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SPINNING DEFORMATIONS OF RATIONAL MAPS

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Abstract. We analyze a real one-parameter family of quasiconformal deformations of a hyperbolic rational map known as *spinning*. We show that under fairly general hypotheses, the limit of spinning either exists and is unique, or else converges to infinity in the moduli space of rational maps of a fixed degree. When the limit exists, it has either a parabolic fixed point, or a pre-periodic critical point in the Julia set, depending on the combinatorics of the data defining the deformation. The proofs are soft and rely on two ingredients: the construction of a Riemann surface containing the closure of the family, and an analysis of the geometric limits of some simple dynamical systems. An interpretation in terms of Teichmüller theory is presented as well.

Contents

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1. Introduction

We analyze a real one-parameter family of quasiconformal deformations of a hyperbolic rational map known as *spinning*. We show that under fairly general hypotheses, the limit of spinning either exists and is unique, or else converges to infinity in the moduli space of rational maps of a fixed degree. When the limit exists, it has either a parabolic fixed point, or a pre-periodic critical point in the Julia set, depending on the combinatorics of the data defining the deformation. The proofs are soft and rely on two ingredients: the construction of a Riemann surface containing the closure of the family, and an analysis of the geometric limits of some simple dynamical systems. An interpretation in terms of Teichmüller theory is presented as well.

Here is an informal introduction to our techniques and results, described in the combinatorially simplest case. Let $d \geq 3$ and let $\mathrm{Poly}_d^\times$ be the complex manifold which is the space of all degree d polynomials $f^\times$ with marked critical points $c_1(f^\times), \ldots, c_{d-1}(f^\times)$. Let $f_0^\times$ be a small, generic perturbation of $z \mapsto z^d$ and $H(f_0^\times)$ be the connected component of hyperbolic maps in $\mathrm{Poly}_d^\times$ containing $f_0^\times$.

As $f^\times$ varies near $f_0^\times$, the attracting fixed point $a(f^\times)$ varies holomorphically, as do its multiplier $\lambda(f^\times)$ and its linearizing coordinates $\varphi_{f^\times}$, suitably normalized. The $d-1$ critical points $c_1(f^\times), \ldots, c_{d-1}(f^\times)$ remain in the immediate basin $B(f^\times)$ of $a(f^\times)$ and therefore their locations $\varphi_{f^\times}(c_i(f^\times))$ in the linearizing coordinates vary holomorphically as well.

Fix the multiplier λ of the attracting fixed point and the locations of all but one, say the first, critical point. It turns out that letting $c = c_1$ vary determines a subset of parameter space which (under reasonable genericity conditions on $f_0^\times$) contains a Riemann surface $X(f_0^\times)$ which intersects the boundary of $H(f_0^\times)$. We give two proofs showing the existence of the Riemann surface $X(f_0^\times)$: a by-hands proof using holomorphic motions, and a more conceptual proof using facts about puncture-forgetting maps of Teichmüller spaces. We may then study the intersection $\partial H(f_0^\times) \cap X(f_0^\times)$.

For example, take a real-analytic path of maps $t \mapsto f_t^\times, t \in [0, +\infty)$ in $X(f_0^\times)$ starting at $f_0^\times$ and chosen so that the location of the free critical point c in the linearizing coordinates tends to infinity. The fact that $X(f_0^\times)$ is one-complex dimensional, combined with the fact that there is only one critical point which can be responsible for changes in dynamical behavior, makes analyzing the limiting behavior of the path $f_t^\times$ more tractable. Under the right conditions, any limit point of $f_t^\times$ is a point on the boundary of $H(f_0^\times)$, and so this analysis gives a mechanism for understanding certain slices of this boundary. Compare Rees [Rees] where similar slices and paths are introduced in the study of quadratic rational maps.

Spinning, the topic of this paper, is a special case of such a path. The term "spin" was introduced by Birman [Bir] to describe certain self-homeomorphisms of surfaces with marked points. Fix, say, a torus T with $d - 1 \geq 2$ marked points $c = c_1, c_2, \ldots, c_{d-1}$. Let γ be a simple closed oriented curve in T based at c and avoiding $c_2, c_3, \ldots, c_{d-1}$. To *spin c around γ*, we simply slide the marked point c continuously around γ by a one-parameter family of homeomorphisms of T which are the identity off of a neighborhood of γ. When T is the quotient torus associated with the attracting fixed point $a(f_0^\times)$, it turns out that spinning c around γ is a special case of the type of deformation introduced in the previous paragraph. We shall show that under suitable hypotheses, spinning paths converge, and parabolic points develop via parabolic implosion. (There is a related phenomena in the setting of Kleinian groups: when T is one of the two ideal boundary components associated to a Fuchsian hyperbolic manifold $\mathbb{H}^3/\Gamma$, spinning provides a means for constructing one-parameter families of quasifuchsian deformations for which the algebraic and geometric limits are distinct [Bro].) The hypotheses are a little involved, combinatorially, and our main results are stated precisely in §1.3 after some necessary definitions.

1.1. **Definition of spinning.** Although we shall only treat the simplest cases in this work, we formally define spinning in a general context.

Notation. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$. The *grand orbit* of a point $z \in \mathbb{P}^1$ is the set of w such that $f^{\circ i}(z) = f^{\circ j}(w)$ for some $i, j \geq 0$. Let $\hat{J}$ denote the closure of the grand orbits of all periodic points and all critical points of f. The set $\hat{J}$ contains the Julia set. Its complement, $\hat{\Omega}$, is therefore contained in the Fatou set.

The set $\hat{\Omega}$ is the union of disjoint open subsets $\Omega^{dis} \sqcup \Omega^{fol}$. The set Ω^{dis} is the subset of points whose grand orbits are discrete subsets of $\mathbb{P}^1$ (these are contained in attracting and parabolic basins), while the set Ω^{fol} is the subset of points lying in basins foliated by real-analytic closed curves which are components of the closure in $\hat{\Omega}$ of grand orbits (these are contained in superattracting basins, Siegel disks, and Herman rings). For generic hyperbolic rational maps, Ω^{fol} is empty.

Quotient surfaces. By the classification of stable regions, the set Ω^{dis} consists of points lying in the basin of attraction of parabolic and (nonsuper)-attracting cycles. The restriction of f to Ω^{dis} is a holomorphic self-covering, so the quotient Ω^{dis}/f of Ω^{dis} by the grand orbit equivalence relation is a one-dimensional, possibly disconnected complex manifold and will be called the *quotient surface* associated to f. The quotient surface is a disjoint, possibly empty union of at-least-once-punctured tori (one for each attracting cycle) and at-least-once-punctured copies of $\mathbb{C}^*$ (one for each parabolic cycle), where in each case the number of punctures is the number of grand orbits of critical points in the corresponding basin. Note that by a theorem of Fatou, the *immediate* basin of an attracting or parabolic cycle contains at least one critical point, and yields therefore at least one puncture in each case.

Let $\Psi : \Omega^{dis} \to \Omega^{dis}/f$ be the canonical projection.

Input data for spinning. Let S be a component of Ω^{dis}/f with at least two punctures. Let $\Psi(c)$ be one of these punctures, where c is a critical point of f (abusing notation, we write $c = \Psi(c)$ if no confusion can arise). The point c we call the *spun critical point*. Let $S^\sharp = S \cup \{c\}$, let γ be a simple closed oriented curve containing c, and let $[\gamma]$ denote its homotopy class in $\pi_1(S^\sharp, c)$.

It turns out that as a discrete process producing a sequence of (Möbius conjugacy classes of) maps f_n, $n = 0, 1, 2, 3, \ldots$, spinning depends only on f, $\Psi(c)$, and the homotopy class of $[\gamma]$ (see §7). However, to make spinning a continuous process producing a path of maps f_t, $t \in [0, +\infty)$, we must introduce restrictions on γ and some noncanonical choices.

We suppose γ is real-analytic. Then there is an annular neighborhood A of γ such that $\overline{A} \subset S^\sharp$ and such that γ is the unique essential curve fixed by an anticonformal involution of A. By the Uniformization Theorem and the Riemann mapping theorem, we may assume that the universal covering space $\widetilde{A}$ is given by

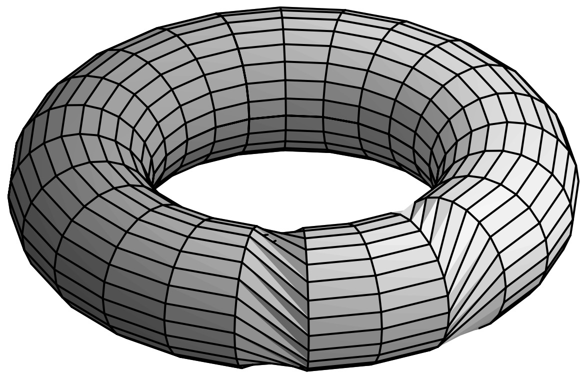

FIGURE 1. Spinning on the torus. The annulus A is the subannulus which includes the two regions where "twisting" occurs. The curve γ is the central curve of A.

$\widetilde{A} = \{x + iy \in \mathbb{C} | -2l < y < 2l\}$ where $l > 0$. Let $p_A : \widetilde{A} \to A$ be the universal covering map, analytic on the interior and continuous on the boundary, such that $p_A(\mathbb{R}) = \gamma$, p_A is orientation-preserving on $\mathbb{R}$, and $p_A(0) = c$. By precomposing with a map of the form $x + iy \mapsto \rho(x + iy), \rho > 0$ we may assume that the map on $\widetilde{A}$ induced by lifting γ is translation by one (to the right). Note that this implies that the modulus of A is $4l$. With these conventions, the covering space and map $p_A : \widetilde{A} \to A$ are unique.

Definition of spinning. Let $f_0 = f$. Define

$$\widetilde{h} : \widetilde{A} \times \mathbb{R} \to \widetilde{A}$$

by linearly interpolating translation to the right by t on the horizontal strip $|y| \leq l$ and the identity map on the boundary of $\widetilde{A}$. More precisely, set

$$\widetilde{h}(x + iy, t) = \begin{cases} x + t + iy & \text{if} \quad 0 \leq |y| \leq l, \\ x + t(2 - |y|/l) + iy & \text{if} \quad l \leq |y| \leq 2l. \end{cases}$$

We set $\widetilde{h}^t = \widetilde{h}(\cdot, t) : \widetilde{A} \to \widetilde{A}$. The key features are:

- For each t, the map $\widetilde{h}^t$ is quasi-conformal, $\widetilde{h}^t|_{\partial \widetilde{A}} = id$, and $\widetilde{h}^t$ commutes with the group of deck transformations, thus giving a well-defined map $h : A \times \mathbb{R} \to A$.
- h extends to a continuous homomorphism from the reals under addition to the group of qc self-homeomorphisms of the Riemann surface $S^\sharp$, such that h^t is the identity on the complement of A. If e.g. S is a torus T with punctures, then h extends to T as well.
- For each t, the map $h^t : S^\sharp \to S^\sharp$ is conformal on the complement of the region $p_A(l < |y| < 2l)$, a union of two parallel subannuli of A. Also $h^t(A) = A$, so that the modulus is unchanged.

Finally, let μ_t be the f-invariant Beltrami differential obtained by pulling back the dilatation of h^t under the canonical projection $\Psi : \Omega^{dis} \to \Omega^{dis}/f$. By the Measurable Riemann Mapping Theorem there is a quasiconformal homeomorphism

$$H_t : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}},$$

unique up to postcomposition with Möbius transformations, whose dilatation agrees with μ_t. If a representative H_t is chosen, then the map

$$f_t = H_t \circ f \circ H_t^{-1}$$

preserves the standard conformal structure and is therefore a rational map. Different representatives for H_t yield Möbius conjugate functions f_t, and so we obtain a map

$$\sigma = \sigma_{\gamma,A} : \mathbb{R} \to \mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$$

from the reals into the space of conjugacy classes of rational maps of degree d, which we call a *spinning path* of γ. The image $\sigma([0,\infty))$ we call a *spinning ray* of γ. It is easily shown that σ is real-analytic.

Visibility. A curve $\gamma \subset S^\sharp$ can lift to the sphere in a variety of combinatorially distinct ways. Let B denote the immediate basin of an attracting cycle of period p, $\hat{B} = B - B \cap \hat{J}$, $S = \hat{B}/\langle f \rangle$ the corresponding quotient surface, in this case an at-least-once-punctured torus, and $\Psi : \hat{B} \to S$ the projection. Let

$$\Gamma = \bigcup_{\delta \subset \Psi^{-1}(\partial A)} \overline{\delta}$$

where the union is over all connected components of the preimage of ∂A under the projection Ψ. Then Γ divides B into various components; see Figures 2, 3 where $\Psi^{-1}(A)$ consists of the (green) lighter colored regions. Observe that if $W_1, W_2, \ldots$ denote the components of $B - \Gamma$ whose closures contain points of the attractor, then the W_k are finite in number, and the restriction $f^p : \bigcup_k W_k \to \bigcup_k W_k$ is proper.

Definition 1.1 (Visible point)**.** A point $w \in B - \Gamma$ is *visible with respect to* γ if the closure of the unique connected component of $B - \Gamma$ containing w contains a point of the attracting cycle. A point $w \in \mathbb{P}^1$ becomes *visible with respect to* γ *after* r *steps* if $f^{\circ r}(w)$ is visible but $f^{\circ i}(w)$ is not visible for $0 \le i < r$.

For example, the critical points c and b in Figure 2 are both visible, while in Figure 3 critical point b is visible but c is visible after one step. Note that a visible critical point b necessarily has infinite forward orbit, hence $\phi(b) \neq 0$ in a linearizing coordinate ϕ.

1.2. **An example.** Consider, for complex $c \neq 0$, the family of cubic critically marked polynomials

$$f(c,z) = -\int_0^z (\zeta - c)(\zeta + \frac{1}{2c})d\zeta = \frac{1}{2}z + az^2 - \frac{1}{3}z^3$$

where $a = \frac{1}{2}(c - \frac{1}{2c})$. This family consists of polynomials with an attracting fixed point of multiplier $1/2$ at the origin, normalized conveniently so the leading coefficient is $-1/3$, and having two marked critical points c and $b = b(c) = -\frac{1}{2c}$. Conjugation by $z \mapsto -z$ preserves this family and replaces c by $-c$.

Let us set

$$c_0 = \frac{\sqrt{2}}{2} \approx .707.$$

For $c = c_0$, the marked map $f_0^\times(z) = f(c_0, z)$ is equal to

$$f_0^\times(z) = -\frac{1}{3}z^3 + \frac{1}{2}z$$

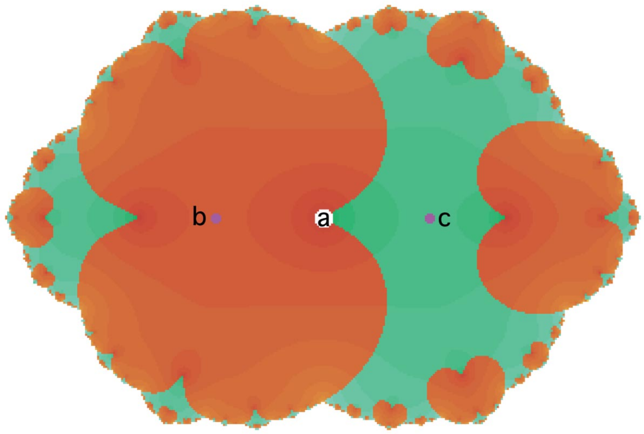

FIGURE 2. The critical point c is visible

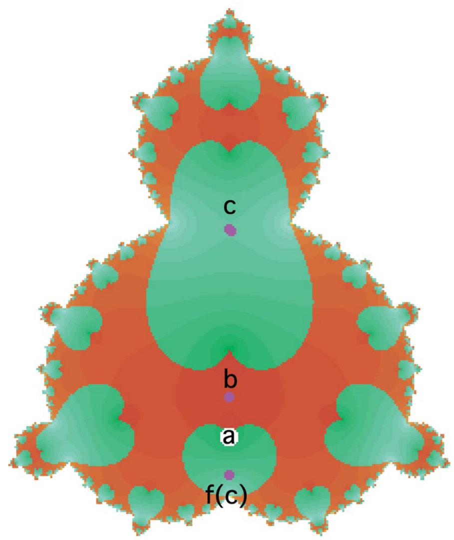

FIGURE 3. The critical point c is not visible

and in particular is odd. Hence both critical points c_0 and $b_0 = -c_0$ lie in the immediate basin of the origin, since by Fatou's theorem the immediate basin must contain at least one critical point.

In the c-parameter plane, consider the locus

$$\{c \mid b \text{ converges to the origin under iteration of } f(c, z)\}.$$

Let $X(f_0^\times)$ be the connected component containing $f_0^\times$; see Figure 4. Then $X(f_0^\times)$ is the locus of maps $f_c^\times$ such that the critical point $b = b(c)$ lies in the immediate basin of the origin. Evidently $X(f_0^\times)$ is not closed in the c-parameter plane.

Let q, r be positive integers. Let $X_{par}^{q,r}(f_0^\times)$ denote the subset of $X(f_0^\times)$ for which the corresponding maps have a parabolic cycle of period less than or equal to q and

FIGURE 4. At left, the annular region shaded in gray is the locus of c-parameters for which both critical points c, b lie in the immediate basin of the attractor at the origin. Superimposed is the path $\sigma(t), t \geq 0$. At right, the unbounded (orange) dark region is the locus $X(f_0^\times)$ of c-parameters for which the critical point b lies in the immediate basin of the attractor at the origin. Both images are the windows $|\mathrm{Re}(c)| \leq 5.6$, $|\mathrm{Im}(c)| \leq 4.2$. Note that the limit of $\sigma(t)$ lies in $X_{par}^{1,0}(f_0^\times) \subset X(f_0^\times)$.

of multiplier an rth root of unity. Let $X_{mis}^{r,q}(f_0^\times)$ denote the set of Misiurewicz-type parameter values, i.e., those for which the critical point c lands on a repelling cycle of period at most q after at most r iterations. Note that $X_{par}^{q,r}(f_0^\times)$ and $X_{mis}^{q,r}(f_0^\times)$ are discrete subspaces of $X(f_0^\times)$.

Let γ be a simple closed curve satisfying the hypothesis of Theorem 1.2. For example, we may take γ to be the image of the positive real axis under the projection $p : \mathbb{C}^* \to T$, i.e., a real curve containing $\Psi(c)$. Choose an annulus A surrounding γ, and consider spinning for the map f_0 about the curve γ. By examining the real graph of f_0 it is easy to see that both b and c are visible with respect to γ; cf. Figure 2 in which $\Psi^{-1}(A)$ for f_0 is shown as the (green) lighter colored region.

A lift $\sigma^\times$ of the spinning ray to $X(f_0^\times)$ is shown in Figure 4. The evolution of the dynamics is shown in Figure 5. The limiting map in the above example is

$$g^\times(z) = f(c_\infty, z)$$

where

$$c_\infty = \frac{\sqrt{2} + \sqrt{14}/2}{\sqrt{3}} \approx 1.897.$$

1.3. **Main results.** We will consider only the case of spinning critical points in attracting basins. That is, we consider spinning a single puncture $\Psi(c)$ around a curve $\gamma \subset S^\sharp$, where S is a complex torus T with punctures. We do not require that c is in the *immediate* basin B of the attractor. (If c is not in B, then c is not visible with respect to γ.) Recall that the torus T has a canonical homology class α represented by a counterclockwise oriented simple closed curve surrounding the attractor which is round in the linearizing coordinates.

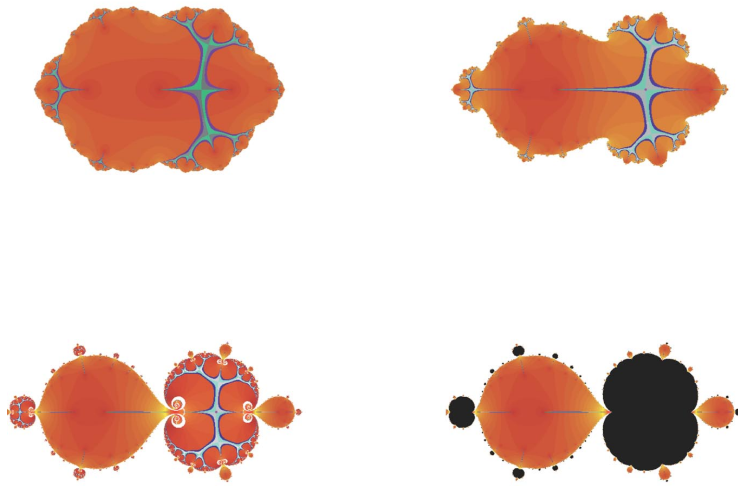

FIGURE 5. Evolution of spinning, from $c = c_0 = .707\ldots$ (top left), $c = 1.4$ (top right), $c = 1.88$ (lower left), and the limit g with $c = c_\infty = 1.897\ldots$ (lower right). The Julia sets are the boundaries of the unbounded white regions. The attracting fixed point a for f_0 is the center of the upper-left figure. Note the apparent collision of the pair of repelling fixed points, the creation of the bottleneck between the spirals, and the resulting parabolic implosion. Legend: orange = complement of $\Psi^{-1}(A)$, dark blue = support of dilatation of H_t; $\Psi^{-1}(\gamma)$ is the boundary between the light green and purple region; at lower right: black = basin of new parabolic point.

Standing assumptions. In each of the results below, we assume that a rational map f is given, and the triple (S, c, γ) for which we spin is chosen as follows:

(A1) a is an attracting *fixed* point, B is its immediate basin, $S = \hat{B}/f$ is the quotient surface, $\Psi : \hat{B} \to S$ is the projection, and c is a critical point whose grand orbit passes through B;

(A2) $\gamma \subset S^\sharp \equiv S \cup \{\Psi(c)\}$ is a simple closed curve containing c such that $[\gamma]\cdot[\alpha] = +1$, where $[\alpha], [\gamma] \in H_1(T, \mathbb{Z})$ are the corresponding classes and "$\cdot$" is the signed homological intersection number.

(A3) The grand orbit of c does not contain other critical points, and there exists a critical point $b \in B$ which is distinct from c and which is visible with respect to γ.

The sign conventions on the intersection number mean, in particular, that if we lift α, γ under the projection $p : \mathbb{C} - \{0\} \to T$ to oriented curves $\widetilde{\alpha}, \widetilde{\gamma}$ in the linearizing coordinate plane for a, then $\widetilde{\alpha}$ winds counterclockwise once about the origin and $\widetilde{\gamma}$ is an infinite ray pointing away from the origin, invariant by $z \mapsto f'(a)z$.

Remarks.

(A1) keeps our exposition free of additional notation.

(A2) implies that the spun critical points are pushed away, rather than toward, attractors.

(A3) keeps the discussion generic and avoids the need for separate consideration of a plethora of combinatorially distinguished special cases.

Genericity. Critical orbit relations. A rational map (respectively, a polynomial) is *critically generic* if every critical point (respectively, every finite critical point) is simple. A rational map (polynomial) has *no critical orbit relations* if the grand orbits of any two distinct (finite) critical points are disjoint, and the forward orbit of every (finite) critical point is infinite.

Theorem 1.2 (Lands or diverges)**.** *Suppose f is a critically generic hyperbolic rational map with no critical orbit relations, and (S, c, γ) satisfies the standing assumptions.*

Then the spinning ray σ_γ either has a unique limit in $\mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$*, or else converges to infinity. Furthermore, the limit depends only on f and the homotopy class of γ in $\pi_1(S^\sharp, c)$.*

(By converging to infinity, we mean that given any compact subset K of $\mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$, there is an $R = R(K)$ such that $[f_t] = \sigma_\gamma(t) \notin K$ whenever $t > R$.)

Theorem 1.3 (Lands for polynomials)**.** *Let f be a hyperbolic polynomial with connected Julia set and no critical orbit relations (but possibly having a multiple critical point or points). Then the spinning ray σ_γ has a unique limit in* $\mathrm{Poly}_d/\mathrm{Aut}(\mathbb{C})$ *which depends only on f and the homotopy class of γ in $\pi_1(S^\sharp, c)$.*

We note that Cui, by very different methods, has announced sufficient criteria for the existence of spinning limits in which *multiple* critical points are spun [Cui].

The following theorem states what dynamical features are inherited by limits of spinning.

Theorem 1.4 (Spinning limit inherits a large part of the dynamics)**.** *Suppose F is an arbitrary rational map and (S, γ, c, a, B) satisfies the standing assumptions. Let $F_{t_n} = H_{t_n} \circ F \circ H_{t_n}^{-1} \in \sigma(t_n)$ be rational maps produced by spinning c around γ, where $\{t_n\}_{n=0}^{\infty}$ is any sequence of real numbers. Suppose $F_{t_n} \to G \in \mathrm{Rat}_d$. Then:*

(1) *Let $\mathcal{W}_0$ denote the union of those Fatou components of F which do not iterate to B or to a Siegel disk. Then there is a holomorphic embedding $\mathcal{J} : \mathcal{W}_0 \to \mathbb{P}^1$ such that $\mathcal{J} \circ F = G \circ \mathcal{J}$.*
(2) *If $a_{t_n} = H_{t_n}(a)$, then the a_{t_n} are attracting fixed points of constant multiplier, and after possibly passing to a subsequence, $a_{t_n} \to a_\infty$, an attracting fixed point of G of the same multiplier.*
(3) *Let B_{t_n} be the immediate basin of a_{t_n} under F_{t_n}, and let B_∞ be the immediate basin of a_∞ under G.*

    There exists an open subset $\mathcal{U}$ contained in the grand orbit of the basin B (the basin of the attractor a for F) and holomorphic embeddings $J_{t_n} : \mathcal{U} \to \mathbb{P}^1$ such that:
    (i) *$F(\mathcal{U}) \subset \mathcal{U}$;*
    (ii) *$\mathcal{U}$ contains the attractor a and all critical points converging to a except those in the grand orbit of c;*
    (iii) *$J_{t_n} \circ F = F_{t_n} \circ J_{t_n}$; (iv) after passing to a subsequence, the embeddings J_{t_n} converge uniformly on compact subsets to an embedding $J : \mathcal{U} \to \mathbb{P}^1$ satisfying $J \circ F = G \circ J$.*

If D is a Siegel disk for F, the conformal radii of $J_{t_n}(D)$ may shrink to zero, and we do not understand what the consequences of this are for dynamics. On the other hand, the modulus of an annulus controls the size of at least one of the complementary components. Hence, if the images under J_{t_n} of a Herman ring for F degenerate, then one side of the Herman ring collapses in the limit. The proof will show this is not possible under the hypotheses of the above theorem.

The following result describes the possibilities for the new dynamics arising in the limit. The orientation conventions on γ imply that the spun critical point "moves away" from the attractor.

Theorem 1.5 (Possibilities for the spun critical point c)**.** *Let F be an arbitrary rational map and (c,γ) satisfy the standing assumptions, where c is visible after r steps. Let $F_{t_n} = H_{t_n} \circ F \circ H_{t_n}^{-1} \in \sigma(t_n)$ be rational maps produced by spinning c around γ, with $t_n \to +\infty$ as $n \to \infty$. Suppose $F_{t_n} \to G \in \mathrm{Rat}_d$ and $H_{t_n}(c) \to c_\infty$.*

Then either $G^r(c_\infty)$ lies in a fixed parabolic basin of multiplier 1, or $G^r(c_\infty)$ is a repelling or indifferent fixed point of G.

Under further assumptions, we can make more precise the connection between visibility and the limiting dynamics. Combined with Theorem 1.4, the previous theorem shows:

Corollary 1.5 (Geometrically finite limits)**.** *Under the hypothesis of Theorem 1.5, assume in addition that F is hyperbolic. Then,*

(1) *if $r = 0$, i.e., c is visible, then c_∞ lies in the immediate basin of attraction of a parabolic fixed point of G with multiplier 1;*
(2) *if $r > 0$, i.e., c is visible after $r \geq 1$ steps, then $G^r(c_\infty)$ is a repelling fixed point of G.*

In particular, the limit G is geometrically finite, possessing a single critical point c_∞ which does not converge to an attractor.

The arguments used to prove the above results do not identify how the parabolic fixed point of the limit G in case (1) is created. With more work, we have the following.

First, some notation. Let A_c be the central subannulus of A on which the spinning map h is holomorphic (see Figure 1). Denote the boundary components of A_c by $\hat{\delta}^{\pm}$. Their lifts to the dynamical space have a unique component $\delta^{\pm}$ with a as one endpoint. More precisely δ^+ joins a to a repelling or parabolic fixed point u^+; similarly δ^- joins a to a point u^- (see Lemma 8.2). The points u^+ and u^- may or may not coincide. In Figure 2, the points $u^{\pm}$ are the points in the Julia set directly above and below, respectively, the attractor a. As we perform spinning along γ, we obtain $F_t = H_t \circ F \circ H_t^{-1}$. Denote by $\delta_t^{\pm}$, $a_t, \ldots$, etc., the images $H_t(\delta^{\pm})$, $H_t(a), \ldots$ etc.

Theorem 1.6 (How the parabolic point is created)**.** *Assume that F is hyperbolic, (S, c, γ) satisfies the standing assumptions, and $F_{t_n} \to G$. Assume, by taking subsequences if necessary, $u_{t_n}^{\pm} \to u_\infty^{\pm}$ and $c_{t_n} \to c_\infty$.*

Assume that c is visible. Then:

(1) *The map G has a unique parabolic basin Fatou component Ω. It is fixed, contains c_∞ and intersects the orbits of no other critical points.*
(2) *The parabolic point v of Ω lies on the boundary of B_∞.*
(3) *$u_\infty^+ = u_\infty^- = v$ and $u^+ \neq u^-$.*

(4) *The multipliers* $\lambda^{\pm}_{t_n} = F'_{t_n}(u^{\pm}_{t_n})$ *satisfy*

$$m < \mathrm{Re}\left(\frac{1}{1-\lambda^{+}_{t_n}} + \frac{1}{1-\lambda^{-}_{t_n}}\right) < 1$$

for some real number $m < 1$, *and therefore* $\lambda_{t_n} \to 1$ *tangentially as* $n \to \infty$.

If c *is visible after* $r \geq 1$ *steps, then* $u^+ = u^-$, $u^{\pm}_{\infty} = G^r(c_\infty)$ *and* $u^{+}_{\infty} \in \partial B_\infty$.

The proof relies on a soft but subtle analysis of certain geometric limits, developed in §8. The proof also shows

Theorem 1.7 (A case of divergence)**.** *Assume that* f *is a hyperbolic rational map (not necessarily critically generic),* (S, c, γ) *satisfies the standing assumptions, and* c *is visible. If* $u^+ = u^-$, *then* σ_γ *converges to infinity.*

1.4. **Outline of the paper.** The proof of Theorem 1.2 comprises the following steps, which include proving Theorems 1.4 and 1.5. Theorem 1.3 is proved in essentially the same way.

Step 1. The spinning path σ lifts to a continuous path $\sigma^\times : [0, \infty) \to X$, where X is a Riemann surface lying in a suitable space of maps $f^\times$ with normalized and marked critical points. We give two constructions of X, one in §2 using holomorphic motions, and a second in §7 using properties of puncture-forgetting maps between Teichmüller spaces.

Step 2. $\sigma(t_n) \to [g]$ in $\mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$ if and only if $\sigma^\times(t_n) \to g^\times$ in X. This follows from Theorem 1.4; see §3.

Step 3. The set of accumulation points of $\sigma^\times(\mathbb{R})$ (with respect to the topology of X) is discrete in X. This follows immediately from Theorem 1.5, proved in §4, since multipliers vary holomorphically.

Step 4. We assemble the results in §5 to prove that the set of accumulation points of $\sigma^\times([0, \infty))$ (with respect to the topology of X) is either empty or one point, using in an essential fashion the connectedness of the image of the spinning ray. In the latter case the spinning ray converges in X. Hence by Step 2, the spinning ray converges in $\mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$.

Step 5. Independence from noncanonical choices is shown in §7, using Teichmüller theory and results of Bers and Nag.

Appendix 8 develops the theory of geometric limits of invariant strips needed for the proof of Theorems 1.6 and 1.7. Appendix 9 contains miscellaneous analytical results used in several places.

1.5. **Acknowledgments.** We are very grateful to Curt McMullen for many useful conversations, and to the Université de Cergy-Pontoise for financial support.

2. Construction of X

The main result of this section is the construction of a certain Riemann surface X consisting of rational maps with normalized marked critical points. The surface X is defined implicitly by fixing the dynamical behavior of all but the spun critical point. It will contain a lift of the spinning path, and any limit of spinning, as we shall later show.

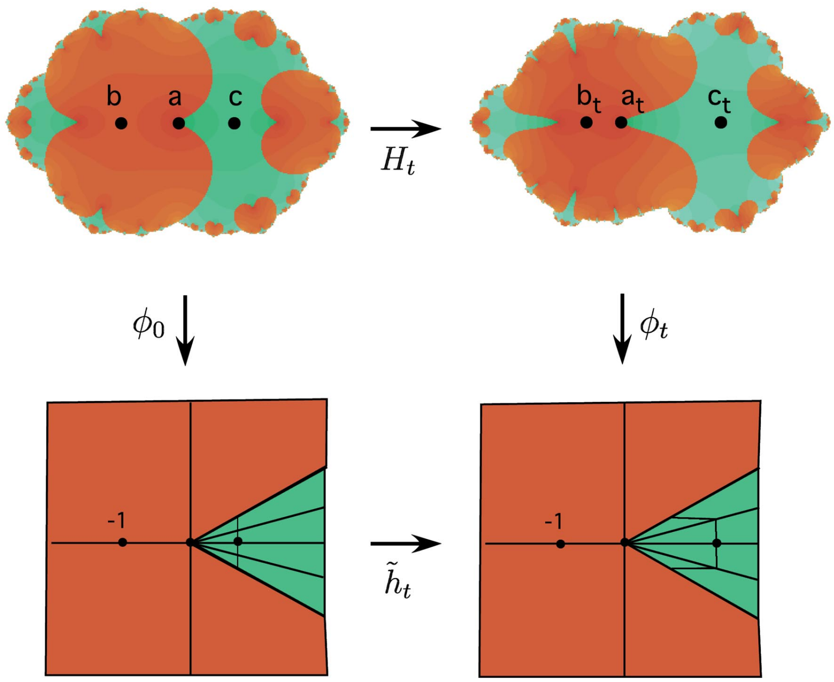

FIGURE 6. The qc map $\widetilde{h}_t$. The region $p^{-1}(A)$ is the large (green) light wedge on the right.

Let S, B, a, c, etc., be as in the setup for spinning. Let $\phi : (B, a) \to (\mathbb{C}, 0)$ be a linearizing coordinate (cf. [Mil], Cor. 18.4) and $p : \mathbb{C} - \{0\} \to T$ be projection from this coordinate to the quotient torus. Since the spinning homeomorphism $\widetilde{h}^t : T \to T$ acts trivially on the homology of the quotient torus, it lifts under p to a map $\widetilde{h}_t : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$. Let $a_t = H_t(a)$ and $B_t = H_t(B)$ be the corresponding attractor and basin for f_t.

Lemma 2.1. *For all $t \in \mathbb{R}$, $\widetilde{h}_t$ is the identity outside of $p^{-1}(A)$, and the map $\phi_t := \widetilde{h}_t \circ \phi|_{\widehat{B}} \circ H_t^{-1} : \widehat{B}_t \to \mathbb{C}$ is a holomorphic linearizing map conjugating f_t to multiplication by λ. In particular, the multiplier of a_t is again λ.*

Proof. See Figure 6. By construction, the map $\widetilde{h}_t \circ \phi \circ H_t^{-1} : B_t \to \mathbb{C}$ is well defined, holomorphic with respect to the standard conformal structure and conjugates f_t to multiplication by λ. □

Our strategy for creating X is to work within a suitable space $\mathrm{GRat}_d^{\times,*}$ in which the critical points are marked and the maps are normalized by e.g. conjugating so three critical points are at $0, 1, \infty$. Given the data defining spinning, label the critical points of f and normalize to get a map $f^\times = f_0^\times \in \mathrm{GRat}_d^{\times,*}$. We will produce spaces

$$Z(f_0^\times) \supset Y(f_0^\times) \supset X(f_0^\times)$$

where $Z(f_0^\times)$ is open in $\mathrm{GRat}_d^{\times,*}$ and $Y(f_0^\times), X(f_0^\times)$ are respectively the connected components containing $f_0^\times$ of the fibers of two holomorphic maps Λ, Φ defined on $Z(f_0^\times)$ and $Y(f_0^\times)$, respectively.

2.1. **Marked generic rational maps. The space** $\mathrm{GRat}_d^{\times,*}$**.** Fix a degree $d \geq 3$. Let $\mathrm{GRat}_d \subset \mathrm{Rat}_d$ denote the subspace of critically generic rational maps (i.e., all critical points are simple). Clearly, this is open and dense. By passing to a finite covering, we may assume that the locations of critical points are globally defined functions. The Lie group $\mathrm{Aut}(\mathbb{P}^1)$ will then act freely on this space, and it is then simple to show that the quotient $\mathrm{GRat}_d^{\times,*}$ is a complex manifold which may be concretely realized as a subset of $\mathrm{Rat}_d \times \left(\mathbb{P}^1\right)^{2d-2-3}$. For the remainder of this section, we will work exclusively within the space $\mathrm{GRat}_d^{\times,*}$.

2.2. **Marking attractors. The space** $Z(f_0^\times)$**.** Here, we describe a general construction of analytic subsets of parameter spaces in which the behaviors of some, but not all, critical points are held fixed.

Data for the definition. Write

$$C \equiv \{1, 2, 3, \ldots, 2d-2\}$$

and choose a decomposition

$$C = I \sqcup J \sqcup K, \quad I \neq \emptyset.$$

Choose a function $\omega : J \to I$ in case J is nonempty. *Throughout the remainder of this section, we assume that these choices have been given.*

Definition of Z**.** Let $Z = Z(I, J, K, \omega)$ denote the following subspace of $\mathrm{GRat}_d^{\times,*}$: a normalized, marked map $f^\times$ belongs to Z if and only if

- for $i \in I$, the critical point c_i is in the *immediate* basin B_i of an attracting, but not superattracting, cycle $\langle a_i \rangle$, c_i has infinite forward orbit, and for $i \neq i' \in I$ the basins of $\langle a_i \rangle$ and $\langle a_{i'} \rangle$ are disjoint;
- for $j \in J$, the critical point c_j is in the basin (not necessarily the immediate basin) of $\langle a_{\omega(j)} \rangle$, where $\omega : J \to I$ the given function; the forward orbit of c_j is infinite, for $j \neq j'$ the grand orbits of c_j and $c_{j'}$ are distinct, and the grand orbits of c_j and $c_{\omega(j)}$ are distinct;
- for $k \in K$, there are no restrictions on the behavior of c_k.

Note that a given $f_0^\times \in \mathrm{GRat}_d^{\times,*}$ for which the underlying map f_0 is hyperbolic and without critical orbit relations can be regarded as an element of Z for a variety of different choices of subsets I, J, and that such a choice determines the function $\omega : J \to I$ uniquely. Also, the space Z contains many combinatorially distinguished connected components, since we have not specified e.g. how many iterates are needed for c_j to map into the basin of $a_{\omega(j)}$.

We let $Z(f_0^\times)$ denote the connected component of Z containing $f_0^\times$. The space Z is open in $\mathrm{GRat}_d^{\times,*}$, and is therefore a complex manifold of dimension $2d-2$. In particular, $Z(f_0^\times)$ is a complex manifold of dimension $2d-2$.

2.3. **Fixing the multipliers. The space $Y(f_0^\times)$.** For $f^\times \in Z$ and $i \in I$, let $a_i(f^\times) \in \mathbb{P}^1$ denote the location of the attracting periodic point whose immediate basin contains c_i, and let $p_i(f^\times)$ be the period of this attractor. Clearly, these are functions of $f^\times$. Hence the location of each point in the attractors $\langle a_i \rangle$, and the multiplier λ_i of this attractor, are in fact a function of $f^\times$.

We denote by Λ the multiplier map:

$$\Lambda : Z \to (\Delta^*)^I, \quad f^\times \mapsto \big(\lambda_i(f^\times)\big)_{i \in I},$$

where $\Delta = \{\lambda : |\lambda| < 1\}$ and $\Delta^* = \Delta - \{0\}$.

Proposition 2.1. *The map Λ is holomorphic and admits local holomorphic sections. That is, given any point $\vec{\lambda}_0 = (\lambda_i)_{i\in I} \in (\Delta^*)^I$, and any map $f^\times$ with $\Lambda(f^\times) = \vec{\lambda}_0$, there is a neighborhood U of $\vec{\lambda}_0$ and a holomorphic map*

$$\Sigma : U \to Z$$

such that

$$\Lambda \circ \Sigma = \mathrm{id}_U.$$

From the theory of holomorphic functions of several complex variables, we immediately obtain (see Corollary C.10, p. 23 of [Gun]):

Corollary 2.1. *Given any $f_0^\times \in Z$, the fiber*

$$\Lambda^{-1}(\Lambda(f_0^\times))$$

is a complex manifold of dimension $2d - 2 - |I|$ which is a closed subset of Z, with respect to the induced topology of Z.

Note that the fibers $\Lambda^{-1}(\Lambda(f_0^\times))$ need not be closed in $\mathrm{GRat}_d^{\times,*}$.

Given $f_0^\times \in Z$, we let $Y(f_0^\times)$ denote the connected component of the fiber $\Lambda^{-1}(\Lambda(f_0^\times))$ which contains $f_0^\times$. This is again closed in Z.

Proof of Proposition 2.1. Denote by $\mathbb{H} = \{x + iy, x > 0\}$ the right half plane. For any $s \in \mathbb{H}$, define a homeomorphism $l_s : \mathbb{C} \to \mathbb{C}$ by

$$l_s(z) = l_s(re^{2\pi i\theta}) = r^s e^{2\pi i\theta} = z \cdot r^{s-1} = z \cdot e^{(s-1)\log r}.$$

It is a quasi-conformal map on z, and it depends holomorphically on s. An easy calculation shows that $l_s(\lambda z) = \lambda|\lambda|^{s-1} l_s(z)$, for any $\lambda \neq 0$.

Suppose $f^\times$ is now given. For $s = (s_i) \in \mathbb{H}^I$, set $\vec{\lambda}(s) = (\lambda_i|\lambda_i|^{s_i-1})_{i\in I}$. Fix temporarily $i \in I$. Let $a_i \in \mathbb{P}^1$ be the attractor (say of period p) whose immediate basin B_i contains the critical point c_i. Let $\widetilde{B}_i$ be the entire basin of a_i. Choose a holomorphic map

$$\psi_i : (B_i, a_i) \to (\mathbb{C}, 0)$$

satisfying

$$\psi_i \circ f^{\circ p_i}(z) = \lambda_i \psi_i(z).$$

Extend it then to $\widetilde{B}_i$ by the following recipe:

$$\psi(z) = \lambda^{-[\frac{n}{p}]} \psi(f^{\circ n}(z))$$

where n is any nonnegative integer for which $f^{\circ n}(z) \in B'$ and $[n/p]$ is the greatest integer less than or equal to n/p. The extended ψ satisfies the same functional equation and maps a grand orbit of f onto a grand orbit of λz.

For any $s = (s_i) \in \mathbb{H}^I$, we define a new complex structure $\sigma(s)$ as follows: for each $i \in I$, $\sigma(s)|_{\widetilde{B}_i}$ is the pull-back of the standard complex structure under $l_{s_i} \circ \psi_i$, and $\sigma(s)|_{\overline{\mathbb{C}}-\bigcup_i \widetilde{B}_i}$ is the standard complex structure. This complex structure is f-invariant by construction. Denote by h_s the integrating map fixing $0, 1, \infty$, by g_s the rational map $h_s f h_s^{-1}$, and by $g_s^\times$ the marking $(g_s, h_s(\vec{c}(f^\times)))$; see the following diagram:

$$\begin{array}{ccc} f,\ \overline{\mathbb{C}} \supset \bigcup_i \widetilde{B}_i \ni a_i & \xrightarrow{h_s} & \overline{\mathbb{C}} \supset \bigcup_i \widetilde{B}_i(s) \ni a_i(s),\ g_s \\ \downarrow_{\prod_i \psi_i} & & \downarrow_{\prod_i (l_{s_i}\psi_i)h_s^{-1}} \\ \mathbb{C}^I,\ \prod_i(\lambda_i z) & \xrightarrow{\prod_i l_{s_i}} & \mathbb{C}^I,\ \prod_i(\lambda_{s_i} z). \end{array}$$

Note that for $s = (1, \cdots, 1)$, $l_{s_i} = id$, $h_s = id$ and $g_s = f$. Otherwise $s \mapsto g_s^\times$ is holomorphic, with $\Lambda(g_s^\times) = \vec{\lambda}(s)$. Note that the map $w : s \mapsto \vec{\lambda}(s)$ is locally bi-holomorphic mapping a small neighborhood of $(1, \cdots, 1)$ onto a neighborhood U of $\vec{\lambda}_0$. Moreover the maps $g_s^\times$ are in Z. As a consequence, $\Sigma : U \to Z$, $\Sigma(\vec{\lambda}) = g^\times_{\omega^{-1}(\vec{\lambda})}$ is a holomorphic section of Λ. □

2.4. **Fixing critical points in linearized coordinates. The space $X(f_0^\times)$.** On the space Z, there is another map defined as follows. Fix temporarily $i \in I$ and $f^\times \in Z$. Let $a_i \in \mathbb{P}^1$ be the point in the attracting cycle $\langle a_i \rangle$ whose immediate basin B_i contains the critical point c_i. There is a unique normalized holomorphic map

$$\psi_i : (B_i, a_i) \to (\mathbb{C}, 0)$$

satisfying

$$\psi_i \circ f^{\circ p_i}(z) = \lambda_i \psi_i(z), \quad \text{and } \psi_i'(a_i) = 1\ .$$

By hypothesis, c_i has infinite forward orbit, so $\psi_i(c_i) \neq 0$. Rescale to set

$$\phi_i(z) = -\frac{1}{\psi_i(c_i)} \psi_i(z).$$

Then

$$\phi_i : (B_i, a_i, c_i) \to (\mathbb{C}, 0, -1).$$

It is a linearizing map. We then extend it to $\widetilde{B}_i$ as in the proof of Proposition 2.1. Recall that ϕ_i maps grand orbits of f onto λ_i-orbits.

From the definition of the subspace Z, recall that:

- If $\omega(j) = i$, then the critical point c_j lies in $\widetilde{B}_i$. Hence the value $\phi_i(c_j)$ makes sense.
- The critical points c_j have infinite forward orbits. Hence if $\omega(j) = i$, then $\phi_i(c_j) \neq 0$.
- For $j \neq j'$, the critical points c_j and $c_{j'}$ have distinct grand orbits. Suppose in addition that $j \neq j'$ and $\omega(j) = \omega(j') = i$. Then $\phi_i(c_j), \phi_i(c_{j'})$ have distinct λ_i-orbits. In particular, $\phi_i(c_j) \neq \phi_i(c_{j'})$.
- The critical points c_i and c_j have distinct grand orbits. Hence if $\omega(j) = i$, then $\phi_i(c_j) \neq -1$, and $\phi_i(c_j)$ and -1 have distinct λ_i-orbits.

Notation. We set

$$\mathbf{C}^i = (\mathbb{C}^* - \{\lambda_i^n(-1), n \in \mathbb{Z}\})^{|\omega^{-1}(i)|} - \text{big diagonal}$$

where the big diagonal is the locus where two or more coordinates have the same λ_i-orbit.

Thus for each $i \in I$ we have a function

$$\Phi_i : Z \to \mathbf{C}^i \quad \text{given by } \Phi_i(f^\times) = \left(\phi_i(c_j(f^\times))\right)_{j\in\omega^{-1}(i)}.$$

Putting these together, we have a function

$$\Phi : Z \to \prod_i \mathbf{C}^i \subset \mathbb{C}^J \quad \text{given by } \Phi(f^\times) = (\phi_i(c_j(f^\times)))_{j\in J,\, i=\omega(j)}$$

which records the locations of the critical points c_j in the linearizing coordinates.

Proposition 2.2. *The map Φ is holomorphic. The restriction of Φ to any fiber of Λ admits local holomorphic sections. That is, given any point $\mathbf{w}_0 \in \prod_i \mathbf{C}^i$, and any map $f^\times$ with $\Phi(f^\times) = \mathbf{w}_0$, there is a neighborhood U of $\mathbf{w}_0$ and a holomorphic map $\Sigma : U \to Y(f^\times)$ such that $\Phi \circ \Sigma = \mathrm{id}_U$.*

Corollary 2.2. *Given $f_0^\times \in Z$, the fiber of the restriction*

$$\left(\Phi\Big|_{Y(f_0^\times)}\right)^{-1}(\Phi(f_0^\times))$$

is a complex manifold of dimension $2d-2-|I|-|J|$ which is closed as a subset of $Y(f_0^\times)$, therefore closed in Z.

Proof of Proposition 2.2. Let $\mathbf{w}_0 = (w_j)_{j\in J}$. For each $i \in I$ and each $j \in \omega^{-1}(i)$, choose a small round disk D_j centered at w_j so that $\lambda_i^n D_j$ are disjoint for distinct $n \in \mathbb{Z}$, and the λ_i-orbits of D_j, of $D_{j'}$ and of -1 are mutually disjoint, for any $j' \neq j$ and $\omega(j') = i$. Set $\Delta_i = \prod_{j,\omega(j)=i} \frac{1}{2}D_j$, and $U = \prod_i \Delta_i = \prod_{j\in J} \frac{1}{2}D_j$.

Fix $i \in I$. For each $j \in \omega^{-1}(i)$, define a holomorphic motion $M_{ij} : \frac{1}{2}D_j \times \overline{D}_j \to \overline{D}_j$ as follows:

> $M_{ij}(u, w_j) = u$, $M_{ij}(u, z) = z$ for any $(u, z) \in \frac{1}{2}D_j \times \partial D_j$, M_{ij} is holomorphic on u and injective on z, and $M_{ij}(w_j, \cdot)$ is the identity.

Let $M_i : \Delta_i \times \mathbb{C} \to \mathbb{C}$ be the following holomorphic motion:

- For each $j \in \omega^{-1}(i)$ and any pair $(\mathbf{u}, z) \in \Delta_i \times \overline{D}_j$ with $\mathbf{u} = (u_j)$, $M_i(\mathbf{u}, z) = M_{ij}(u_j, z)$. In particular $M_i(\mathbf{u}, w_j) = u_j$.
- $M_i(\mathbf{u}, z) = \lambda_i^n M_i(\mathbf{u}, z/\lambda_i^n)$ for any $n \in \mathbb{Z}$ and any pair $(\mathbf{u}, z) \in \Delta_i \times \lambda_i^n \overline{D}_j$.
- $M_i(\mathbf{u}, z) = z$, for $z \in \bigcup_j \partial D_j$ and for z outside of the λ_i-orbits of $\bigcup_j D_j$.

Note that by construction $M_i(\mathbf{u}, \cdot)$ commutes with the multiplication by λ_i.

Do this for every $i \in I$.

Fix now $\mathbf{u} \in U$. We define a new complex structure $\sigma(\mathbf{u})$ as follows: for each $i \in I$, $\sigma(\mathbf{u})|_{\widetilde{B}_i}$ is the pull-back by $M_i(\mathbf{u}|_{\Delta_i}, \cdot)$ and then by ϕ_i of the standard structure, $\sigma(\mathbf{u})|_{\overline{\mathbb{C}}-\bigcup_i \widetilde{B}_i}$ is the standard structure. Such structure is f-invariant by construction, and is holomorphic on $\mathbf{u}$. Let $h_{\mathbf{u}}$ be the unique integrating map fixing $0, 1, \infty$. Let $g_{\mathbf{u}} = h_{\mathbf{u}} f h_{\mathbf{u}}^{-1}$ and $g_{\mathbf{u}}^\times = (g_{\mathbf{u}}, h_{\mathbf{u}}(\vec{c}(f^\times)))$. Then $\Lambda(g_{\mathbf{u}}^\times) \equiv \Lambda(f^\times)$, and $\Phi(g_{\mathbf{u}}^\times) = \mathbf{u}$. Therefore $\Sigma(\mathbf{u}) = g_{\mathbf{u}}^\times$ is a holomorphic section of $\Phi|_{Y(f^\times)}$. $\square$

Note that the same proof can be adapted to get similar results in a parabolic basin or a rotation domain.

Given $f_0^\times$, we let $X(f_0^\times)$ denote the connected component of the fiber in the above corollary containing $f_0^\times$, which is again closed in Z.

Corollary 2.3 (One-dimensional)**.** *If $|K| = 1$, i.e., if there is a single free critical point, then $X(f_0^\times)$ is one-dimensional.*

2.4.1. *Polynomial case.* We briefly sketch the construction of analogous spaces for polynomials, having possibly multiple critical points.

A polynomial of degree d has $d-1$ finite critical points, counted with multiplicity. Given a partition $\mathcal{D}$ of $d-1$:

$$d - 1 = d_1 + d_2 + \cdots + d_M;$$

define

$$\mathrm{Poly}_d^\times(\mathcal{D}) = \left\{ f^\times(z) \equiv d \int_0^z \prod_{m=1}^M (\zeta - c_m)^{d_m} d\zeta \right\} \leftrightarrow \{(c_m) \in \mathbb{C}^M\}.$$

Note that elements of $\mathrm{Poly}_d^\times$ are polynomials which fix the origin, are monic, and whose critical points are labelled. The projection $\mathrm{Poly}_d^\times(\mathcal{D}) \to \mathrm{Poly}_d^\times/\mathrm{Aut}(\mathbb{C})$ is thus finite-to-one. We then use this space in place of $\mathrm{GRat}_d^{\times,*}$, and proceed to define Z, Y, X as before.

The *connectedness locus* is the subspace of $\mathrm{Poly}_d^\times(\mathcal{D})$ consisting of maps whose Julia set is connected. Equivalently, the orbit of every critical point c_m is bounded. Later, we will need the following result:

Lemma 2.2. *The connectedness locus is a bounded subset of* $\mathrm{Poly}_d^\times(\mathcal{D})$.

Proof. Suppose $f^\times \in \mathrm{Poly}_d^\times(\mathcal{D})$ has connected Julia set. Let K_f be the filled-in Julia set of f. By a theorem of Böttcher, there is a unique Riemann map

$$(\mathbb{P}^1 - \overline{\Delta}, \infty) \to (\mathbb{P}^1 - K_f, \infty)$$

which is tangent to the identity at infinity and which conjugates $w \mapsto w^d$ to f. Note that $0 \in K_f$. By the Koebe $\frac{1}{4}$-theorem (applied in the $1/z$ coordinates), the image of Σ contains a spherical disk centered at infinity whose radius is independent of f ($\{|z| > 4\}$). Thus K_f is contained in $\{|z| \leq 4\}$, independent of f. Since the critical points c_m of f are contained in K_f, the lemma follows. □

2.5. **Application to spinning.** Let $f_0 \in \mathrm{GRat}_d$, γ be as in the setup for spinning, and σ be the corresponding spinning path. Assume that $f_0^\times \in \mathrm{GRat}_d^{\times,*}$ is a representative of f_0. In other words, we choose a labelling $c_1, \cdots, c_{2d-2}$ of the critical points of f_0, and we normalize f_0 so that $c_1 = 0$, $c_2 = 1$ and $c_3 = \infty$.

Proposition 2.3 (Spinning path lifts to $\mathrm{GRat}_d^{\times,*}$)**.** *The spinning path*

$$\sigma : (\mathbb{R}, 0) \to (\mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1), [f_0])$$

is continuous and is the projection of a continuous path

$$\sigma^\times : (\mathbb{R}, 0) \to (\mathrm{GRat}_d^{\times,*}, f_0^\times).$$

Proof. Choose the quasiconformal conjugacies H_t to fix $0, 1, \infty$. Then H_t, as well as $H_t f_0 H_t^{-1}$, depend continuously (actually, real analytically) on t. Set

$$\sigma^\times(t) = f_t^\times \equiv (H_t f_0 H_t^{-1}, H_t(\vec{c})).$$

By construction, the image of $\sigma^\times$ lies in $\mathrm{GRat}_d^{\times,*}$ and projects to σ. So σ is itself continuous. □

Assume further that we have written the set of indices of critical points of $f_0^\times$ as $I \sqcup J \sqcup K$ as above, such that $f_0^\times \in Z$ and that the spun critical point c has its index in K (this places restrictions on the behavior of the critical points). This then determines the function $\omega : J \to I$.

Proposition 2.4 (Spinning path lifts to $X(f_0^\times)$)**.** *Suppose*

$$f_0^\times \in Z = Z(I, J, K, \omega).$$

Then the lift $\sigma^\times$ of the spinning path lies in $X(f_0^\times)$.

Proof. Let $i \in I$ and c_i denote the ith critical point of $f_0^\times$. Then c_i is in the immediate basin B_i of the attractor a_i.

By conjugacy $H_t(c_i)$ is in the immediate basin of $H_t(a_i)$ and any $i \in I$, and $H_t(c_j)$ is in the basin of $H_t(a_{\omega(j)})$ for any $j \in J$. So $\sigma^\times \subset Z(f_0^\times)$.

By Lemma 2.1, the multiplier of $H_t(a_i)$ is independent of t. Thus the multiplier map Λ is constant on the spinning path $\sigma^\times$, and so $\sigma^\times \subset Y(f_0^\times)$.

Recall that

$$\phi_i : (B_i, a_i, c_i) \to (\mathbb{C}, 0, -1)$$

is the normalized linearizing map on B_i.

Assume that $i \in I$ and a_i does not attract the spun critical point c. Then for $\phi_{i,t} = \phi_i \circ H_t^{-1}$, we have $\phi_{i,t}(c_{j,t}) \equiv \phi_i(c_j)$, in particular $\phi_{i,t}(c_{i,t}) \equiv -1$. So $\phi_{i,t}$ coincides with the normalized lineariser in the definition of Φ.

Assume now $i \in I$ such that $a = a_i$ does attract the spun critical point c. Then for $\phi_{i,t} = \widetilde{h}_t \circ \phi_i \circ H_t^{-1}$, we have $\phi_{i,t}(c_{j,t}) = \widetilde{h}_t(\phi_i(c_j)) = \phi_i(c_j)$, by Lemma 2.1, and the fact that $c_{j,t}$ is not in the grand orbit of c. In particular $\phi_{i,t}(c_{i,t}) \equiv -1$. So, again, $\phi_{i,t}$ coincides with the normalized lineariser in the definition of Φ.

It follows that the function Φ is constant on $\sigma^\times$. Thus

$$\sigma^\times \subset X(f_0^\times)$$

and the Proposition is proved. □

Remark. Let $f_0^\times, \gamma, \tilde{\sigma}$ be as in the example in §1.4, but now let $t \to -\infty$. Using the symmetry $z \mapsto -z$ of f_0 it is straightforward to verify that the resulting path is the same as the path defined by spinning the other critical point b outward along a curve which is the image of the negative real axis under projection from the linearizing coordinate. In the limit, the critical point b lies in a parabolic basin (by Theorem 1.5) and so the locus $X(f_0^\times)$ of points for which b converges to the origin is indeed not closed.

3. Limits of spinning, I

In this section, we prove Theorem 1.4, which explains what dynamical features are preserved when passing to a limit of spinning.

Proof. Conclusion (1) follows by Lemma 9.4 below. Conclusion (2) follows by Lemma 2.1. We now prove (3).

Let $\phi : (B, a, b) \to (\mathbb{C}, 0, -1)$ be the normalized linearizing map for the attractor a. Extend ϕ to the grand orbit of B. Let $p : \mathbb{C}^* \to T$ be projection onto the quotient torus (i.e., identifying z to λz). In the linearizing coordinate plane $\mathbb{C}$, the

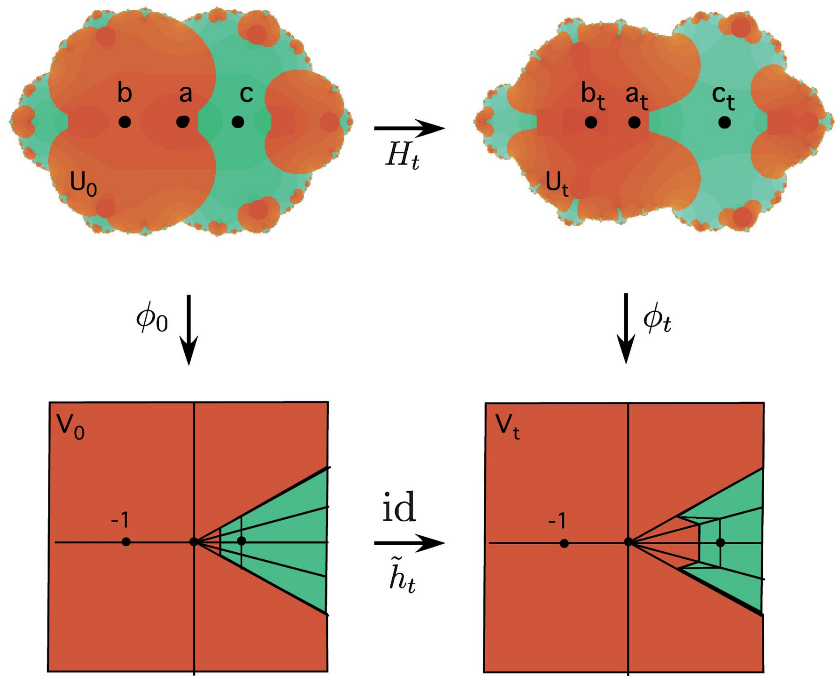

FIGURE 7. The identity map $\mathrm{id}|V_0 : V_0 \hookrightarrow V_t$ is homotopic to $\widetilde{h}_t|V_0 : V_0 \to V_t$ and therefore lifts.

set $p^{-1}(A)$ is a single, thickened logarithmic spiral emanating away from the origin (due to the standing assumption A2 on γ). The map

$$p^{-1} \circ p_A : (\{x + iy : |y| \leq 2l\}, 0) \to (\mathbb{C}^*, \phi(c))$$

conjugates translation by -1 to multiplication by $\lambda = F'(a)$. Consider the domain in the linearizing coordinate plane given by

$$V_0 = V = \mathbb{C} - \overline{p^{-1} \circ p_A(\{x + iy : x \geq -1\})}.$$

(See Figure 7.) Note that V is open, contains the origin, is forward-invariant under multiplication by λ, omits $\phi(c)$ and $\phi(F(c))$, and contains the images under ϕ of all other critical points in the grand orbit of B. Let $\mathcal{U} = \phi^{-1}(V)$. Then conditions (i)–(ii) of the conclusion (3) hold.

To prove (iii), let U_0 be the component of $\mathcal{U}$ containing a. Then U_0 is forward-invariant. Let $\phi_t, \widetilde{h}_t$ be as in the proof of Proposition 2.4, let $V_t = \widetilde{h}_t(V_0)$, and let $U_t = H_t(U_0)$; see Figure 7.

Fix $t \in \mathbb{R}$. If $s < t$, then $V_s \subset V_t$. Thus, the family of maps

$$\widetilde{h}_s : V_0 \to V_t, \quad 0 \leq s \leq t$$

provides an isotopy from the restriction $\mathrm{id}|_{V_0}$ of the identity map to the map $\widetilde{h}_t : V_0 \to V_t$. Let B' be the complement in B of the critical points of F and all of their backward orbits; see Lemma 9.1. Let $U_0' = U_0 \cap B'$, and put $V_0' = \phi_0(U_0')$. Then by Lemma 9.1 the restriction $\phi_0 : U_0' \to V_0'$ is a covering map. Similarly, with the corresponding notation, $\phi_t : U_t' \to V_t'$ is also a covering map.

For each s in $0 \le s \le t$, the map $\widetilde{h}_s$ is the identity on $\mathbb{C} - p^{-1}(A)$. In particular, it is the identity on each puncture of V'_0. Since $\widetilde{h}_t : V'_0 \to V'_t$ lifts under ϕ_0, ϕ_t to $H_t : U'_0 \to U'_t$, by lifting of isotopies we have that $\mathrm{id}|V_0 : V'_0 \to V'_t$ lifts to a holomorphic embedding

$$J_t : U'_0 \to U'_t$$

such that $\phi_t \circ J_t = \mathrm{id} \circ \phi$ on U. Since $\phi \circ F = \lambda\phi$ and $\phi_t \circ F_t = \lambda\phi_t$ also, this implies $J_t \circ F = F_t \circ J_t$.

Using similar reasoning, one can inductively extend J_t to each component of $\mathcal{U}$ to obtain an embedding $J_t : \mathcal{U} \to \mathbb{P}^1$ such that $J_t \circ F = F_t \circ J_t$. Note that this implies that $J_t(b)$ is a critical point of F_t.

To prove (iv), apply Lemma 9.3 with $W = U_0$ to conclude that after passing to subsequences, the holomorphic maps $J_t|U_0$ converge locally uniformly to a map J which is either an embedding, or else is a constant map with value in $\mathrm{Fix}(G)$. Since U_0 contains both the visible critical point b and the fixed point a, if J were constant, then $J(b) = \lim_{n\to\infty} J_{t_n}(b)$ would be a fixed critical point of G, i.e., a fixed point of multiplier zero. On the other hand, $J(b) = J(a) = \lim_{n\to\infty} J_{t_n}(a) = a_{t_n} = a_\infty$ is a fixed point of multiplier $\lambda \neq 0$ by (2). This is not possible. Hence J is an embedding of U_0 into $\mathbb{P}^1$.

The maps J_{t_n} are actually defined on all of $\mathcal{U}$; as before we may assume that $J_{t_n} : \mathcal{U} \to \mathbb{P}^1$ converges locally uniformly to a map $J : \mathcal{U} \to \mathbb{P}^1$ satisfying $J \circ F = G \circ J$. On each component W of $\mathcal{U}$, the map J is either an embedding, or is constant. Suppose $W \subset F^{-k}(U_0)$ is a component of $\mathcal{U}$. If $J|W$ is constant, then $G^k \circ J = J \circ F^k$ is constant on W. Since F^k is open, and $F^k(W) \subset U_0$, this implies $J|U_0$ is constant; a contradiction. □

4. Limits of spinning, II

Here, we study what new dynamical features develop in limits of spinning. These are summarized in Theorem 1.5, which we now prove.

Proof. Theorem 1.4 implies that after passing to subsequences, $a_{t_n}, b_{t_n}, c_{t_n} \to a_\infty, b_\infty, c_\infty$ where $a_\infty, b_\infty, c_\infty$ are distinct. Let $M_{t_n} \in \mathrm{Aut}(\mathbb{P}^1)$ be the unique map sending $a_{t_n}, b_{t_n}, c_{t_n} \mapsto 0, -1, +1$. Then $M_{t_n} \to M_\infty$, and so $M_{t_n} \circ F_{t_n} \circ M_{t_n}^{-1} \to M \circ G \circ M^{-1}$. Hence we may assume $a_{t_n}, b_{t_n}, c_{t_n} = 0, -1, +1$.

Let $\phi : (B, 0, b = -1) \to (\mathbb{C}, 0, -1)$ be the unique normalized linearization map for the basin B. Recall that $\Psi = p \circ \phi : B \to B/F = T$, a complex torus, and that A is an annulus on T with core γ. The statement below makes precise the idea that, under these assumptions, points which get "spun" move off to infinity from the point of view of the attractor at the origin.

Proposition 4.1 (Spun points tend to infinity)**.** *Let $z \in \Psi^{-1}(A)$, and suppose z_∞ is any limit point of $z_{t_n} = H_{t_n}(z)$. Then z_∞ is not in B_∞.*

Proof. By Lemma 2.1,

$$\phi_{t_n}(z_{t_n}) = \phi_{t_n}(H_{t_n}(z)) = \widetilde{h}_{t_n}(\phi(z)) \to \infty.$$

For example, if $\Psi(z)$ lies in the central subannulus of A where the conjugacies h_{t_n} are conformal then $\widetilde{h}_{t_n}(\phi(z)) = \lambda^{-t_n}\phi(z)$ which tends to infinity as $t_n \to +\infty$. Here, we have made nontrivial use of the hypotheses that γ is oriented outward, and $t_n \to +\infty$.

Let $J : \mathcal{U} \to \mathbb{P}^1$ be the partial conjugacy from F to G given by Theorem 1.4. We now use the assumption (A3) that B contains a visible critical point b. Since $b \in \mathcal{U}$, $J(b)$ makes sense. Let $b_\infty = J(b) = \lim_{n\to\infty} J_{t_n}(b) \in B_\infty$. Since b is visible, it has an infinite forward orbit, hence so does $b_\infty = J(b)$. If ψ_{t_n}, ψ_∞ denote the linearizing maps normalized to have derivative one at the origin, then by Lemma 9.2, $\psi_{t_n} \to \psi_\infty$ uniformly on compact subsets of B_∞. In particular,

$$\psi_{t_n}(b_{t_n}) \to \psi_\infty(b_\infty) \neq 0$$

since b_∞ has infinite forward orbit. Hence as $n \to \infty$, the maps

$$\phi_{t_n}(z) = -\frac{1}{\psi_{t_n}(b_{t_n})}\psi_{t_n}(z)$$

converge as well. Thus $z_\infty \in B_\infty$ would imply that $\phi_{t_n}(z_{t_n})$ converges to a finite value, which by the previous paragraph is impossible. □

Proof of Theorem 1.5. Let A_c be the central subannulus of A on which the spinning map h is holomorphic (see Figure 1). Let W be the component of $\Psi^{-1}(A_c)$ whose closure contains the attractor a. By the standing assumption (A2) W exists, is unique, and $F(W) \subset W$. (In Figure 6, the subset W is a slightly skinnier version of the prominent (green) light region on the right-hand side of the upper left image, and c is visible.) Suppose c is visible after $r \geq 0$ steps. Then r is the smallest nonnegative integer for which $F^r(c) \in W$. The conjugacies H_{t_n} are holomorphic on W. Lemma 9.3 implies that after passing to a subsequence, the maps $H_{t_n}|_W$ converge to a limit H_∞ which is either univalent and conjugates $F|W$ to G, or is a constant map with image a' equal to a fixed point of G.

Case $H_\infty(W) = a'$ is constant. Then $G^r(c_\infty) = a'$ is a fixed point of G. If a' is attracting or superattracting, then the basin of a' contains $G^r(c_\infty)$. By Proposition 4.1, the basin of a' is disjoint from B_∞. This is impossible, since then for n large, F_{t_n} would have $F^r_{t_n}(c_{t_n})$ and a_{t_n} in disjoint basins; a contradiction. Thus, $G^r(c_\infty) = a'$, a repelling or neutral fixed point.

Case $H_\infty|_W$ is univalent. It follows easily that $H_\infty(W)$ is contained in Fatou component Ω of G, $G^r(c_\infty) \in \Omega$, and $G(\Omega) = \Omega$. We claim that Ω must be parabolic; the condition $G(\Omega) = \Omega$ implies that the multiplier is one. The same argument as that given in the previous paragraph shows that $\Omega \neq B_\infty$ and that Ω cannot be an attracting or superattracting basin. The map $F|W$ has the property that $F(W) \subset W$ and that under iteration, every orbit leaves any compact subset. The same is true therefore for $G|_{H_\infty(W)}$. So Ω cannot be a Siegel disk or Herman ring either. By the classification of Fatou components, Ω is a parabolic basin. □

Proof of Corollary 1.5. Now F is hyperbolic, i.e., all critical points c_i are in attracting or superattracting basins. Assume, by taking a subsequence if necessary, $H_{t_n}(c_i) \to \xi_i$ as $n \to \infty$. Clearly each ξ_i is a critical point of G and G has no other critical points. Recall that B denotes the immediate attracting basin containing the attractor a of the spun critical point. Let $\widetilde{B}$ denote the full basin of a.

Suppose $c_i \notin \widetilde{B}$. Then $c_i \in \mathcal{W}_0$, where $\mathcal{W}_0$ is as in Theorem 1.4. This theorem implies that $\xi_i = \mathcal{J}(c_i)$ lies in an attracting or superattracting basin of G.

Suppose $c_i \in \widetilde{B}$ and $c_i \neq c$. Again, Theorem 1.4 implies that $\xi_i = J(c_i)$ is attracted by a_∞.

Suppose $c_i = c$, the spun critical point. Then $\xi_i = c_\infty$. Theorem 1.5 implies that either $G^r(c_\infty)$ is in a fixed parabolic basin Ω, or $G^r(c_\infty)$ is a repelling or neutral fixed point.

In the former case, c_∞ must be itself in Ω as well, since Ω must contain a critical point, and by assumption (A3) the orbit of c does not contain other critical points. All other critical points converge to attractors, so G has no other parabolic basins and is therefore geometrically finite.

In the latter case, $G^r(c_\infty)$ cannot be neutral, since this would require the existence of another critical point having infinite orbit and not converging to an attracting cycle. Hence $G^r(c_\infty)$ is a repelling fixed point, G has no parabolic basins, and G is subhyperbolic, hence geometrically finite. □

5. Proof of Theorems 1.1 and 1.2

Let f be a critically generic hyperbolic rational map without critical orbit relations, or a hyperbolic polynomial with connected Julia set and without critical orbit relations. Let a, B, γ, c, b be as in Theorem 1.2, and suppose c is visible after r steps. Let $\sigma : [0, +\infty) \to \mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$ be the spinning ray.

In case f is a rational map, we make the assumption that the spinning ray has at least one limit point in $\mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$. In the polynomial case, existence of a limit point follows since the connectedness locus is bounded (Lemma 2.2).

If f is a rational map, conjugate it and label its critical points to produce $f_0^\times \in \mathrm{GRat}_d^{\times,*}$. If f is a polynomial, use the orders of its critical points to determine a partition $\mathcal{D}$ of $d-1$. Conjugate f and label its critical points to produce $f_0^\times \in \mathrm{Poly}_d^\times(\mathcal{D})$.

Since f is hyperbolic, and has no critical orbit relations, there exists a decomposition of indices of critical points

$$C = \{1, 2, \ldots, 2d-2\} = I \sqcup J \sqcup \{k\}$$

in the rational case and

$$C = \{1, 2, \ldots, M\} = I \sqcup J \sqcup \{k\}$$

in the polynomial case, such that (i) $c_k = c$, the spun critical point, and (ii) the conditions in §4 for the definition of the subspace Z are satisfied. By Corollary 2.3, the subspace $X(f_0^\times)$ is a Riemann surface. By Proposition 2.4, the spinning ray lifts to $X(f_0^\times)$.

We now verify the assertion in Step 2 of the outline in §1.4. Let Ξ denote the set of limit points of the spinning ray in $\mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$, and let $\widetilde{\Xi}$ denote the set of limit points of its lift in $X(f_0^\times)$. Let $\pi^{\times,*} : \mathrm{GRat}_d^{\times,*} \to \mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$ be the natural projection which forgets the labelling of critical points and records the conjugacy class.

Let $f_t^\times = \sigma^\times(t)$. Suppose $\sigma(t_n) \to \xi_\infty \in \mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$. We must show that after passing to subsequences, $f_{t_n}^\times \to g^\times$ for some $g^\times \in X(f_0^\times)$. Since the possible labellings are finite in number, it suffices to prove that the underlying maps f_{t_n} converge to g. The definition of the quotient topology on $\mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$ implies that there exist $F_{t_n}, G \in \mathrm{Rat}_d$ such that $F_{t_n} \to G$, $\pi^{\times,*}(F_{t_n}) = \sigma(t_n)$, and $\pi^{\times,*}(G) = \xi_\infty$. By Theorem 1.4, the critical points of G are distinct. We may choose labelling of critical points so that $F_{t_n}^\times \to G^\times$ in $\mathrm{GRat}_d^\times$, since the set of possible labelling is finite. We may then write $f_{t_n} = M_n F_{t_n} M_n^{-1}$ where $M_n(\{0, 1, \infty\})$ is contained in the set of critical points of F_{t_n}. Since the critical points of F_{t_n} converge to those of G, which are distinct, after passing to a subsequence we have $M_n \to M$. Thus $f_{t_n} = M_n F_{t_n} M_n^{-1} \to MGM^{-1}$. Putting $g = MGM^{-1}$, we have produced a limit point $g^\times$ of $f_{t_n}^\times$. By Theorem 1.4, for $i \in I$, the attractors $a_i(g^\times)$ have the same multiplier, the critical points $c_i(g^\times)$ have linearizing position -1, and for $j \in J$, the critical points $c_j(g^\times)$ have the same linearizing coordinate as $c_j(f_0^\times)$. Thus $g^\times \in Z$. However, $X(f_0^\times)$ is closed in Z. So $g^\times \in X(f_0^\times)$, and Step 2 is shown.

By Corollary 1.5, any limit point $g^\times$ of $\sigma^\times(t)$ has either a 1-parabolic fixed point, or else $g^r(c_\infty)$ is a repelling fixed point. That is, $g \in X_{par}^{1,1}$ or $g \in X_{mis}^{r,1}$ in the notation of §1.2. Since $f_0^\times$ is hyperbolic, $X_{par}^{1,1}, X_{mis}^{r,1}$ are not all of $X(f_0^\times)$. Since $X(f_0^\times)$ is one-dimensional, it follows that $X_{par}^{1,1}, X_{mis}^{r,1}$ are discrete subsets of $X(f_0^\times)$. Since the spinning ray is connected, if two distinct limit points exist, then a continuum of limit points exists, which violates discreteness. Hence the spinning ray σ has a unique limit point in $\mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$.

This proves most of Theorems 1.2 and 1.3. The proof of the conclusion that the limit is independent of the representative γ and the annulus A, requires a bit of technology from Teichmüller theory and is given in §7. $\square$

6. Limits of spinning, III

In this section, we prove Theorems 1.6 and 1.7.

We first recall some notation. $A-\gamma$ is the union of two annuli $A^\pm$, with core curve $\hat{\gamma}^\pm$ (see Figure 1). Denote by $\delta^\pm \subset \Psi^{-1}(\hat{\delta}^\pm)$ (respectively, $\gamma^\pm \subset \Psi^{-1}(\hat{\gamma}^\pm)$, $W \subset \Psi^{-1}(A_c)$, $S^\pm \subset \Psi^{-1}(A^\pm)$) the unique lifts whose closures contain the attractor a.

Assume c is visible. Conclusion (1) follows immediately from Corollary 1.5.

Taking subsequences if necessary, we may assume $H_{t_n}|W \to H_\infty$ locally uniformly, $\overline{\gamma_{t_n}^\pm} \to \Gamma^\pm$ in the Hausdorff topology, $u_{t_n}^\pm \to u_\infty^\pm$, $a_{t_n} \to a_\infty$ and $c_{t_n} \to c_\infty$. Note that $H_\infty(\gamma^\pm) \subset \Gamma^\pm$. We will make use of the structure theorem for geometric limits of invariant strips (Appendix 8).

In our setting $a_\infty \in \Gamma^\pm$ is an attracting vertex. It follows by Theorem 8.6 that $\Gamma^\pm$ has exactly one edge $\zeta^\pm$ contained in B_∞. By Theorem 1.5 and Corollary 1.5, $c_\infty \in H_\infty(W) \subset \Omega$, with Ω a fixed parabolic basin, H_∞ is univalent, G has no other parabolic basins, Ω contains no other critical points. Therefore $G|\Omega$ is conformally conjugate to the "cauliflower" $z^2 + \frac{1}{4}$. The set $H_\infty(W)$ represents an annulus in Ω/G and contains the unique puncture (corresponding to c_∞). Thus $H_\infty(\gamma^\pm)$ are two loops in Ω ending at the parabolic point v, symmetric under an anticonformal involution of $H_\infty(W)$. By Theorem 8.6, every other edge of $\Gamma^\pm$ is contained in Ω, is therefore a loop based on v. So $\Gamma^\pm$ has only two vertices: a_∞ and v. As the backward end of $\zeta^\pm$ is a nonattracting vertex of $\Gamma^\pm$, it must be v. So $v \in \partial B_\infty$. In summary, $\Gamma^+ \cup \Gamma^-$ looks like Figure 8, possibly with infinitely many loops. This proves conclusion (2).

Conclusion (3) is more delicate.

Claim 1. The point v is split into two fixed points for nearby maps. This is due to the fact that v has multiplicity 1.

Denote by $v_{t_n}^1, v_{t_n}^2$ the corresponding fixed points for F_{t_n}. As F_{t_n} is hyperbolic, $v_{t_n}^1$, $v_{t_n}^2$ are repelling, and thus distinct.

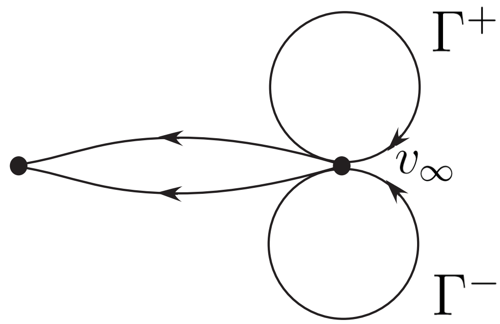

FIGURE 8. Structure of the Hausdorff limits $\Gamma^\pm$.

Claim 2. $u^\pm_{t_n} \in \{v^1_{t_n}, v^2_{t_n}\}$. This is because $u^\pm_\infty$ are nonattracting vertices of $\Gamma^\pm$, must be equal to v so $u^\pm_{t_n}$ are close to v, and the only fixed points of F_{t_n} close to v are $v^1_{t_n}$ and $v^2_{t_n}$.

Now choose P, a repelling petal of v. ∂P contains a subarc $I \subset B_\infty$ connecting ζ^+ to ζ^-, and $\partial P - (I \cup \{v\})$ has two components (call them 'sides'), each intersects one of $H_\infty(\gamma^+)$, $H_\infty(\gamma^-)$.

Claim 3. Γ^+ (resp. Γ^-) intersects $\partial P - \{v\}$ only on one side, e.g. the side of $H_\infty(\gamma^+)$ (resp. $H_\infty(\gamma^-)$).

Proof by contradiction. Take the outermost edges ι_1, ι_2 of Γ^+ on each side. Adjust P so that each ι_i intersects $\partial P - \{v\}$ at only one point, transversally, and points outwards. Enlarge I into an arc $I' \subset \partial P$ so that it intersects only ζ^+, ι_1 and ι_2 among the edges of Γ^+. Choose U a small neighborhood of $\overline{\zeta^+}$. Adjust U and P so that $\kappa = \partial(U - P) \cup I'$ is a Γ^+-transversal graph, intersecting Γ^+ only on I', and at three points. By Theorem 8.5, for large n, $\gamma^+_{t_n} \cap \kappa$ has essentially the same oriented structure. However, $\gamma^+_{t_n}$ is an embedded arc without self-intersections. This is not possible by the Jordan curve theorem.

Claim 4. For some large n, $u^+_{t_n} \neq u^-_{t_n}$.

Proof by contradiction. Denote by $\iota^\pm$ the outermost loop of $\Gamma^\pm - (\zeta^\pm \cup \{a_\infty\})$. Adjust P and I' as above. Choose U a small disk containing $\overline{\zeta^+} \cup \overline{\zeta^-}$. Define κ as above. Now $\Gamma^+ \cup \Gamma^-$ intersects κ at exactly four points, all pointing outwards. Again, for large n, $\gamma^+_{t_n} \cup \gamma^-_{t_n} \cap \kappa$ has essentially the same oriented structure. Assume $u^+_{t_n} = u^-_{t_n}$. Then $\overline{\gamma^+_{t_n}} \cup \overline{\gamma^-_{t_n}}$ is a Jordan curve. This is not possible. (Note that the same idea proves also that ζ^+ and $H_\infty(\gamma^+)$ are on the same side.)

Claim 5. $u^+ \neq u^-$ (conclusion (3)), as $u^+_{t_n} \neq u^-_{t_n}$ for some n, and $u^\pm_{t_n} = H_{t_n}(u^\pm)$.

For conclusion (4), we need the holomorphic index formula. The only fixed points of F_{t_n} near v are $u^+_{t_n}, u^-_{t_n}$. If we integrate around a small loop going once around v, then for n sufficiently large,

$$\frac{1}{1-\lambda^+_{t_n}} + \frac{1}{1-\lambda^-_{t_n}} = \frac{1}{2\pi i}\int \frac{dz}{z - F_{t_n}(z)} \to \frac{1}{2\pi i}\int \frac{dz}{z - G(z)}.$$

The first inequality now follows with m equal to slightly less than the real part of the index of G at v. Since $|\lambda^\pm_{t_n}| > 1$, the second inequality is trivial.

Case c not visible is much easier. We omit the details. $\square$

Theorem 1.7 follows immediately from Theorem 1.6, since the conclusion (3) (i.e., $u^+ \neq u^-$) of Theorem 1.6 is violated.

7. Interpretation via Teichmüller theory

In this section, we give an alternative construction of the complex manifold $X(f_0^\times)$ presented in §4, and we prove the uniqueness assertion of Theorems 1.2 and 1.3.

Moduli spaces. Fix

- a complex torus T,
- a nonempty set $\{z_1, \ldots, z_l\}$ of distinct points on T,
- a nonzero primitive homology class $\alpha \in H_1(T, \mathbb{Z})$.

We let $S = T - \{z_1, \ldots, z_l\}$, so that $\overline{S} = T$. We recall that a quasiconformal homeomorphism $h : S \to S'$ extends uniquely to a quasiconformal homeomorphism $\overline{h} : T \to T'$.

Recall that the *modular group* $\mathrm{Mod}(S) = QC(S)/QC_0(S)$, where $QC(S)$ is the group of quasiconformal homeomorphisms of S to itself and $QC_0(S)$ is the normal subgroup of those maps which are isotopic to the identity through qc maps which leave the punctures fixed. The group $\mathrm{Mod}(S)$ (anti)-acts properly discontinuously by holomorphic automorphisms on the Teichmüller space via

$$h.(\psi : S \to S') \equiv \psi \circ h^{-1} : S \to S'.$$

Define $P\mathrm{Mod}(S, \alpha)$ to be the subgroup of $\mathrm{Mod}(S)$ represented by those maps $h : S \to S$ for which $\overline{h}(z_j) = z_j, j = 1, \ldots, l$ and for which $\overline{h}_*(\alpha) = \alpha$, where $\overline{h}_* : H_1(T, \mathbb{Z}) \to H_1(T, \mathbb{Z})$ is the induced map on homology. It is a subgroup of the pure modular group consisting of maps which fix each puncture, but is not a normal subgroup.

Let

$$\mathcal{M}(S, \alpha) = \mathrm{Teich}(S)/P\mathrm{Mod}(S, \alpha).$$

Proposition 7.1. *The space $\mathcal{M}(S, \alpha)$ is a complex manifold of dimension l.*

Proof. It is enough to prove that the action is fixed-point free. Let $h \in P\mathrm{Mod}(S, \alpha)$, and let $(\psi : S \to S')$ represent a fixed point for h. Then there exists a conformal isomorphism $g : S' \to S'$ such that h is isotopic to $\psi^{-1} g \psi$. Moreover, $\overline{g} : T' \to T'$ fixes the primitive nonzero homology class $\overline{\psi}_*(\alpha) \in H_1(T', \mathbb{Z})$.

By assumption T' has at least one marked point z_1 which must fixed by $\overline{g}$. Let $p : (\mathbb{C}, 0) \to (T', z_1)$ denote a universal cover, Λ its deck group acting by translations, and let $\tilde{g} : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a lift of $\overline{g}$ under p. Then $\tilde{g}(w) = \omega w$ for some $\omega \in \mathbb{C}^*$. Since $\overline{g}$ fixes a nonzero homology class, $\tilde{g}$ fixes a nonzero element of Λ. Hence $\omega = 1$, i.e., $\tilde{g}$ is the identity. Thus $g : S' \to S'$ is the identity and h is isotopic to the identity. $\square$

Puncture-forgetting maps. As above, fix a torus T, a set $\{z_1, \ldots, z_l\} \subset T$, and let $S = T - \{z_1, \ldots, z_l\}$. Suppose $l > 1$ and write $l = n + k$ where $n, k > 0$. Let $S^\sharp = T - \{z_1, \ldots, z_n\}$.

Suppose a point in $\mathrm{Teich}(S)$ is represented by $\psi : S \to S'$. Let $S'^\sharp = T - \overline{\psi}(\{z_1, \ldots, z_n\})$ be the surface in which the last k punctures are filled in. This induces a holomorphic "puncture-forgetting" map

$$\nu^\sharp : \mathrm{Teich}(S) \to \mathrm{Teich}(S^\sharp).$$

Theorem 7.1.

(1) *The map* $\nu^\sharp : \mathrm{Teich}(S) \to \mathrm{Teich}(S^\sharp)$ *is a holomorphic fibration; in particular it is a holomorphic submersion.*

(2) *If* $k = 1$*:*

   (a) *the fiber* $\mathcal{F}$ *above a point* $S^\sharp$ *is a properly embedded holomorphic disk which is naturally identified with the universal cover of* $S^\sharp$*, and*

   (b) *there is a natural embedding*

$$\pi_1(S^\sharp, z_{n+1}) \overset{\iota}{\hookrightarrow} \mathrm{Mod}(S)$$

   such that the restriction of $\iota(\gamma)$ *to* $\mathcal{F}$ *coincides with the action of* γ *on the universal cover of* $S^\sharp$*.*

Proof. See e.g. [Nag], §5.3 for (1) and [Kra] for (2). □

New construction of $X(f_0^\times)$**.** We consider only the case of rational maps; the polynomial case is entirely analogous. Assume a decomposition $C = \{1, 2, 3, \ldots, 2d-2\} = I \sqcup J \sqcup K$ and a function $\omega : J \to I$ are given as in §4, and suppose $f_0^\times \in Z$.

Fix $i \in I$. For $f^\times \in Z$, let $S_i(f^\times)$ denote the quotient surface $\hat{B}_i/f$ corresponding to the ith attractor of $f^\times$ (i.e., S_i is the corresponding torus T_i punctured at the orbits of *all* critical points in the basin $\hat{B}_i$). Let $S_i^\sharp(f^\times)$ be the surface $S_i(f^\times)$ with the punctures corresponding to free critical points $c_k, k \in K$ of $f^\times$ filled in. Let $\alpha_i(f^\times) \in H_1(T_i(f^\times), \mathbb{Z})$ denote the canonical homology class. Abusing notation let us denote by $c_i(f^\times), c_j(f^\times)$ $(j \in \omega^{-1}(i))$ both the critical points and their images under projection to T_i.

Lemma 7.2. *The correspondence* $f^\times \mapsto (S_i^\sharp(f^\times), \alpha_i(f^\times))$ *determines a well-defined holomorphic map* $\varphi_i^\sharp : Z \to \mathcal{M}(S_i^\sharp(f_0^\times), \alpha_i(f_0^\times))$*.*

Proof. Up to composition with an element of $P\mathrm{Mod}(S_i^\sharp(f_0^\times), \alpha_i(f_0^\times))$, there is a unique isotopy class of quasiconformal map $\psi : S_i^\sharp(f_0^\times) \to S_i^\sharp(f^\times)$ such that the extension $\overline{\psi} : T_i(f_0^\times) \to T_i(f^\times)$ sends $c_i(f_0^\times)$ to $c_i(f^\times)$, sends $c_j(f_0^\times)$ to $c_j(f^\times)$, and for which $\overline{\psi}_*(\alpha_i(f_0^\times)) = \alpha_i(f^\times)$. Hence $\varphi_i^\sharp$ is well defined.

To show that $\varphi_i^\sharp$ is holomorphic, use the fact that the multiplier $\lambda_i(f^\times)$ and the locations $\phi_i(c_j)$ of the critical points in the linearizing coordinates vary holomorphically in f; see §4. □

Denoting $\mathcal{M}(S_i^\sharp(f_0^\times), \alpha_i(f_0^\times))$ by $\mathcal{M}_i^\sharp$, we have therefore a map

$$\varphi^\sharp \equiv (\varphi_i^\sharp) : Z \to \prod_i \mathcal{M}_i^\sharp.$$

Lemma 7.3. *The map* $\varphi^\sharp : Z \to \prod_i \mathcal{M}_i^\sharp$ *is a holomorphic submersion.*

Proof. Let $f^\times \in Z$. By ([MS], Thm. 6.2), the Teichmüller space of any rational map f is naturally isomorphic to

$$\mathrm{Teich}(\Omega^{dis}/f) \times M_1(J, f) \times \mathrm{Teich}(\Omega^{fol}, f).$$

The second factor is a polydisk corresponding to invariant line fields supported on the Julia set (conjecturally, this occurs only in the case of Lattès examples). The third is a polydisk corresponding to deformations supported in Siegel disks, Herman rings, and superattracting basins. The first factor is in turn isomorphic to a product of Teichmüller spaces of quotient surfaces. Hence the Teichmüller spaces of quotient surfaces appear naturally as factors in the complex manifold which is the Teichmüller space of f. Moreover, there is a canonically defined holomorphic map

$$\eta : \mathrm{Teich}(f) \to \mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$$

obtained by straightening. By taking the qc conjugacy to fix zero, one, and infinity, we get a lift $\eta^{\times}$ whose image lies in Z by construction.

We have the following commutative diagram of pointed complex manifolds:

$$\begin{array}{ccccc}
\prod_i \mathrm{Teich}(S_i(f^{\times})) & \overset{\iota}{\hookrightarrow} & \mathrm{Teich}(f) & \overset{\eta^{\times}}{\to} & (Z, f^{\times}) \\
(\nu_i^{\sharp}) = \nu^{\sharp} \downarrow & & & & \downarrow \varphi^{\sharp} = (\varphi_i^{\sharp}) \\
\prod_i \mathrm{Teich}(S_i^{\sharp}(f^{\times})) & & \longrightarrow & & \left(\prod_i \mathcal{M}_i^{\sharp}, (S_i^{\sharp}(f^{\times}))\right)
\end{array}$$

Here, $\nu_i^{\sharp} : \mathrm{Teich}(S_i) \to \mathrm{Teich}(S_i^{\sharp})$ is the map induced by forgetting punctures corresponding to free critical points, and ι is the inclusion map.

The map on the bottom is a (universal) holomorphic covering map. By Theorem 7.1, each $\nu_i^{\sharp} : \mathrm{Teich}(S_i) \to \mathrm{Teich}(S_i^{\sharp})$ is a holomorphic submersion, therefore the product is a submersion as well. Since the diagram commutes, it follows that the derivative of $\varphi^{\sharp}$ is surjective when evaluated at $f^{\times}$, and the proof is complete. $\square$

Fix again $i \in I$ and let $n_i = \#\omega^{-1}(i)$. Recall from §4 that

$$\mathbf{C}^i = (\mathbb{C}^* - \{(-1)\lambda_i^n, n \in \mathbb{Z}\})^{\omega^{-1}(i)} - \text{big diagonal}$$

where the big diagonal is the locus where two or more coordinates have the same λ_i-orbits. The map $\varphi_i : Z \to \mathcal{M}_i^{\sharp}$ can be written as a composition

$$Z \overset{(\lambda_i, \Phi_i)}{\longrightarrow} \Delta^* \times \mathbf{C}^i \overset{\overline{\varphi_i^{\sharp}}}{\longrightarrow} \mathcal{M}_i^{\sharp}.$$

The second map $\overline{\varphi^{\sharp}}_i$ is the one induced by sending the pair $(\lambda, (w_1, \ldots, w_{n_i}))$ to the quotient torus $\mathbb{C}^*/\langle w \mapsto \lambda w \rangle$ punctured at the images of -1 and at the w_j's. Using this it is easy to see that the map $\overline{\varphi^{\sharp}}_i$ is in fact an infinite covering. It follows that the $|K|$-dimensional analytic set $X(f_0^{\times})$ constructed in §4 coincides with the connected component of $(\varphi^{\sharp})^{-1}(\varphi^{\sharp}(f_0^{\times}))$ containing $f_0^{\times}$, which by the previous proposition is a complex submanifold.

When $|K| = 1$. Suppose $|K| = 1$ and the unique free critical point $c = c_k$ lies in the basin of attraction of the ith attracting cycle of $f_0^{\times}$. Let $S_i = S_i(f_0^{\times})$ denote the corresponding quotient surface, regarded as the basepoint in $\mathrm{Teich}(S_i)$. Set $T_i = T_i(f_0^{\times})$.

The inclusion

$$S_i \hookrightarrow S_i^{\sharp} = S_i \cup \{c\}$$

induces the puncture-forgetting map

$$\nu_i^{\sharp} : \mathrm{Teich}(S_i) \to \mathrm{Teich}(S_i^{\sharp}).$$

By Theorem 7.1, the fiber $\mathcal{F}$ above the basepoint $S_i^\sharp$ is canonically identified with the universal cover of $S_i^\sharp$ and is a properly embedded holomorphic disk. Spinning continuously about γ defines a map $\hat{\sigma} : \mathbb{R} \to \mathcal{F}$ such that $\hat{\sigma}(0)$ is the basepoint S_i. Identifying factors with their images under the canonical inclusion (i.e., using basepoints corresponding to f_0 in the factors other than i) we have the following diagram of pointed manifolds:

$$
\begin{array}{cccccc}
 & (\mathbb{R}, 0) & \xrightarrow{\text{id}} & (\mathbb{R}, 0) & \\
 & \hat{\sigma} \downarrow & & \downarrow \sigma^\times & \\
\text{Teich}(S_i) \supset & (\mathcal{F}, S_i) & \xrightarrow{\eta^\times \circ \iota} & (X(f_0^\times), f_0^\times) & \subset Z(f_0^\times) \\
 & \nu_i^\sharp \downarrow & & \downarrow \varphi_i^\sharp & \\
\text{Teich}(S_i^\sharp) \ni & \{S_i^\sharp\} & \longrightarrow & \{(S_i^\sharp, \alpha_i)\} & \in \mathcal{M}_i^\sharp
\end{array}
$$

Let $t \in \mathbb{Z}$, let i be the index of the attractor in which spinning takes place, and let $h : S_i \to S_i$ denote the quasiconformal homeomorphism as in the definition of spinning (§1). Then h represents an element of the pure modular group $P\text{Mod}(S_i)$. Since $\text{Mod}(S_i)$ anti-acts on $\text{Teich}(S_i)$ by *pre*composition on the marking maps, if we set

$$\tau \equiv [h_{-1}],$$

then

$$\tau.\hat{\sigma}(t) = \hat{\sigma}(t+1) \quad \text{for all} \quad t \in \mathbb{R}.$$

The lemma below follows from Theorem 7.1.

Lemma 7.4. *The infinite cyclic group $\langle \tau \rangle$ acts on* $\text{Teich}(S_i)$ *by biholomorphic maps preserving the holomorphic disk $\mathcal{F}$. With respect to the Poincaré metric on $\mathcal{F}$, the map τ is a hyperbolic translation whose length is the length of the unique simple closed geodesic on $S_i^\sharp$ freely homotopic to γ.*

To set up the next statement, let $H : \mathbb{P}^1 \to \mathbb{P}^1$ be a quasiconformal conjugacy between $f_0^\times$ and another map $F_0^\times \in Z$ such that $c_m(F_0^\times) = H(c_m(f_0^\times))$, $m = 1, 2, \ldots, 2d-2$. The pair (F_0, H) represents an element of $\text{Teich}(f_0)$, by definition.

Theorem 7.5. *Suppose $(F_0, H) \in \mathcal{F}$. Let $f_n^\times = (\eta^\times \circ \iota)(\tau^n.(f_0, \text{id}))$ and let $F_n^\times = (\eta^\times \circ \iota)(\tau^n.(F_0, H))$. Then*

$$\lim_{n\to\infty} F_n^\times = \lim_{n\to\infty} f_n^\times$$

if the latter limit exists in the space $X(f_0^\times)$.

Proof. Suppose the latter limit is $g^\times$. Consider the hyperbolic surface Ω obtained by puncturing the Riemann surface $X(f_0^\times)$ at $g^\times$ and equipping it with the hyperbolic metric, denoted d_Ω. Let $d_{\mathcal{F}}$ denote the hyperbolic metric on $\mathcal{F}$. Since τ acts by hyperbolic isometries,

$$d_{\mathcal{F}}(\tau^n.(f_0, \text{id}), \tau^n.(F_0, H)) = d_{\mathcal{F}}((f_0, \text{id}), (F_0, H)) = D,$$

which is independent of n. By hypothesis, the sequence $\{f_n^\times\}$ exits the cusp of Ω corresponding to the puncture at $g^\times$. The map $(\eta^\times \circ \iota) : \mathcal{F} \to \Omega$ is holomorphic, so it is distance nonincreasing with regard to $d_{\mathcal{F}}$ and d_Ω. Hence for all n,

$$d_\Omega(f_n^\times, F_n^\times) \leq D.$$

By choosing a local chart centered at $g^\times$ and comparing Euclidean and hyperbolic metrics, we see that the sequence $\{F_n^\times\}$ must exit the cusp at $g^\times$ as well. $\square$

Proof of uniqueness in Theorems 1.1 and 1.2. Consider spinning, starting with the map $F_0^\times$, and using the curve $H(\gamma)$ and the annulus $H(A)$ instead. Then

$$\sigma_{\gamma,A},\ \ \sigma_{H(\gamma),H(A)} : \mathbb{R} \to \mathrm{Rat}_d/\mathrm{Aut}(\mathbb{P}^1)$$

are two spinning paths, which yield lifts

$$\sigma^\times_{\gamma,A},\ \ \sigma^\times_{H(\gamma),H(A)} : \mathbb{R} \to X(f_0^\times) = X(F_0^\times).$$

Since

$$(\eta^\times \circ i)(\tau^n.(F_0, H)) = \sigma^\times_{H(\gamma),H(A)}(n), \quad n \in \mathbb{Z},$$

the previous theorem and Step 2 of the proof of Theorem 1.1 in §5 yields

Corollary 7.1. *The limits of spinning rays*

$$\lim_{t\to+\infty} \sigma_{\gamma,A}(t) \quad \text{and} \quad \lim_{t\to+\infty} \sigma_{H(\gamma),H(A)}(t)$$

coincide if one (hence the other) exists. In particular, the limit of spinning depends only on the homotopy class of γ in $\pi_1(S_i^\sharp, c)$.

8. Appendix: Geometric limits

In this section, we analyze the possible geometric limits of restrictions of rational maps to certain forward-invariant open disks, called *invariant strips.*

Invariant strips. Let L be a positive real number. The *standard strip* S is the domain

$$\{x + iy \mid |y| < L\}.$$

The *standard translation* $T : S \to S$ is given by $T(x+iy) = x+iy-1$, i.e., translation by one unit to the left. The *standard central line* $\mathbb{R}$ is conformally distinguished as the unique geodesic (with respect to the hyperbolic metric on S) stabilized by T. In the following, we shall be concerned exclusively with the case when the parameter L is fixed.

Definition 8.1 (Invariant strip)**.** Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map. An *invariant strip of f* is an open, simply-connected set $A \subset \mathbb{P}^1$ such that the restriction $f|A$ is holomorphically conjugate to the standard translation on the standard strip. The *central line* γ of A is the unique hyperbolic geodesic stabilized by $f|A$.

The following lemma follows from a normal families argument and the Snail Lemma [Mil], Lemma 16.2.

Lemma 8.2 (End points of an invariant strip)**.** *An invariant strip A for a rational map f has a unique* forward endpoint *a given by $a = \lim_{n\to+\infty} f^{\circ n}(z)$, where $z \in A$ is arbitrary. The forward endpoint is fixed under f, and the multiplier $f'(a)$ satisfies $f'(a) = 1$ or $0 < |f'(a)| < 1$. Similarly, it has a unique fixed* backward endpoint *u where either $|f'(u)| > 1$ or $f'(u) = 1$. It is possible for the forward and backward ends to coincide; this occurs if and only if $a = u$ is a 1-parabolic fixed point of f.*

Assume, for the remainder of this section, that $f_n \to g$ uniformly, $A_n \subset \mathbb{P}^1$ are invariant strips for f_n with central lines γ_n, and $\overline{\gamma_n} \to \Gamma$ in the Hausdorff topology of compact subsets. We now analyze the structure of Γ.

Denote by a_n, u_n the forward and backward ends of γ_n, respectively.

Theorem 8.3 (Tameness of central lines)**.** *Suppose, taking subsequences if necessary, $a_n \to a_\infty$ and $u_n \to u_\infty$. Then, except when Γ reduces to a single point, there is a nonempty collection $\{S_e\}_{e \in E}$ of disjoint invariant strips for g, indexed by a (finite or) countable set E, with central lines γ_e, such that*

$$\Gamma - \mathrm{Fix}(g) = \bigcup_{e \in E} \gamma_e \quad \textit{and} \quad \Gamma = \bigcup_{e \in E} \overline{\gamma_e}.$$

Proof. Let $\mathcal{H}_n$ denote the set of all holomorphic conjugacies from (T, S) to (f_n, A_n) and put $\mathcal{H} = \bigcup \mathcal{H}_n$. Then $\mathcal{H}$ is a normal family of univalent functions, since the images of S under elements of $\mathcal{H}_n$ must avoid three repelling periodic points x_n, y_n, z_n of f_n, with x_n, y_n, z_n tending to x, y, z, three distinct repelling periodic points of g. We may assume one of these points, say x_n, is the point at infinity. Therefore, any sequence (h_n) with $h_n \in \mathcal{H}_n$ has a subsequence converging uniformly on compact subsets. Such a limit of univalent functions is either univalent or constant. In either case, any limiting function H satisfies $H \circ T = g \circ H$. Moreover, suppose $h_{n_k} : (S, 0) \to (A_{n_k}, y_k)$ is any sequence of conjugacies which converges to a limiting conjugacy $h : (S, 0) \to (S_y, y)$. Then h is nonconstant iff $g(y) \neq y$, and iff $(A_{n_k}, y_k) \to (S_y, y)$ in the Carathéodory topology (see e.g. [McM2], Ch. 5).

Lemma 8.4. *Let $\{U_n\}$ be a sequence of open disks in $\mathbb{C}$ and let $z_n, z_n' \in U_n$. Suppose $(U_n, z_n) \to (U, a)$ and $(U_n, z_n') \to (U', a')$ in the Carathéodory topology. Then either $U = U'$ or $U \cap U' = \emptyset$.*

Note that the basins U_n are the same in both sequences; only the basepoints differ. The proof is omitted.

Proof of Theorem 8.1 (continued). Our goal is the construction of a sequence (S_m, y_m) which contains all possible nonconstant limits of pointed strips (A_n, y_n) where $y_n \in A_n$.

Choose a countable dense subset $\{y_m\}_{m=1}^\infty$ of $\Gamma - \mathrm{Fix}(g)$.

Since $\overline{\gamma_n} \to \Gamma$ and $y_m \in \Gamma$, for each m, we may find a sequence $\{y_{m,n}\}_{n=1}^\infty$ such that $y_{m,n} \in \gamma_n$ and $y_{m,n} \to y_m$ as $n \to \infty$.

Consider now the following array:

$$\begin{array}{cccc}
(A_1, y_{1,1}) & (A_2, y_{1,2}) & (A_3, y_{1,3}) & \dots \\
(A_1, y_{2,1}) & (A_2, y_{2,2}) & (A_3, y_{2,3}) & \dots \\
(A_1, y_{3,1}) & (A_2, y_{3,2}) & (A_3, y_{3,3}) & \dots \\
\dots & & &
\end{array}$$

From row 1, choose a subsequence $\{n(1,k)\}_{k=1}^\infty$ for which the pointed strips $(A_{n(1,k)}, y_{1,n(1,k)})$ converge to a strip (S_1, y_1).

From row 2, using only those columns used in the subsequence just constructed, choose a subsequence $\{n(2,k)\}_{k=1}^\infty$ for which the pointed strips $(A_{n(2,k)},\ y_{2,n(2,k)})$ converge to a strip (S_2, y_2).

Continue this process inductively, each time using only column indices that have already been used. We obtain, for each row m, a sequence $\{n(m,k)\}_{k=1}^\infty$ of column indices such that

$$(A_{n(m,k)},\ y_{m,n(m,k)}) \to (S_m, y_m).$$

This construction has the following property. Suppose $m < m'$, i.e., that row m is above row m'. Then

$$(A_{n(m,k)},\ y_{m,n(m,k)}) \to (S_m, y_m) \tag{8.1}$$

while also

$$(A_{n(m',k)},\ y_{m',n(m',k)}) \to (S_{m'}, y_{m'}). \tag{8.2}$$

However, by construction the sequence $\{n(m',k)\}_{k=1}^{\infty}$ of column indices appearing in row m' is in fact a subsequence of $\{n(m,k)\}_{k=1}^{\infty}$, the indices for row m, so we may substitute $m = m'$ in the expression $n(m,k)$ appearing in equation (8.1) to deduce that

$$(A_{n(m',k)},\ y_{m,n(m',k)}) \to (S_m, y_m). \tag{8.3}$$

By applying Lemma 8.4 to the two converging pointed strips in (8.2) and (8.3), we conclude that either $S_m = S_{m'}$ or else S_m and $S_{m'}$ are disjoint.

Now suppose $y \in \Gamma - \mathrm{Fix}(g)$. We will show that y lies on a central line γ_m of S_m for some m. Choose a subsequence $\{y_{m_i}\}_{i=1}^{\infty}$ with $y_{m_i} \to y$ as $i \to \infty$. Then from the array shown above we may extract a diagonal sequence of pointed basins

$$(A_{n(m_i,k_i)},\ y_{m_i,n(m_i,k_i)}) \to (S_y, y) \tag{8.4}$$

where S_y is an invariant strip containing y.

We will now show that S_y is one of the limiting invariant strips already constructed. Fix a row, say m. By construction, the sequence of column entries occurring in (8.4) is eventually a subsequence of the column entries chosen for row m, i.e., $n(m_i,k_i) \in \{n(m,k)\}_{k=1}^{\infty}$ for all i sufficiently large. Hence, in addition to (8.4) we may obtain, by substituting $n(m,k) = n(m_i,k_i)$ in (8.1) that

$$(A_{n(m_i,k_i)},\ y_{m,n(m_i,k_i)}) \to (S_m, y_m). \tag{8.5}$$

Comparing (8.4) and (8.5) we see that Lemma 8.4 applies again. So, for any fixed m, either $S_m = S_y$ or else they are disjoint.

However, S_y is open and $\{y_m\}$ is dense in $\Gamma - \mathrm{Fix}(g)$, so S_y cannot be disjoint from all of the strips S_m. Hence $y \in S_m$ for some m. Since y is on the central line γ_y for $S_y = S_m$ and central lines are unique, we have $y \in \gamma_m$. We now enumerate the distinct central lines which arise as $\{\gamma_e\}_{e\in E}$. Clearly

$$\Gamma - \mathrm{Fix}(g) = \bigcup_{e\in E} \gamma_e.$$

In case Γ reduces to a single fixed point, E is empty. Otherwise Γ is compact, connected, and contains infinitely many points. Note that $\mathrm{Fix}(g)$ is finite.

We now show $\Gamma = \bigcup_{e\in E} \overline{\gamma_e}$ by contradiction:

Assume $y \in \Gamma \cap \mathrm{Fix}(g)$ and $y \notin \overline{\gamma_e}$ for any e. Choose a small closed disk D centered at y such that D does not contain other fixed points of g, and $\Gamma - D \neq \emptyset$.

There are only finitely many edges γ_i intersecting ∂D (as they belong to disjoint strips). If $y \notin \overline{\gamma_i}$, then the ends of γ_i, as fixed points of g, are outside D. Using the fact that Γ is connected, one gets easily a contradiction. □

The previous result does not exploit the constraints imposed by plane topology. A C^1-smooth arc κ is called a Γ-*transversal* if its ends do not meet Γ, it does not meet the vertices V and it intersects any edge transversally (if not empty). One defines a Γ-*transversal* graph (with finitely many arcs) similarly.

Theorem 8.5 (Stability of transversal)**.** *Let κ be a Γ-transversal graph. Then taking a subsequence if necessary,*

(1) *$\Gamma \cap \kappa$ is finite.*
(2) *There is n', for any $n \geq n'$, $\#\overline{\gamma_n} \cap \kappa = \#\gamma_n \cap \kappa = \#\Gamma \cap \kappa$, $\overline{\gamma_n} \cap \kappa \to \Gamma \cap \kappa$ in the Hausdorff topology. Furthermore, γ_n is transversal to κ, in the same orientation as $\Gamma \cap \kappa$.*

Proof. Each γ_e is a real-analytic arc. By transversality it meets κ in at most finitely many points. Only finitely many edges of Γ can meet κ. So $\Gamma \cap \kappa = \bigcup\{y_i\}$ is finite.

For each i choose $y_{i,n} \in \gamma_n$ converging to y_i. Let $h_{i,n} : S \to A_n$ be a conjugacy from the standard translation T to f_n, mapping 0 to $y_{i,n}$. Taking subsequences if necessary, we may assume, for every i, $h_{i,n}|_S \to l_i|_S$ locally uniformly. As $l_i(0) = y_i$ and y_i is not fixed by g, l_i is univalent. Therefore $l_i(\mathbb{R})$ is an edge of Γ.

Cover now $\Gamma \cap \kappa$ by finitely small disks $\bigcup D_i$, with boundary transversal to both Γ and κ. We may choose D_i small enough so that $D_i \subset l_i(\Delta)$, with Δ some fixed small closed neighborhood of 0. For n large, $\overline{\gamma_n} \cap \kappa \subset \bigcup D_i$ by Hausdorff convergence. Note that, as analytic functions, the derivatives $h'_{i,n}$ converges to l'_i locally uniformly as well, in particular uniformly on Δ. By transversality with κ, $(h_{i,n}(\mathbb{R} \cap \Delta)) \cap \kappa$ is a single point, is contained in D_i, and the intersection is transversal, in the same direction as $l_i(\mathbb{R})$. Now $\gamma_n = h_{i,n}(\mathbb{R})$ cannot cross D_i again, as $D_i \subset h_{i,n}(\Delta)$ and $h_{i,n}$ is univalent. $\square$

Theorem 8.6 (Structure theorem for limits of strips)**.** *Under the hypotheses of Theorem 8.3, the set Γ admits the structure of a connected directed planar graph $\Gamma = (V, E)$ (possibly with edges joining a vertex to itself, and possibly with infinite valence), where the edges point in the direction of the dynamics, subject to the following restrictions:*

(1) *$V = \Gamma \cap \mathrm{Fix}(g)$ is nonempty and finite, and contains a_∞, u_∞.*
(2) *$\Gamma - V = \bigcup_{e \in E} \gamma_e$. Each edge γ_e is the central line of an invariant strip S_e (recall that these strips are disjoint), so*
(3) *the edges are isolated;*
(4) *every vertex is the end of at least one edge*
(5) *a_∞ is either an attracting or a parabolic fixed point of g. In the former case, $u_\infty \neq a_\infty$ and there is a unique edge pointing to a_∞ (and no edges pointing out from a_∞). (There is a symmetric statement for u_∞.)*
(6) *If the forward end of an edge γ_e is not a_∞, then S_e, γ_e are contained in a fixed parabolic basin.*

Proof. Assume that $w \neq a_\infty$ is the forward end of a γ_e. By the Snail Lemma, w is either attracting or 1-parabolic. It cannot be attracting as otherwise by stability some point of γ_n would be in a distinct attracting basin than that of a_n. So w is 1-parabolic.

Assume a_∞ is attracting and there is more than one edge ending at a_∞. Let D be a small disk centered at a_∞, contained in the attracting basin, and whose boundary is a Γ-transversal. By Theorem 8.5, for large n, γ_n intersects transversally ∂D with the same number of points, all pointing inside D. However, γ_n is the central line of a single invariant strip. This is not possible by the stability of the attractors. $\square$

9. Appendix: Analytical lemmas

Lemma 9.1. *Let B be the full basin of attraction of a fixed point a of multiplier λ with $0 < |\lambda| < 1$ for a rational map f, and let $\psi : (B, a) \to (\mathbb{C}, 0)$ be a linearizing function for B. Then:*

(1) *The set of critical points $\mathrm{Crit}(\psi)$ of ψ is $\bigcup_{n \geq 0} f^{-n}(\mathrm{Crit}(f) \cap B)$ and is invariant under f^{-1}.*
(2) *The set of critical values of ψ is $\psi(\mathrm{Crit}(\psi)) = \bigcup_{n\geq 0} \lambda^{-n}\psi(\mathrm{Crit}(f) \cap B)$ and is invariant under multiplication by λ^{-1}.*
(3) *If $B' = B - \psi^{-1}(\psi(\mathrm{Crit}(\psi)))$, then the restriction $\psi|_{B'} : B' \to \psi(B')$ is a covering map onto its image.*

Proof. These properties follow immediately from the existence of a local biholomorphic conjugacy on a neighborhood of the attractor a, and the extension of this conjugacy to B by setting $\psi(z) = \lambda^{-1}\psi(f(z))$. □

Recall that a *holomorphic family of rational maps over X* is a holomorphic map

$$f : X \times \mathbb{P}^1 \to \mathbb{P}^1$$

where X is a connected complex manifold.

Lemma 9.2 (Holomorphic dependence of Königs functions)**.** *Let*

$$f(x, z) = f_x(z) = \lambda(x)z + a_2(x)z^2 + \dots$$

be a holomorphic family of rational maps over X for which $0 < |\lambda(x)| < 1$, and let $g \in X$. Then there exists a neighborhood $\mathcal{U}$ of g and a disk $D \subset \mathbb{C}$ containing the origin such that

(1) *for all $f \in \mathcal{U}$, we have $f(\overline{D}) \subset D$, $f|D$ is univalent, and $D \subset B_f(0)$, the immediate basin of attraction of the origin for f;*
(2) *the map $\psi : \mathcal{U} \times D \to \mathbb{C}$ given by $\psi(x, z) = \psi_{f_x}(z) = z + b_2(x)z^2 + \dots$ (where ψ_{f_x} is the unique linearizing function for f_x normalized to have derivative one at the origin), is holomorphic.*

Furthermore, given any compact subset K of $B_g(0)$, there is a neighborhood $\mathcal{U}$ of g in X, a neighborhood $\mathcal{N}$ of K in $B_g(0)$ containing the origin, and an extension of ψ to a holomorphic map $\psi : \mathcal{U} \times \mathcal{N} \to \mathbb{C}$.

The proof is omitted. There is also a similar statement in the superattracting case, assuming that all $f(x, z)$ have the same order at 0.

Lemma 9.3 (Limits of models)**.** *Suppose f, f_n, g are rational maps with $f_n \to g$ uniformly on $\mathbb{P}^1$. Suppose W is a connected open hyperbolic subset of $\mathbb{P}^1$, with $f^k(W) \subset W$, and suppose $j_n : W \to W_n$ are holomorphic embeddings such that $j_n \circ f^k = f_n^k \circ j_n$. Then, after passing to subsequences, the maps j_n converge uniformly on compact subsets of W to a holomorphic map j such that $j \circ f^k = g^k \circ j$. Moreover, either j is univalent, or $j(W)$ is a single fixed point of g^k.*

Proof. Let x, y, z be three distinct repelling periodic points of g of periods p_x, p_y, p_z. By the stability of repelling periodic points, we may find repelling periodic points x_n, y_n, z_n of f_n of periods p_x, p_y, p_z, respectively, such that $x_n, y_n, z_n \to x, y, z$ as $n \to \infty$. Since $j_n(W) \subset \mathbb{P}^1 - \{x_n, y_n, z_n\}$, we have that $\{j_n\}$ is a normal family. Thus after passing to subsequences we have $j_n \to j$ uniformly on compact subsets. Moreover $j \circ f^k = g^k \circ j$. □

Lemma 9.4 (Partial persistence of the dynamics)**.** *Suppose f, f_n, g are rational maps with $f_n \to g$ uniformly on $\mathbb{P}^1$, and with $f_n = h_n \circ f \circ h_n^{-1}$, where $h_n : \mathbb{P}^1 \to \mathbb{P}^1$ are homeomorphisms. Suppose W is a periodic Fatou component of f that is not a Siegel disk, and suppose that h_n are holomorphic on the grand orbit $\widetilde{W}$ of W.*

Then after passing to subsequences, the maps $j_n = h_n|\widetilde{W}$ converge uniformly on compact subsets to a univalent $j : \widetilde{W} \to \mathbb{P}^1$ satisfying $j \circ f = g \circ j$. If W is attracting or superattracting, then $j(W)$ is a Fatou component of g. In general, $j(W)$ is contained in the Fatou set of g.

Proof. Denote by $\mathcal{L}$ the finite set of critical points and neutral periodic points of f. For any $w \in \mathcal{L}$, assume (by taking subsequences if necessary) $h_n(w) \to w_\infty$. If w is a critical point, so is w_∞, possibly with higher order. If w is eventually periodic, so is w_∞. If w is p-periodic point of multiplier $\lambda \in S^1 - \{1\}$, so is w_∞. If w is of multiplier 1, so is w_∞, but with possibly a smaller period.

On a given component, j is either univalent or constant. By the functional equation, it suffices to show that j is univalent on periodic Fatou components. By considering iterates, it is enough to prove that $j|W$ is nonconstant, where W is a forward-invariant component of f.

Case (a). W contains a superattracting fixed point 0 of order k. Then f_n has a superattracting fixed point at $h_n(0)$ of the same order. Assuming $h_n(0) \to 0$. Then 0 is a superattracting fixed point of g, possibly with a higher order.

Assume by contradiction that $j|W$ is constant. Then $B_g(0)$ has no other critical points than 0, for if $c_g \in B_g(0)$, and $c_g \neq 0$, then there are c_n critical points of f_n contained in $h_n(W)$ and away from $h_n(0)$. There is therefore $c \in W$ such that $j_n(c) = c_n$ for infinitely many n. This contradicts $j|W \equiv 0$.

Therefore $B_g(0)$ is simply connected.

Take a large disk $K \subset B_g(0)$ such that $g(K) \subset \textit{interior}(K)$ and K contains all critical points in $B_g(0)$. This is stable for f_n. Therefore $h_n(W)$ is simply connected. So is W. This then in turn contradicts the fact that j_n must preserve the modulus of $W - L$ for any L compact in W.

Case (b). W contains an attracting but not superattracting fixed point a. Then $a_n = j_n(a)$ is an attractor for f_n with the same multiplier, and after passing to a subsequence we have $a_n \to a$, an attractor for g of the same multiplier. However, W contains a critical point c'. So if j were constant, $j(a) = j(c')$ must be a superattracting fixed point. This is not possible.

Case (c). W is a parabolic basin. Then W contains a critical point c'. If j is constant, then after passing to subsequences we have as $n \to \infty$ that $c'_n = j_n(c') \to c'_g$, a fixed critical point of g. But then for n sufficiently large, the critical point c'_n converges under iteration of f_n to an attracting fixed point a_n, which is not the case.

Case (d). W is a Herman ring. Let K be a leaf of the foliation of the Herman ring. If j is constant, then the spherical diameter of $j_n(K)$ tends to zero. Hence the diameter of a component $D_n = h_n(D)$ of the complement of γ tends to zero as well. However, $D \cap J \neq \emptyset$ and this yields a contradiction to Lemma 3.1 of [Tan]. The same lemma also yields that $j(W)$ is in fact contained in a Herman ring (but not a Siegel disk). $\square$

Remark. A. Chéritat ([Ché], Part II) proved in a more general setting that if W is a Siegel disk (resp. Herman ring) of Brjuno rotation number, then $j|_{\widetilde{W}}$ is univalent and $j(W)$ is again a Siegel disk (resp. Herman ring).

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