacy is conformal on any parts of Λ_+ and Λ_- that have interiors.
- K_P contains an invariant subset $K_{P,c}$ on a neighbourhood of which a restriction of P is hybrid equivalent to q. Thus, in particular, the restriction of P to $K_{P,c}$ is topologically conjugate to q acting on its filled Julia set and the conjugacy is conformal on interiors if there are any.
- There exists a completely invariant set of curves $\mathcal{C}$ in Ω such that $\Omega - \mathcal{C}$ is simply connected and such that the action of F on $\Omega - \mathcal{C}$ is conformally conjugate to that of G on a simply connected subset $\mathcal{D}$ of its regular set. To be precise, if $G = \langle \sigma, \rho \rangle$, then F is conformally conjugate to the $4:4$ correspondence $z \to w$ defined by

$$\left(w - \sigma\rho z\right)\left(w - \sigma\rho^2 z\right)\left(w - \sigma\rho^3 z\right)\left(w - \sigma\rho^4 z\right) = 0.$$

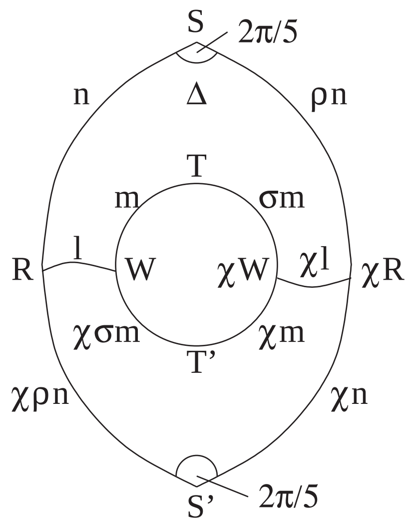

FIGURE 3. A fundamental domain for G.

We will have a closer look at the ingredients G and q in the following two sections.

4. The group

We consider a non-contact Hecke group G. For any such group G, there exists a (unique) involution χ which exchanges the two fixed points of σ and the two fixed points of ρ and such that $\chi\sigma = \sigma\chi$ and $\chi\rho = \rho^{-1}\chi$. Write H for the group $\langle \sigma, \rho, \chi \rangle$.

Figure 3 illustrates the fundamental domain of G: S and S' are the fixed points of ρ, T and T' are the fixed points of σ, R is a fixed point of $\chi\rho$ and W is a fixed point of $\chi\sigma$. We choose the lines l, m, and n, joining R to W, W to T and R to S so that their images on $\Omega(H)/H$ do not intersect (here, $\Omega(H)$ is the regular set of H): the quotient $\Omega(H)/H$ is a sphere with four cone points (coming from the fixed points of ρ, σ, $\chi\sigma$ and $\chi\rho$). We can always connect these points by non-intersecting lines in the desired order and take l, m and n to be the inverse images under the projection $\Omega(H) \to \Omega(H)/H$.

A fundamental domain for ρ is the region bounded by $n, \rho n, \chi n$ and $\chi\rho n$ and a fundamental domain for σ is the region exterior to the loop made up of $m, \sigma m, \chi m$ and $\chi\sigma m$. By the Klein combination theorem, the intersection of these two fundamental domains is a fundamental domain for the action of G on $\Omega(G)$. The part Δ of that fundamental domain bounded by $n, l, m, \sigma m, \chi l$ and ρn is a fundamental domain for the action of H. The union of all translates of Δ under elements of G is a topological disk $\mathcal{D}$ which is a fundamental domain for the action of χ. The complement $\Lambda(G)$ of $\Omega(G) = \mathcal{D} \cup \chi(\mathcal{D})$ in $\hat{\mathbb{C}}$ is a Cantor set.

The quotient $\hat{\mathbb{C}}/\chi$ is a sphere and the covering map $\pi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}/\chi$ sends $\mathcal{D}$ homeomorphically onto its image. Consider the region $\Delta^\rho = \bigcup_{0 \le i \le 4} \rho^i(\Delta)$ (see Figure 4). Under π, this maps onto a *double annulus* B (see Figure 5).

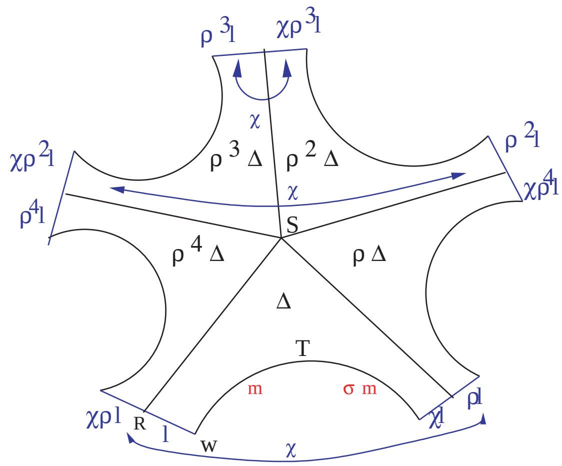

FIGURE 4.

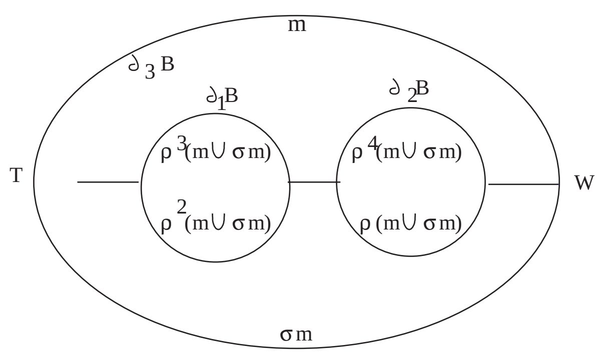

FIGURE 5. The double annulus B.

The action of G now projects to B: Let

- $\partial_3 B = \pi\,(m \cup \sigma m)$,
- $\partial_1 B = \pi\left(\rho^3 m \cup \rho^3 \sigma m \cup \rho^2 m \cup \rho^2 \sigma m\right)$,
- $\partial_2 B = \pi\left(\rho^4 m \cup \rho^4 \sigma m \cup \rho m \cup \rho \sigma m\right)$.

The projection of ρ is no longer well defined, but the projection of $\rho \cup \cdots \cup \rho^4$ gives a $4:4$ correspondence. We note that the points $\pi(R) = \pi(\rho(R))$, $\pi(\rho^3(R)) = \pi(\rho^4(R))$ and $\pi\rho^3\,(R)$ are singular in the sense that they each have less than four images.

We get the following maps on the boundary components of B:

The involution σ projects to an orientation reversing involution σ on $\partial_3 B$. The projection of the maps

$$\sigma\rho^2 : \left(\rho^3 m \cup \rho^3 \sigma m\right) \to m \cup \sigma m$$

and

$$\sigma\rho^3 : \left(\rho^2 m \cup \rho^2 \sigma m\right) \to m \cup \sigma m$$

is an orientation preserving $2:1$ map from $\partial_1 B$ to $\partial_3 B$ which we shall denote by δ_1.

Similarly, the projection of the maps

$$\sigma\rho : \big(\rho^4 m \cup \rho^4 \sigma m\big) \to m \cup \sigma m$$

and

$$\sigma\rho^4 : (\rho m \cup \rho\sigma m) \to m \cup \sigma m$$

is an orientation preserving $2:1$ map from $\partial_2 B$ to $\partial_3 B$ which we shall denote by δ_2.

The maps

- $\rho^4 : \big(\rho^3 m \cup \rho^3 \sigma m\big) \to \big(\rho^2 m \cup \rho^2 \sigma m\big)$,
- $\rho : \big(\rho^2 m \cup \rho^2 \sigma m\big) \to \big(\rho^3 m \cup \rho^3 \sigma m\big)$,

together project to a homeomorphism ι_1 from $\partial_1 B$ to itself, exchanging points with the same image under δ_1. Similarly, the pair

- $\rho^3 : (\rho m \cup \rho\sigma m) \to \big(\rho^4 m \cup \rho^4 \sigma m\big)$,
- $\rho^2 : \big(\rho^4 m \cup \rho^4 \sigma m\big) \to (\rho m \cup \rho\sigma m)$,

gives a homeomorphism ι_2 from $\partial_2 B$ to itself, which is the deck transformation of δ_2.

The pairs of maps

- $\rho : \big(\rho^3 m \cup \rho^3 \sigma m\big) \to \big(\rho^4 m \cup \rho^4 \sigma m\big)$,
- $\rho^4 : \big(\rho^2 m \cup \rho^2 \sigma m\big) \to (\rho m \cup \rho\sigma m)$,

and

- $\rho^2 : \big(\rho^2 m \cup \rho^2 \sigma m\big) \to \big(\rho^4 m \cup \rho^4 \sigma m\big)$,
- $\rho^3 : \big(\rho^3 m \cup \rho^3 \sigma m\big) \to (\rho m \cup \rho\sigma m)$,

together give a $2:2$ correspondence γ from $\partial_1 B$ to $\partial_2 B$, which exchanges pairs of points with the same image under δ_1 and δ_2.

5. The quadratic polynomial

Given a quadratic polynomial q in the Mandelbrot set, we can always find two open disks V and U, such that $q(V) = U$ and $\bar{V} \subset U$. By definition, the filled Julia set K_q of q is

$$\bigcap_{n=0}^{\infty} q^{-n}(U),$$

and contains the critical point 0. Thus the boundary of V gets mapped $2:1$ onto that of U, and there exists an involution on ∂V interchanging points with the same image under q. Moreover, we have an involution on the boundary of U, coming from external rays: if $z \in \partial U$ lies on an external ray of argument $2\pi\theta$, then the involution sends z to the point on ∂U that lies on an external ray of argument $2\pi(1-\theta)$.

6. Generalised polynomial-like maps

In this section we will show that given any quadratic q in the Mandelbrot set, there exists a degree 4 polynomial P with the following properties: there exist simply connected regions U, V_1 and V_2 in $\mathbb{C}$, with $\bar{V}_i \subset U$ for each i, and $\bar{V}_1 \cap \bar{V}_2 = \emptyset$, and such that P restricted to each of the V_i is quadratic-like and maps V_i onto U. Moreover, $P_1 = P|_{V_1} : V_1 \to U$ is hybrid-equivalent to q. Hence $U - (\bar{V}_1 \cup \bar{V}_2)$ is a double annulus carrying similar data on its boundary components as the double annulus B. This fact will enable us to construct a mating. In order to prove the existence of the polynomial P, we need a slightly more general version of the Straightening Theorem:

Proposition 6.1. *Let U, V_1, V_2 be simply connected open subsets of $\mathbb{C}$ with smooth Jordan curves as boundaries such that $\bar{V_1} \subset U$ and $\bar{V_2} \subset U$ and $\bar{V_1} \cap \bar{V_2} = \emptyset$. Let $f : (V_1 \cup V_2) \to U$ be a proper 4:1 holomorphic map such that $f|_{V_1} : V_1 \to U$ and $f|_{V_2} : V_2 \to U$ are quadratic-like.*

Then f is hybrid equivalent to a polynomial of degree 4.

Proof. Let Q be the degree 4 polynomial $Q(z) = z^4 - 2z^2 + 2$. Then Q has finite critical points ± 1 and 0, where 0 escapes to infinity.

$Q(1) = Q(-1) = 1$, so 1 and -1 don't escape to infinity. It's easy to check that Q has the following properties:

(i) there exists an open simply connected domain $S \subset \mathbb{C}$ with smooth boundary and $\bar{S} \subset Q(S)$;
(ii) $Q^{-1}(S)$ consists of two open simply connected components T_1 and T_2, such that $\bar{T_1} \cup \bar{T_2} \subset S$ and $\bar{T_1} \cap \bar{T_2} = \emptyset$;
(iii) $Q : T_1 \to S$ and $Q : T_2 \to S$ are quadratic-like.

Now let $R : \hat{\mathbb{C}} - \bar{U} \to \hat{\mathbb{C}} - \bar{S}$ be a Riemann map with $R(\infty) = \infty$. Then R extends smoothly to $R : \hat{\mathbb{C}} - U \to \hat{\mathbb{C}} - S$, because the boundaries of R and S are smooth simple curves.

We observe that f describes ∂U twice as it runs through ∂V_1 and similarly Q describes ∂S twice as it runs through ∂T_1. Therefore we can define a smooth map $\phi_{V_1} : \partial V_1 \to \partial T_1$ such that $Q \circ \phi_{V_1}(x) = R \circ f(x)$.

In the same way we can define a smooth map $\phi_{V_2} : \partial V_2 \to \partial T_2$, respecting the action of f and Q.

An elementary argument shows that there exists a quasi-conformal homeomorphism

$$\phi : \bar{U} - (V_1 \cup V_2) \to \bar{S} - (T_1 \cup T_2)$$

which extends ϕ_{V_1}, ϕ_{V_2} and $R|_{\partial U}$.

Let

$$\psi(z) = \begin{cases} R(z) & \text{on } \mathbb{C} - U, \\ \phi(z) & \text{on } U - (V_1 \cup V_2). \end{cases}$$

We define a new map $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) = \begin{cases} f(z) & \text{on } V_1 \cup V_2, \\ \psi^{-1} \circ Q \circ \psi(z) & \text{on } \mathbb{C} - (V_1 \cup V_2). \end{cases}$$

Now g is continuous everywhere and holomorphic on $V_1 \cup V_2$ and $\mathbb{C} - \bar{U}$. Let $A = U - (\bar{V_1} \cup \bar{V_2})$.

We now define a g-invariant almost complex structure σ_1 on $\mathbb{C}$, by defining an infinitesimal field of ellipses with bounded ratio of axes, such that the ellipses are invariant under the tangent map of g:

For $x \in \mathbb{C} - \bar{U}$, we define E_x to be a circle.

For $x \in g^{-m}(A)$, where $m \geq 0$, we define

$$E_x = \left(T_x g^{m+1}\right)^{-1} \left(E_{g^{m+1}(x)}\right),$$

where $T_x g$ is the tangent map of g at x. Now by definition, $E_{g^{m+1}(x)} = S^1$, and the map g^{-m-1} is a composition of a quasi-regular map $g^{-1} = (\psi^{-1} Q \psi)^{-1}$ and a holomorphic map f^{-m}. Therefore the ratio of the axes of the ellipse E_x is bounded. For

$$x \in \bigcap_{n=0}^{\infty} g^{-n}\left(V_1 \cup V_2\right) = K_g,$$

we define E_x to be a circle.

It is easy to check that g respects this almost complex structure: K_g is invariant under g which is holomorphic on K_g sending infinitesimal circles to infinitesimal circles. For x in $\mathbb{C} - \bar{U}$, $g(x)$ is also contained in $\mathbb{C} - \bar{U}$ and g is holomorphic. For x in the rest of $\mathbb{C}$, g respects σ_1 by definition. An application of the Measurable Riemann Mapping Theorem now gives that g is conjugate to a polynomial of degree 4 via a quasi-conformal homeomorphism, which has dilatation zero on K_g. □

Lemma 6.2. *Given any c in the Mandelbrot set we can find a polynomial P with the following properties:*

(i) *There exists a topological disk U, with $P^{-1}(U) = V_1 \cup V_2$, where the V_i are topological disks with disjoint closures and are entirely contained in U.*
(ii) *Each restriction $P_i = P|_{V_i} : V_i \to U$ is quadratic-like, and the restriction P_1 is hybrid-equivalent to $q : z \to z^2 + c$, so K_P is made up of infinitely many copies of K_q and point components.*

Proof. As mentioned before, q comes equipped with two topological disks V_1 and U such that $\bar{V}_1 \subset U$ and $q(V_1) = U$. Choose an open simply connected region V_2 contained in U such that $\bar{V}_2 \subset U$ and $\bar{V}_1 \cap \bar{V}_2 = \emptyset$ and such that the boundary of V_2 satisfies the conditions of Proposition 6.1. We can now find a holomorphic homeomorphism M from V_2 onto V_1.

Define $f : V_1 \cup V_2 \to U$ by

$$f\left(z\right) = \begin{cases} q\left(z\right) & \text{for } \ z \in V_1, \\ q \circ M\left(z\right) & \text{for } \ z \in V_2. \end{cases}$$

Now $f|_{V_1}$ and $f|_{V_2}$ are holomorphic proper maps of degree 2, and we can apply Proposition 6.1.

Hence there exists a polynomial P of degree 4 to which f is hybrid equivalent via a quasi-conformal homeomorphism ϕ. Taking $\phi(V_i)$ and $\phi(U)$ to be the disks V_i and U in the statement of the Lemma, it is easy to see that P satisfies properties (i) and (ii). □

Remark 6.3. Actually one can prove that given any q there exists a degree 4 polynomial P, giving rise to regions V_i and U as above, but with an *iterate* P^n which, restricted to a certain region, is hybrid-equivalent to q. This is not hard, but tedious to write down. The proof for $n = 2$ is included in section 11, and is left to the reader for $n > 2$.

Now suppose q is in the Mandelbrot set. The corresponding polynomial P as constructed in Lemma 6.2 gives rise to the double annulus $A = U - (\bar{V}_1 \cup \bar{V}_2)$ and the following maps:

- A homeomorphism $M : V_2 \to V_1$ (corresponding to $\phi M \phi^{-1}$ in the proof of Lemma 6.2), which extends to the boundary of V_2.
- On each of the $\bar{V_j}$ we have an involution $i_j : \bar{V_j} \to \bar{V_j}$ sending each point to the point that has the same image under P.
- Two quadratic-like maps $P_j = P|_{V_j} : V_j \to U$, with a common critical value t. The maps give $2 : 1$ maps from ∂V_j to ∂U. Moreover, M and $i_1 \circ M$ are two homeomorphisms from $\bar{V_2}$ to $\bar{V_1}$ which, post-composed with P_1 give P_2.
- On ∂U there is an involution j defined on external angles as described in section 5.

Thus A carries the same data on its boundary components as the double annulus B coming from the group G.

Let

- $\partial_3 A = \partial U$,
- $\partial_1 A = \partial V_1$,
- $\partial_2 A = \partial V_2$.

7. A bijection between the double annuli

Lemma 7.1. *There exists a quasi-conformal bijection h from $\bar{A}$ to $\bar{B}$ such that:*

- $\partial_1 A \to \partial_1 B$.
- $\partial_2 A \to \partial_2 B$.
- $\partial_3 A \to \partial_3 B$.
- *For $x \in \partial_1 A$,*
$$P(x) = h^{-1} \delta_1 h(x).$$
- *For $x \in \partial_2 A$,*
$$P(x) = h^{-1} \delta_2 h(x).$$
- *For $x \in \partial_3 A$,*
$$j(x) = h^{-1} \sigma h(x).$$

Proof. We choose a smooth homeomorphism $h_3 : \partial_3 A \to \partial_3 B$ conjugating j to σ. Using the same procedure as in the proof of Proposition 6.1 we can construct smooth homeomorphisms,

$$h_2 : \partial_2 A \to \partial_2 B,$$
$$h_1 : \partial_1 A \to \partial_1 B,$$

conjugating P to the required maps. Now there exists a quasi-conformal homeomorphism $h : A \to B$ extending h_1 , h_2 and h_3. □

Remark 7.2. We could have chosen h to send $\partial_1 A$ to $\partial_2 B$ and $\partial_2 A$ to $\partial_1 B$. This would not make any difference for the surgery construction, but give a different mating (see section 11).

8. Constructing a correspondence

Let G be a non-contact Hecke group, q a quadratic polynomial in the Mandelbrot set, and P be a degree 4 polynomial constructed as in Lemma 6.2. We have seen that there exists a simply connected region U carrying both the action of the degree 4 polynomial P and, on the subset A of U, the action of the group G by quasi-conformal conjugation. Now take a second copy U^* of U. Given a point $u \in U$, we denote by u^* the corresponding point in U^*. Let Σ be the quotient space $(U \cup U^*)/\sim$, where $\sim$ identifies $z \in \partial U$ to $j(z)^* \in \partial U^*$. The object Σ equipped with the quotient topology is a topological sphere carrying an involution j given by

$$j|_U : u \to \begin{cases} j(u) & \text{if } u \in \partial U, \\ u^* & \text{if } u \in U, \end{cases}$$

and

$$j(u^*) = j^{-1}(u^*).$$

It is on this sphere that we will construct the $4:4$ correspondence realising our mating. This correspondence will actually be the product of two $2:2$ correspondences, one for each of the pairs of generators $\{\sigma\rho, \sigma\rho^4\}$ and $\{\sigma\rho^2, \sigma\rho^3\}$ of the group G.

Let

- $V_1^* = j(V_1)$,
- $V_2^* = j(V_2)$,
- $A^* = j(A)$,
- $P^* = jPj : V_1^* \cup V_2^* \to U^*$,
- $i_1^* = ji_1j : V_1^* \to V_1^*$,
- $i_2^* = ji_2j : V_2^* \to V_2^*$,
- $M^* = jMj : V_2^* \to V_1^*$.

We have a $2:2$ correspondence g on $\bar{A}$ defined by $g : z \to w$ if

$$w \in \left\{ h^{-1}\pi\rho^2\pi^{-1}h(z), h^{-1}\pi\rho^3\pi^{-1}h(z) \right\}.$$

Restricted to ∂V_1, g splits into two branches:

$$g_1 : \partial V_1 \to \partial U, \quad z \to \begin{cases} h^{-1}\pi\rho^2\pi^{-1}h(z) & \text{if } \pi^{-1}h(z) \in \rho^3 m \cup \rho^3\sigma m, \\ h^{-1}\pi\rho^3\pi^{-1}h(z) & \text{if } \pi^{-1}h(z) \in \rho^2 m \cup \rho^2\sigma m, \end{cases}$$

$$g_2 : \partial V_1 \to \partial V_2, \quad z \to \begin{cases} h^{-1}\pi\rho^3\pi^{-1}h(z) & \text{if } \pi^{-1}h(z) \in \rho^3 m \cup \rho^3\sigma m, \\ h^{-1}\pi\rho^2\pi^{-1}h(z) & \text{if } \pi^{-1}h(z) \in \rho^2 m \cup \rho^2\sigma m. \end{cases}$$

Note that $g_1 = j \circ P|_{\partial V_1}$. It is easy to check that either

(i) $g_2|_{\partial V_1} \equiv M^{-1}$

or

(ii) $g_2|_{\partial V_1} \equiv M^{-1} \circ i_1 = i_2 \circ M^{-1}$.

Define $N : V_1 \to V_2$ to be $N = M^{-1}$ if (i) holds and $N = M^{-1} \circ i_1$ if (ii) holds. Define a $2:2$ correspondence $f_1 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by putting together:

- $j \circ g : A \to A^*$ $(2:2)$,
- $P|_{V_1} : V_1 \to U$ $(2:1)$,
- $j \circ N : V_1 \to V_2^*$ $(1:1)$,
- $j \circ i_2 : V_2 \to V_2^*$ $(1:1)$,
- $j \circ N^{-1} : V_2 \to V_1^*$ $(1:1)$,
- $P^{*-1} : U^* \to V_1^*$ $(1:2)$.

It is easy to check that this correspondence is continuous everywhere.

The following points are "singular":

- The point $h^{-1} \circ \pi (S)$ has a unique image $j \circ h^{-1} \circ \pi (S)$ and is the only pre-image of this point.
- The point $h^{-1} \circ \pi \circ \rho^3 (R)$ has a unique image.
- The point $j(t)$, where t is the critical value of P, has a unique image.
- The point $j \circ h^{-1} \circ \pi \circ \rho^3(R)$ has a unique pre-image.
- The point t has a unique pre-image.

We now define a second $2 : 2$ correspondence $f_2 : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by $f_2 : z \to w$ if $w \in \{f_1 \circ j \circ f_1 (z)\} - \{j (z)\}$. Checking through the maps we have:

Lemma 8.1.

(i) $j \circ f_2|_A = \bar{g}$, *where* g *is the* $2 : 2$ *correspondence defined by* $\bar{g} : z \to w$ *if* $w \in \left\{h^{-1}\pi\rho\pi^{-1}h (z), h^{-1}\pi\rho^4\pi^{-1}h (z)\right\}$.

(ii) *For* $z \in V_2$ *we have* $P (z) \in f_2 (z)$.

(iii) *For* $z \in U^*$ *we have* $f_2 = (P^*)^{-1} : U^* \to V_2^*$.

(iv) *The singular points are those of* f_1.

9. Constructing an almost complex structure

Clearly the $4 : 4$ correspondence $f = f_1 \cup f_2$ carries the action of the degree 4 polynomial P and that of the group G by quasi-conformal conjugation. We will "straighten" this action by defining an almost complex structure σ_1 on the sphere which is respected by f and then using the Measurable Riemann Mapping Theorem to obtain a holomorphic correspondence F conjugate to f.

First, we define a circle field on B. Recall that the quasi-conformal map h sends A to B. Now for x belonging to A, we define

$$E_x = (T_x h)^{-1} \left(S^1\right).$$

For $x \in P^{-n} (A)$, for some $n \in \mathbb{N}$, we define

$$E_x = (T_x P^n)^{-1} \left(E_{P^n(x)}\right).$$

For $x \in \bigcap_{n=0}^{\infty} P^{-n} (V_1 \cup V_2)$, we define

$$E_x = S^1,$$

and, for x belonging to U^*, we define

$$E_x = (T_x j) \left(E_{j(x)}\right).$$

One can check that σ_1 is bounded everywhere: on U it arises from pulling back the circle field on B via a quasi-conformal map and then pulling the resulting ellipse field on A back using a holomorphic map and its iterates. On U^*, σ_1 arises from pulling back the bounded structure on U using a holomorphic map.

Similarly, one can show that the tangent map of each branch θ_i of f, for $1 \leq i \leq 4$, sends the ellipse E_x at a point x to the ellipse $E_{\theta_i(x)}$ at the point $\theta_i(x)$. Thus f respects σ_1.

Using the measurable Riemann mapping theorem, we can find a quasi-conformal homeomorphism $\phi : \left(\hat{\mathbb{C}}, \sigma_1\right) \to \left(\hat{\mathbb{C}}, \sigma_0\right)$, where σ_0 is the standard complex structure defined by circles. Then the correspondence $F = \phi \circ f \circ \phi^{-1}$ respects σ_0 and therefore is holomorphic. It is known that every holomorphic correspondence on the sphere is algebraic [5] and therefore our correspondence F can be expressed as

$z \to w$ if $p(z, w) = 0$, where p is some polynomial relation of degree four in both z and w. Clearly each of f_1 and f_2 pass to holomorphic correspondences F_1 and F_2, while j passes to a conformal involution J.

Lemma 9.1. *The* $4:4$ *correspondence* $F : z \to w$ *is defined by*

$$\left[\left(\frac{aw-1}{w-1}\right)^2 + \lambda\left(\frac{aw-1}{w-1}\right)\left(\frac{az+1}{z+1}\right) + \left(\frac{az+1}{z+1}\right)^2 - k\right]$$
$$\left[\left(\frac{aw-1}{w-1}\right)^2 + (1-\lambda)\left(\frac{aw-1}{w-1}\right)\left(\frac{az+1}{z+1}\right) + \left(\frac{az+1}{z+1}\right)^2 - \lambda^2 k\right] = 0$$

where $\lambda = -2cos(2\pi/5)$, *for some* $a, k \in \mathbb{C}$.

Proof. Consider the two $2:2$ correspondences $J \circ F_1$ and $J \circ F_2$. We observe:

- Both are symmetric, i.e. $z \to w$ iff $w \to z$.
- Each has one *double point*; that is, a point with a unique image which in turn has a unique pre-image, and two *forward singular points*, i.e. points that have unique images which themselves have two distinct pre-images.
- Let $(JF_i)^{[n]}$ denote iteration without *back-tracking*: if z_0 maps to w_0 under JF_i we do not allow w_0 to map back to z_0 under the next iterate unless z_0 is the *unique* image of w_0 under JF_i. Then $(JF_i)^{[5]}(z) = z$ for all z.
- $(J \circ F_1)^2 = J \circ F_2$.

A general $2:2$ correspondence is conjugate to a correspondence of the form $z \to w$ defined by

$$q_1(z)w^2 + q_2(z)w + q_3(z) = 0$$

where the q_i are polynomials in z of degree at most two. The first two points above give that if we change the coordinates such that the double point of JF_1 lies at ∞ and the two singular points at $\pm x$ for some x, then $JF_i : z \to w$ is defined by

$$z^2 + czw + w^2 - 1 = 0$$

for some constant c. The third point gives that $c = \lambda$ or $c = 1 - \lambda$ (see [9] for details). Then, by the last point, the $4:4$ correspondence $J \circ F$ turns out to be conjugate to $z \to w$ defined by

$$(z^2 + \lambda zw + w^2 - 1)(z^2 + (1-\lambda)zw + w^2 - \lambda^2) = 0.$$

Composing with the involution J and normalising so that J becomes $z \to -z$ gives the result. □

10. Statement and proof of the main theorem

Theorem 10.1. *The* $4:4$ *correspondence* F *constructed in Section 9 is a mating between the quadratic map* $q : z \to z^2 + c$ *and the group* G.

Proof. Clearly

$$\bigcap_{n=0}^{\infty} F^{-n}(\phi(V_1) \cup \phi(V_2)) = \phi\left(\bigcap_{n=0}^{\infty} P^{-n}(V_1 \cup V_2)\right) = \phi(K_P).$$

Let

$$\Lambda_- = \phi\left(\bigcap_{n=0}^{\infty} P^{-n}(V_1 \cup V_2)\right).$$

If K_P has interior, then ϕ is conformal on it and it is easy to see that Λ_- satisfies the requirements in the definition of a mating. The involution J sends Λ_- homeomorphically and conformally onto

$$\Lambda_+ = \phi\left(\bigcap_{n=0}^{\infty} P^{*-n}\left(V_1^* \cup V_2^*\right)\right)$$

and conjugates F to F^{-1}, so Λ_+ also has the required properties.

Let $\Lambda = \Lambda_+ \cup \Lambda_-$, $\Omega = \hat{\mathbb{C}} - \Lambda$,

$$L = \bigcup_{0 \le i \le 4} h^{-1}\pi\left(\rho^i(l \cup \chi\rho l)\right)$$

and

$$\mathcal{C}_0 = \{\bigcup_{n=0}^{\infty} P^{-n}\left(L\right)\} \cup \{\bigcup jP^{-n}\left(L\right)\}$$

(see Figures 4 and 5). One can check that $\mathcal{C} = \phi\left(\mathcal{C}_0\right)$ is completely invariant under F, and that $\Omega - \mathcal{C}$ is simply connected and completely invariant under F.

We will show that the action of F on $\Omega - \mathcal{C}$ is conformally homeomorphic to that of the group G on the disk $\mathcal{D}$, defined in section 4.

We define the following regions:

$B_0 = B - \pi(\bigcup_{0 \le i \le 4}\{\rho^i(l \cup \chi\rho l)\})$,
$\tilde{B_0} = \pi^{-1}\left(B_0\right)$,
$A_0 = h^{-1}(B_0)$.

Now $\tilde{B_0}$ consists of the five images under powers of ρ of a fundamental domain of G with respect to $\mathcal{D}$. Hence $\mathcal{D}$ is equal to the union of images of the closure of $\tilde{B_0}$ (with respect to $\mathcal{D}$) under words of the form

$$\sigma^{\epsilon}\rho^{i_1}\sigma \ldots \rho^{i_n}\sigma$$

where $i_k \in \{1, \ldots, 4\}$ and $\epsilon \in \{1, 0\}$.

We have a quasi-conformal homeomorphism $H = \pi^{-1}h : A_0 \to \tilde{B_0}$. By the definition of the correspondence f, the set $\phi^{-1}(\Omega - \mathcal{C})$ is tessilated by images of A_0 under mixed iteration of f. In fact, each tile must be of the form $f^{-n}(A_0)$ or $jf^{-n}(A_0)$. The boundary components $\partial_i A$ of A, for $i = 1, 2, 3$ translate to 5 boundary components b_i, $i = 0, 1, 2, 3, 4$, of A_0, because each of $\partial_1 A$ and $\partial_2 A$ get split into two connected parts by the removal from A of the curves in $\mathcal{C}_0$. The b_i map under H to the boundary components $\rho^i(m) \cup \rho^i\sigma(m)$, for $i = 0, 1, 2, 3, 4$, of $\tilde{B_0}$. Now $f^{-1}(A_0) = P^{-1}(A_0)$ consists of 4 pieces, each having one b_i, $1 \le i \le 5$ as part of their boundary. Similarly, each of the images $\rho^i\sigma(\tilde{B_0})$, $1 \le i \le 4$ has $\rho^i(m) \cup \rho^i\sigma(m)$ as part of its boundary. Thus we can extend H to each component of $f^{-1}(A)$ in a way that ensures continuity and such that H conjugates f to one of $\sigma\rho^i$. Repeating this process we can extend H to all the components of $f^{-n}(A_0)$ for all n. The image of H will consist of all images of $\tilde{B_0}$ under words $\rho^{i_1}\sigma \ldots \rho^{i_n}\sigma$.

We extend H to tiles of the form $jf^{-1}(A_0)$ by the map $H(x) = \sigma Hj(x)$. Clearly the image of H is now all of $\mathcal{D}$. The map $H\phi^{-1} : \Omega - \mathcal{C} \to \mathcal{D}$ now takes the standard complex structure to the standard complex structure and therefore is holomorphic. It follows from the construction of H, that $H\phi^{-1}$ conjugates the action of F on $\Omega - \mathcal{C}$ to that of the group G on $\mathcal{D}$.

This completes the proof of the Theorem. $\square$

11. More complicated matings

By remarks 7.2 and 6.3, there are various ways in which to construct a correspondence representing a mating between a given group G and quadratic q. In each case, the surgery construction and proof of the main theorem go through in exactly the same way, however we'll see that the matings we get give rise to limit sets Λ with a different structure in each case.

11.1. **First variation.** In the construction of a mating above, we mapped the annulus A quasi-conformally to B, so that $\partial_1 A = \partial V_1$ was mapped to $\partial_1 B$ (see Figures 4 and 5). Recall that the disk V_1 contains the critical value of the restriction of the correspondence to $V_1 \cup V_2$.

Of course we could have chosen to map $\partial_1 A$ to $\partial_2 B$ instead and continue the construction in order to obtain a second mating between G and q.

Lemma 11.1. *A mating constructed by the first method is not conjugate to a mating constructed by the second method.*

Proof. Suppose that F_1 and F_2 are holomorphic correspondences conjugate via a homeomorphism ϕ. Associated to each F_i we have an involution J_i with the property that J_i sends a point z to the unique point w such that $w \notin F_i(z)$ but $w \in F_i F_i^{-1} F_i(z)$. Thus the J_i are conjugate via ϕ. Let A_i be the double annulus associated to F_i coming from the degree 4 polynomial, with boundary components $\partial_j A_i$ for $j = 1, 2, 3$ (here the notation is as in section 5). Since $\partial_3 A_i$ contains the fixed points of J_i for each i, we have that $\phi(\partial_3 A_1) \cap \partial_3 A_2 \neq \emptyset$. Moreover, for each i, $\partial_1 A_i$ denotes the boundary of the disk containing the critical value t_i of F_i restricted to the two bounded components of the complement of A_i. Clearly, $\phi(t_1) = t_2$. This, together with the fact that $F_i^{-1}(\partial_3 A_i) = \partial_1 A_i \cup \partial_2 A_i$ implies that $\phi(\partial_j A_1) \cap \partial_j A_2 \neq \emptyset$ for $j = 1, 2$. On the other hand, let p_i denote the singular point of F_i which is the projection of the fixed point S of the group element ρ (see section 4 for notation), and let Δ_1 be the projection of the fundamental domain Δ to the F_1-sphere. Clearly, $\phi(p_1) = p_2$ and inspection of a neighbourhood of the p_i shows that a branch of F_1 restricted to Δ_1 is conjugate to the group element $\sigma\rho^j$ if and only if its conjugate corresponds to $\sigma\rho^j$ or $\sigma\rho^{5-j}$. Consequently, since $\phi(\partial_1 A_1) \cap \partial_1 A_2 \neq \emptyset$, we have that the two branches of F_i which send $\partial_1 A_1$ to $\partial_3 A_1$ correspond to $\{\sigma\rho^j, \sigma\rho^{5-j}\}$ if and only if the branches of F_2 which send $\partial_1 A_2$ to $\partial_3 A_2$ correspond to $\{\sigma\rho^j, \sigma\rho^{5-j}\}$. Thus the two correspondences have been constructed by the same method. □

One can picture the difference between the two matings as follows: In each case the double annulus A coming from the degree 4 polynomial corresponds to the region $\Delta^\rho = \bigcup_{0 \le i \le 4} \rho^i(\Delta)$ with some of its boundary curves identified. The inverse image of the double annulus A under the correspondence corresponds to images of Δ^ρ under $\rho^i\sigma$, $i = 1, 2, 3, 4$, again with the images of the boundary curves $\rho^i(l \cup \chi\rho l)$ identified. The way in which these are identified is stipulated by the method of construction of the mating: if $\partial_1 A$ corresponds to $\partial_1 B$, then the lines must be identified so that the inverse images of $\partial_1 B$ under the map $\delta_1 \cup \delta_2$ form two circles. However, if $\partial_1 B$ corresponds to $\partial_2 A$, then the lines must be identified so that the inverse images of $\partial_1 B$ under the map $\delta_1 \cup \delta_2$ form four circles. If we imagine the images of the set Δ^ρ under the group to be flexible puzzle pieces, then

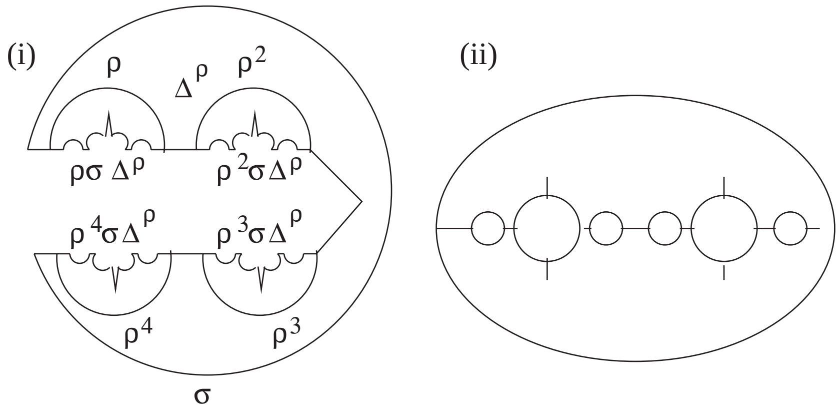

FIGURE 6. Figure (i) shows the set Δ^ρ, and its 4 images under δ_i^{-1} for $i = 1, 2$. A boundary piece of Δ^ρ is labelled by ρ^i if it is equal to $\rho^i m \cup \rho^i \sigma m$. The boundary piece $\sigma m \cup m$ is labelled by σ. Figure (ii) shows the object we get when gluing together curves $\rho^i(l \cup \chi\rho l)$ in the manner imposed by the mating.

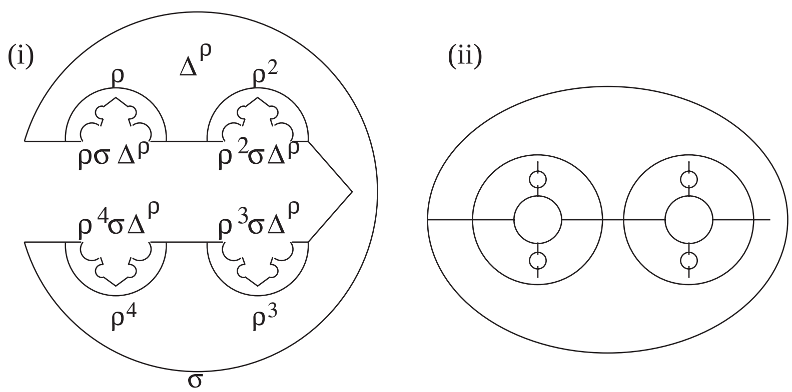

FIGURE 7. Figure (i) is the same as Figure 6(i), but the images of Δ^ρ are drawn differently, in order to illustrate the fact that their boundaries are glued together differently. The resulting object is shown in Figure (ii).

the two different matings described here correspond to different ways of bending and fitting together the puzzle pieces (see Figures 6 and 7).

Figures 8 and 9 show computer plots of the two situations.

11.2. **Second variation.** The next variation is to mate a given group G with a degree 4 polynomial P which has the property that a restriction of an *iterate* P^m of P, for $m > 1$ is hybrid-equivalent to q. Such a P can be constructed for any m,

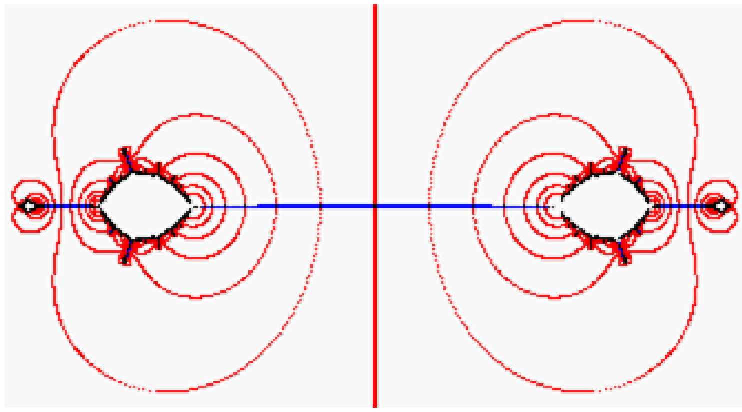

FIGURE 8. This is a computer plot of the set Λ of a mating between a non-contact Hecke group G and $z \to z^2$, which arises from mapping $\partial_1 A$ to $\partial_2 B$. We see the parts Λ_+ and Λ_-. The black part is the boundary of Λ; the red lines are images of $\partial_1 A$ which correspond to $m \cup \sigma(m)$ and the blue lines are elements of the invariant family $\mathcal{C}$ of curves.

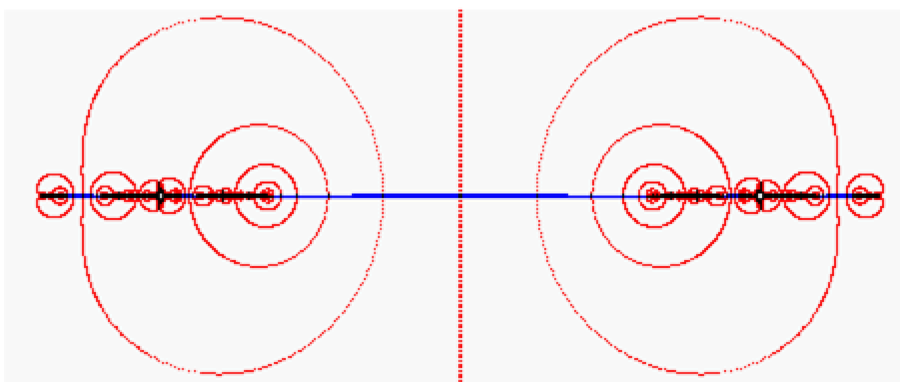

FIGURE 9. This represents a mating between a group G and $z \to z^2$, this time arising from mapping $\partial_1 A$ to $\partial_1 B$.

but we will only prove this fact for $m = 2$:

Proposition 11.2. *Given any quadratic polynomial q in the Mandelbrot set, there exists a degree 4 polynomial P such that a restriction of P^2 is hybrid-equivalent to q.*

Proof. Let Q be the degree 4 polynomial defined in the proof of Proposition 6.1. As before, Q comes with a double annulus $\mathcal{A} = S - (\bar{T}_1 \cup \bar{T}_2)$, and quadratic-like restrictions $Q_i : T_i \to S$. Suppose that the fixed critical point 1 lies in T_1. Then each $D_i = T_i - Q_i^{-1}(\mathcal{A})$ is a quadruply connected region with boundary components ∂T_i, $\partial Q_i^{-1}(T_1)$ and the boundaries of the two connected components of $Q_i^{-1}(T_2)$. The map Q_i maps $C_i = Q_i^{-1}(T_1)$ $2:1$ onto T_1 and each of the two components A_i and B_i of $Q_i^{-1}(T_2)$ $1:1$ onto T_2 (see Figure 10).

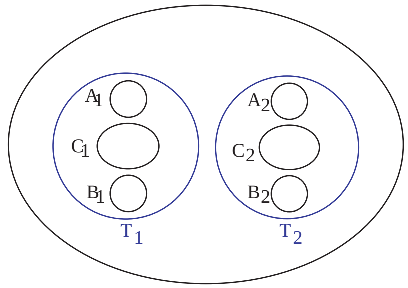

FIGURE 10.

For the quadratic q we find regions A_1^q and U^q such that $q(A_1^q) = U^q$ and $\bar{A}_1^q \subset U^q$. Let V_1^q be a simply connected region such that $q^{-1}(\bar{V}_1^q) \subset A_1^q \subset V_1^q \subset U$. Let V_2^q be a simply connected region such that $\bar{V}_2^q \subset U^q$ and $\bar{V}_1^q \cap \bar{V}_2^q = \emptyset$. Let P_1^q be a Riemann map from A_1^q to V_2^q. Define $P_2^q : V_2^q \to U^q$ by $P_2^q = q \circ (P_1^q)^{-1}$.

Now $(P_2^q)^{-1}(V_2^q)$ consists of two components A_2^q and B_2^q contained in V_2^q. Let $C_2^q = P_2^{-1}(V_1^q) = P_1^q(q^{-1}(V_1^q))$. Then the map $P_2^q : C_2^q \to V_1^q$ is a 2:1 map.

Find disks C_1^q and B_1^q whose closures are contained in V_1^q and don't intersect each other or A_1^q. Let $M_1 : C_1^q \to C_2^q$ and $M_2 : B_1^q \to B_2^q$ be Riemann maps, and define $P_1^q : C_1^q \to V_1^q$ by $P_1^q = P_2^q \circ M_1$ and $P_1^q : B_1^q \to V_2^q$ by $P_1^q = P_2^q \circ M_2$. Note that on A_1 we have $P_2 \circ P_1 = q$. Let $\mathcal{A}^q = U^q - (\bar{V}_1^q \cup \bar{V}_2^q)$ and $D_i^q = V_i^q - (A_i^q \cup B_i^q \cup C_i^q)$.

We now have the same set-up as for the polynomial Q (see Figure 10). Let $R : \hat{\mathbb{C}} - U^q \to \hat{\mathbb{C}} - S$ be a Riemann map fixing infinity. This extends to the boundary of U^q. Now there exists a quasi-conformal homeomorphism $H_1 : \mathcal{A}^q \to \mathcal{A}$ such that $H_1(\partial U^q) = R(\partial U^q) = \partial S$ and such that on the boundary of V_2^q we have $P_2^q(x) = H_1^{-1} Q_2 H_1(x)$. Moreover, one can show that there exists a quasi-conformal homeomorphism $H_2 : D_1^q \to D_1$ such that $H_2(\partial V_1^q) = H_1(\partial V_1^q) = \partial T_1$ and such that on the boundary of C_1^q we have $P_1^q(x) = H_2^{-1} Q_1 H_2(x) \in \partial V_1$, and such that on the boundaries of A_1^q and B_1^q we have $P_1^q(x) = H_1^{-1} Q_1 H_2(x) \in \partial V_2$.

We now define a new map f by

$$f = \begin{cases} P_1^q \text{ on } A_1^q \cup B_1^q \cup C_1^q, \\ P_2^q \text{ on } V_2^q, \\ H_1^{-1} Q_1 H_2 \text{ on } D_1^q, \\ R^{-1} Q H_1 \text{ on } \mathcal{A}^q, \\ R^{-1} Q R \text{ elsewhere.} \end{cases}$$

Now f is a proper holomorphic map of degree 4 with quadratic-like restrictions $f_i : V_i^q \to U^q$, whose critical points map to the same critical value. Moreover, on A_1^q the composition $f_2 \circ f_1$ is equal to the quadratic q. By performing quasi-conformal surgery as in Proposition 6.1, we obtain a degree 4 polynomial P with the following properties: as before there exists a double annulus $A = U - (\bar{V}_1 \cup \bar{V}_2)$ and quadratic-like restrictions $P_i : V_i \to U$ for $i = 1, 2$. Moreover, the sets $P_i^{-1}(V_1)$ are connected. We shall denote them by C_i for $i = 1, 2$. The sets $P_i^{-1}(V_2) \subset V_i$ for $i = 1, 2$ consist of two connected components. By construction of P, P^2 restricted to one of the connected components is hybrid-equivalent to q. Adjusting notation, we shall

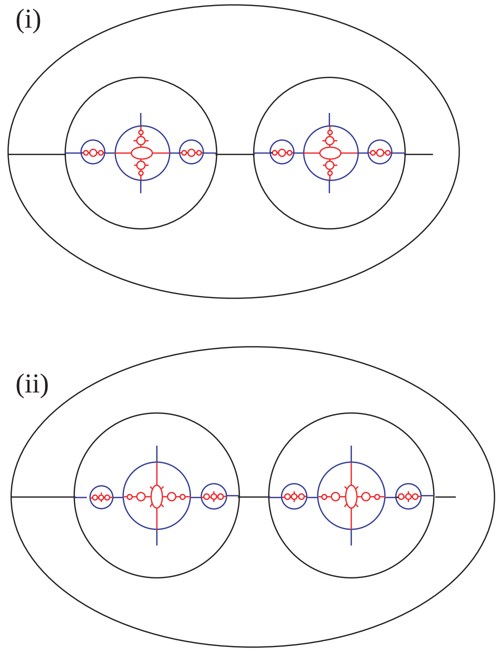

FIGURE 11. These two figures show the projection of $\Delta^\rho = \bigcup_{i=0}^{4} \rho(\Delta)$ and its images under δ_1^{-1} and δ_2^{-1} as in Figure 6, and the two possibilities for the arrangement of the projection of $\rho^j\sigma\rho^i\sigma(\Delta^\rho)$, $1 \leq i, j, \leq 4$.

denote the connected components of $P_i^{-1}(V_2)$ by A_i and B_i, so that $P^2|_{A_1} : A_1 \to U$ is hybrid-equivalent to q. $\square$

Now suppose we mate a degree 4 polynomial as constructed in the previous proposition with a group G. Firstly, we have to choose a homeomorphism $h : A \to B$, mapping $\partial_1 A$ to $\partial_k B$ for k either 1 or 2. Recall that h was constructed by first defining it on the boundary component $\partial_3 A$, and then pulling it back to the $\partial_i A$ so that the actions of the relevant maps are respected, and then extending it to the interior of A. As described above, the images of $\rho^i(l \cup \chi\rho l)$ under δ_1^{-1} and δ_2^{-1} are identified so that each of the images of $\partial_k B$ under δ_1^{-1} and δ_2^{-1} forms a circle, and such that each of the images of $\partial_j B$ (where $j \neq k$) under δ_1^{-1} and δ_2^{-1} forms a pair of circles. These two pairs of circles correspond to the pairs of connected components A_i and B_i for $i = 1, 2$ of $P_i^{-1}(V_2)$. In particular, one of the circles coming from $\delta_k^{-1}(\partial_j(B))$ (where $j \neq k$) corresponds to A_1. If we would like the other circle to correspond to A_1, then we replace the homeomorphism $h : A \to B$ by the homeomorphism h^* defined as follows: let $h^* = h$ on $\partial_3 A$ and $\partial_2 A$. On $\partial_1 A$ we define h^* by pre-composing h with the involution on $\partial_1 A$ exchanging points that have the same image under P. We then extend h^* to the interior of A. (See remark 11.3 below for further explanation.)

Altogether we have four ways of mating G with q in this way, two for each choice of k. For the same reasons as those explained in subsection 11.1, these four matings are genuinely different (see Figures 11 and 12).

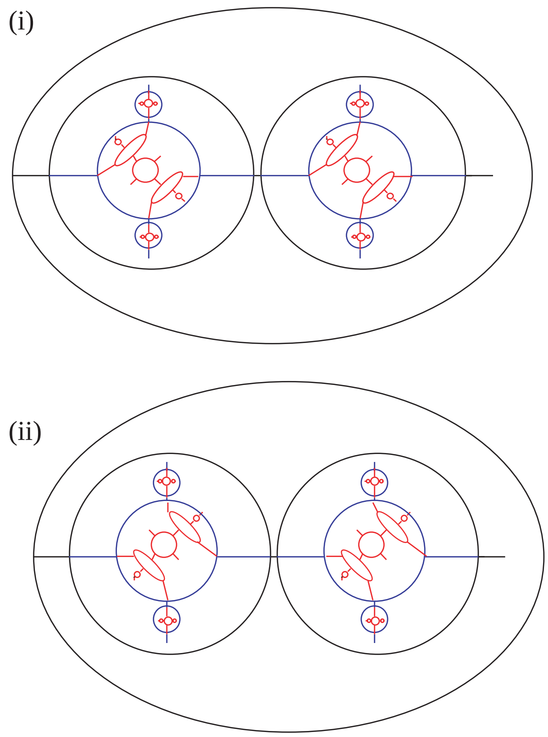

FIGURE 12. The two possibilities for matings corresponding to Figure 7.

Remark 11.3. Of course there is another way to construct h^* arising from defining h^* to be equal to h on $\partial_3 A$, and to be equal to h pre-composed with the said involution on *both* $\partial_1 A$ and $\partial_2 A$. However one can check that this does not affect the way in which images of Δ^ρ are fitted together, and therefore would not give a genuinely different mating. In fact, one can show that the two resulting correspondences are conjugate.

12. Conclusion

We have seen that given a non-contact Hecke group G of degree 5 and a quadratic polynomial q in the Mandelbrot set, there are infinitely many correspondences mating the two. Each of them arises via a degree 4 polynomial P, and belongs to the 2-parameter family of correspondences mentioned above. For each such correspondence F, it is proved in [9], that if we choose one of the parameters so that the set Λ becomes connected (i.e. so that the curves in the family $\mathcal{C}$ shrink to points), then F is a mating between the Hecke group G_5 and q. The structure of the limit set Λ of F, and the way in which the different matings between a given q and G_5 fit together is investigated in detail in [9] and [11] (for matings involving G_4).

For any $n > 3$ an analogous construction to the one given here for $n = 5$ will give matings between non-contact Hecke groups of degree n and quadratic polynomials via a degree $n-1$ polynomial P. The only difference in the surgery constructions for $n \neq 5$ is that one has to 'glue together' $(n-1)/2$-tuply or $n/2$-tuply connected regions A and B corresponding to n being odd or even. As before, various choices will arise and give infinitely many matings between a non-contact Hecke group G and a quadratic q.

It is natural to ask about the nature of the parameter space of such matings. For each n, the non-contact Hecke groups of degree n come from a one (complex) parameter family. The Teichmüller space T_n of non-contact Hecke groups of degree n is a topological disk with fractal boundary, reminiscent of the filled Julia set of $z \to z^2 + 1/4$. Each $G(r)$ with r in the interior of T_n can be mated with any quadratic q_c in the Mandelbrot set by a correspondence $F_{a,k}$ coming from a 2-parameter family. Moreover it can be mated in countably many ways. Let M_n denote the set of pairs (a,k) in $\mathbb{C} \times \mathbb{C}$ for which the correspondence $F_{a,k}$ is a mating between some group in the interior of T_n and a quadratic polynomial in the Mandelbrot set. We expect M_n to consist of the direct product of the interior of T_n and a countably infinite union of copies of the Mandelbrot set. Moving to the boundary of T_n corresponds to shrinking along some, possibly very complicated, line and "pinching". The shape of the set of pairs (a,k) such that $F_{a,k}$ is a mating between some group in the boundary of T_n and a quadratic is therefore very hard to predict.

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