CONFORMAL GEOMETRY AND DYNAMICS
An Electronic Journal of the American Mathematical Society
Volume 8, Pages 143–157 (October 14, 2004)
S 1088-4173(04)00116-X

CONFORMAL DEHN SURGERY AND THE SHAPE OF MASKIT'S EMBEDDING

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Abstract. We study the geometric limits of sequences of loxodromic cyclic groups which arise from conformal Dehn surgery. The results are applied to obtain an asymptotic description of the shape of the main cusp of the Maskit embedding of the Teichmüller space of once-punctured tori.

1. Introduction

The Maskit embedding of the Teichmüller space of once-punctured tori is the set $\mathcal{M}$ of parameters $\tau \in \mathbb{H}$ such that the group G_τ generated by the Möbius transformations

$$g = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad h_\tau = \begin{pmatrix} i\tau & i \\ i & 0 \end{pmatrix} \tag{1.1}$$

is a free Kleinian group which has a simply connected invariant component and which uniformizes a punctured torus and a thrice-punctured sphere. $\mathcal{M}$ and the corresponding embeddings of other Teichmüller spaces were introduced in [11], and treated extensively in [10]. The shape of $\mathcal{M}$ has been studied by various authors; see [7], [10], [17], [18], [21].

Computer-generated images in [21] of the boundary of $\mathcal{M}$ suggested that the boundary is a Jordan curve which is cusped at a dense set of points (see Figure 1). These observations have been verified by Minsky [16] and Miyachi [17], [18] using Minsky's classification of punctured torus groups. In this note we show how the shape of the cusp at $2i$ can be computed in a way which gives explicit bounds on the shape. Our computation is based on a study of geometric limits of the cyclic groups generated by h_τ as $\tau \to 2i$ on curves determined by conformal Dehn surgery sequences of [6] (see (1.2)–(1.3) and (5.1)). The basic principle, informally stated, is that $\mathcal{M}$ is also the deformation space of the geometric limits of the groups G_{τ_n} where the parameters τ_n correspond to conformal Dehn surgery sequences. Thus, the 'finer shape' of the cusp at $2i$ is determined by the shape of $\mathcal{M}$.

More precisely, we show that for any $\epsilon > 0$ the boundary of $\mathcal{M}$ is eventually contained between the curves

$$\left\{ 2i \cosh\left(\frac{\pi i}{t + (2+\epsilon)i} \right) : |t| > M \right\} \cup \{\infty\} \tag{1.2}$$

Received by the editors September 10, 2002 and, in revised form, June 17, 2003.
2000 *Mathematics Subject Classification.* Primary 30F40, 30F60.
The author was supported by Academy of Finland grants 42948 and 41320.

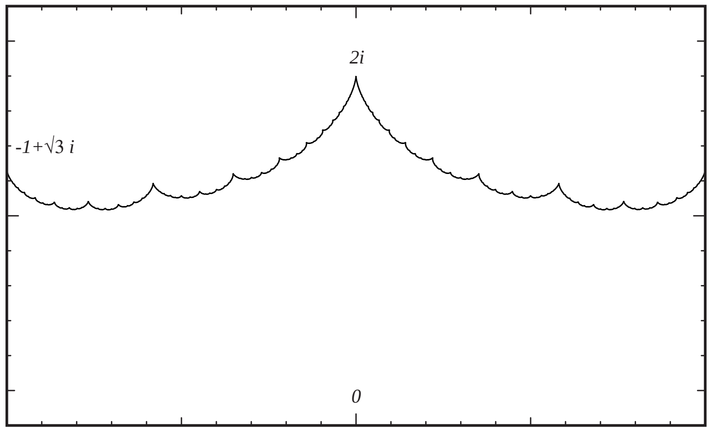

FIGURE 1. The boundary of $\mathcal{M}$ in the strip $|\operatorname{Re}\mu| \leq 1$. This figure was drawn using *kleinian* by David J. Wright.

and

$$\left\{2i\cosh\left(\frac{\pi i}{t+(b-\epsilon)i}\right) : |t| > M\right\} \cup \{\infty\}. \tag{1.3}$$

Here

$$b = \inf_{\mu\in\mathcal{M}} \operatorname{Im}\mu. \tag{1.4}$$

The content of the estimates (1.2) and (1.3) is that the asymptotic shape of the main cusp of $\mathcal{M}$ is determined by the maximal half-plane contained in $\mathcal{M}$ and the minimal half-plane which contains $\mathcal{M}$. We refer to (1.2) and (1.3) as the inside and outside estimates of the cusp. The estimates are proved in Theorems 6.2 and 7.3. We give an elementary lower bound $b \geq 1 + \sqrt{3}/4 \approx 1.433$ in Lemma 7.4. The bounds are illustrated in Figure 2. It is conjectured in [21], Conjecture 6.1, that the minimum b is attained at a point $\mu_J \approx 1.2943 + 1.6169i \in \partial\mathcal{M}$.

Theorem 7.3 is a corollary of Theorem 7.1 where we use the classification of punctured torus groups to prove that any conformal Dehn surgery sequence which corresponds to a parameter $\mu \in \mathbb{H}\setminus\overline{(\mathcal{M}/2+i)}$ is eventually contained in the complement of $\mathcal{M}$. The corresponding result for sequences with parameters $\mu \in \mathcal{M}/2+i$, Proposition 8.3 is weaker. See Section 8 for a more detailed discussion.

The technique used in this paper is a modification of the technique used in [19] to study the boundary of the deformation space of Schottky groups. The method can be applied in certain other situations, such as the main cusps of the Riley slice. The constructions of the fundamental domains in the proofs of the discreteness and structure theorems are different in each case. We have chosen to restrict to the example presented here to illustrate the method in a fairly simple situation.

The paper is organized as follows. In Sections 2 and 3 we review the basic objects in the theory of Kleinian groups and Teichmüller spaces used in the paper and give references to background material. In Section 4 we discuss the problem of determining the shape of the boundary and briefly review earlier work on the subject. In Section 5 we consider the geometric limits of sequences of groups in $\mathcal{M}$

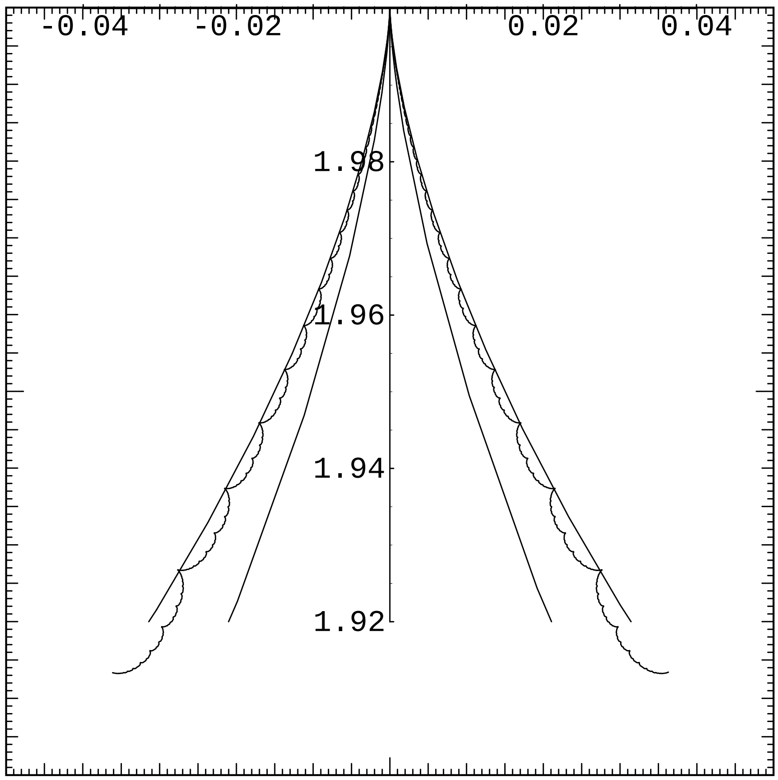

FIGURE 2. An approximation of the boundary of $\mathcal{M}$ at $2i$ by the curves given by Theorem 6.2 and Corollary 7.5. The boundary of $\mathcal{M}$ was drawn using *kleinian* by David J. Wright. The approximating curves were drawn using Mathematica.

which arise from conformal Dehn surgery sequences. The estimates (1.2) and (1.3) are proven in Sections 6 and 7. We conclude the paper with a conjectural picture of the fine structure of the cusp in Section 8.

2. Notation and definitions

We collect here the definitions of some of the basic objects which appear in this paper. We refer to [6], [8], [12], [13], and [20] for background on Kleinian groups and geometric convergence, and to [7] and [10] for more specific results on $\mathcal{M}$.

A Kleinian group is a discrete group of Möbius transformations of $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ or, equivalently, of orientation-preserving isometries of hyperbolic three-space. We identify the group of Möbius transformations with $\mathrm{PSL}_2\,\mathbb{C}$ and write the transformations as 2×2 matrices. Discreteness can be thought of in terms of uniform convergence on compact subsets of $\widehat{\mathbb{C}}$, the discreteness of orbits in $\mathbb{H}^3$, or the natural topology induced from $\mathrm{SL}_2\,\mathbb{C}$.

If g and h are Möbius transformations, their composition is denoted by gh. In our calculations we occasionally simplify the expressions without mention by multiplying the matrices by $-\,\mathrm{id}$, which corresponds to the identity map as a Möbius transformation. The trace $\mathrm{tr}(g)$ of a Möbius transformation g is the trace of one of its representatives in $\mathrm{PSL}_2\,\mathbb{C}$, and thus it is well defined up to multiplication by -1. A Möbius transformation $g \neq \mathrm{id}$ is said to be loxodromic if $\mathrm{tr}\, g \notin [-2, 2]$, parabolic if $\mathrm{tr}\, g = \pm 2$, and elliptic otherwise.

We denote by $\langle g_1, g_2, \ldots, g_n \rangle$ the group generated by the Möbius transformations $g_1, g_2, \ldots, g_n$. The HNN extension of a group G by f is denoted by $G*_f$. For more details on HNN extensions and related combination theorems we refer to [12], Chapter VII.

The set of discontinuity $\Omega(G) \subset \widehat{\mathbb{C}}$ of a Kleinian group G is the set of points which have an open neighborhood which intersects only a finite number of its translates by elements of G. The complement $\Lambda(G) = \widehat{\mathbb{C}} \setminus \Omega(G)$ is the limit set of G.

A set $F \subset \Omega(G)$ is a fundamental set of G if it contains exactly one point from each G-orbit in Ω. An open topological polygon $D \subset \Omega(G)$ whose sides are paired by a set of generating elements of G is a fundamental domain of G if $g(D) \cap D = \emptyset$ for all $g \in G \setminus \{\mathrm{id}\}$ and $\overline{D}$ contains a fundamental set of G. If a fundamental domain D is bounded by line segments and circular arcs it is called a fundamental polygon of G.

We denote the isometric circle of a Möbius transformation γ by $\mathrm{Isom}(\gamma)$. Let G be a Kleinian group such that $\infty \in \Omega(G)$. Let $\mathrm{ext}(g)$ denote the unbounded component of $\widehat{\mathbb{C}} \setminus \mathrm{Isom}(g)$ for $g \in G$. The isometric fundamental polygon (or Ford domain) of G is

$$\mathrm{Ford}(G) = \bigcap_{g \in G \setminus \{\mathrm{id}\}} \mathrm{ext}(g). \tag{2.1}$$

$\mathrm{Ford}(G)$ is a fundamental domain of G; see [12], II.H.3. We will study the isometric fundamental polygons of loxodromic cyclic groups and of parabolic groups of ranks 1 and 2. The isometric fundamental polygon of a loxodromic cyclic group has 2, 4 or 6 sides, and the combinatorics of these sides determine a tessellation of the trace plane $\mathbb{C} \setminus [-2, 2]$ by triangles which are bounded by real algebraic curves with endpoints in $[-2, 2]$; see [3] and [5].

Let $X \subset \widehat{\mathbb{C}}$. The subgroup

$$\mathrm{Stab}(X) = \mathrm{Stab}_G(X) = \{g \in G : g(X) = X\} \tag{2.2}$$

is the stabilizer of X in G. A component Ω_0 of $\Omega(G)$ such that $\mathrm{Stab}_G(\Omega_0) = G$ is called an invariant component of G. A Kleinian group G is called a torsion-free terminal b-group of type (g, n) if it has a simply connected invariant component $\Omega_0(G) \subset \Omega(G)$ such that Ω_0/G is a Riemann surface of genus g with n punctures, and if $(\Omega \setminus \Omega_0)/G$ is a collection of $2g - 2 + n$ thrice punctured spheres.

A sequence G_n of Kleinian groups converges geometrically to G if they converge in the Chabauty topology of closed subgroups of $\mathrm{PSL}_2\,\mathbb{C}$. Geometric convergence can be characterized in terms of the convergence of the quotient three-manifolds and the Dirichlet polyhedra of the groups G_n. For more details we refer to [6], [8], [13], and [15]. Our main references for results on geometric convergence are [6], and Chapter 7 of [13].

3. Deformation spaces of terminal b-groups

The deformation space or Teichmüller space of a Kleinian group G is

$$\mathcal{T}(G) = \left\{ w\, \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \text{ quasiconformal: } w \circ g \circ w^{-1} \in \mathrm{PSL}_2\,\mathbb{C} \text{ for all } g \in G \right\} / \sim, \tag{3.1}$$

where $w_1 \sim w_2$ if there is a Möbius transformation $A \in \mathrm{PSL}_2\,\mathbb{C}$ such that

$$w_1 \circ g \circ w_1^{-1} = A \circ w_2 \circ g \circ w_2^{-1} \circ A^{-1} \quad \forall g \in G. \tag{3.2}$$

If G is finitely generated, then $\mathcal{T}(G)$ is a finite-dimensional complex manifold. For more details see [1] or [10].

If G is a torsion-free terminal b-group of type (g, n), then $\mathcal{T}(G)$ is naturally isomorphic to the Teichmüller space of Riemann surfaces with finite area, genus g, and n punctures. The complex dimension of this space is $3g - 3 + n$. We will be interested in deformation spaces of torsion-free terminal b-groups of types $(1, 1)$ and $(0, 4)$, and of groups without an invariant component such that their deformation space is naturally identified with $\mathcal{M}$ (see (5.2), (5.4), and Proposition 5.3). Such groups appear as geometric limits of conformal Dehn surgery sequences in $\mathcal{M}$ in Section 5. All deformation spaces treated in this note are one-dimensional. We refer to [2], [9], and [11] for background on Teichmüller spaces of Riemann surfaces and Kleinian groups.

We will use horocyclic coordinates introduced in [10] to embed these deformation spaces in the upper half plane $\mathbb{H}$. The following facts are proven in Sections 6 and 8 of [10]: The deformation space of torsion-free terminal b-groups of type $(1, 1)$ is complex analytically equivalent to the set $\mathcal{M}$ of parameters $\tau \in \mathbb{H}$ such that the group G_τ generated by the Möbius transformations

$$g = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad h_\tau = \begin{pmatrix} i\tau & i \\ i & 0 \end{pmatrix} \tag{3.3}$$

is a torsion-free terminal b-group of type $(1, 1)$. Similarly, the deformation space of torsion-free terminal b-groups of type $(0, 4)$ is complex analytically equivalent to the set $\mathcal{M}'$ of parameters $\alpha \in \mathbb{H}$ such that the group G'_α generated by g,

$$b = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \text{ and } b_\alpha = \begin{pmatrix} 1 - 2\alpha & 2\alpha^2 \\ -2 & 1 + 2\alpha \end{pmatrix} \tag{3.4}$$

is a torsion-free terminal b-group of signature $(0, 4)$. $\mathcal{M}$ and $\mathcal{M}'$ are called the Maskit embeddings of the respective Teichmüller spaces.

It is also shown in [10] that for a fixed $\alpha \in \mathbb{H}$ the normalizers in $\mathrm{PSL}_2\,\mathbb{C}$ of $G_{2\alpha}$ and G'_α are the same group. In particular, this implies that $\mathcal{M} = 2\mathcal{M}'$ and that $G_{2\alpha}$ is discrete if and only if G'_α is discrete (see [10], Theorem 8.6). This connection will be useful in Section 7.

4. The shape of cusps

A point $\tau_0 \in \partial\mathcal{M}$ is called a cusp if a group element which corresponds to a simple closed curve on the punctured torus for any $\tau \in \mathcal{M}$ is parabolic at τ_0. Here the groups G_τ and G_{τ_0} are identified by the natural homomorphism which maps g to g and h_τ to h_{τ_0}. Equivalently, $\tau_o \in \partial\mathcal{M}$ is a cusp if G_{τ_0} has three conjugacy classes of parabolic elements. For more details we refer the reader to [7].

It was conjectured in [21] that

(1) the boundary of $\mathcal{M}$ is a Jordan curve,
(2) cusps are dense in $\partial\mathcal{M}$, and
(3) the boundary curve is cusp-shaped at each cusp.

Conjectures (1) and (2) are proven in [16], pp. 621–622. In the case of quasifuchsian groups, the result analogous to (2) is proved in [14]. Conjecture (3) on the shape of the cusps was proven for a number of cusps, including the point $2i \in \partial\mathcal{M}$ which is studied here, in [17], and for all cusps in [18]. The proof uses the pivot theorem of [16] and the fact that the trace function of the group element which defines the cusp has nonzero derivative at the cusp. The corresponding result is proven for Schottky groups whose Ford domains satisfy a technical condition in [19].

It is shown in [18], Theorem 1, that there are constants $a_1 > a_2 > 0$ such that close to $2i$ the boundary of $\mathcal{M}$ is contained between the curves defined by the equations

$$(\operatorname{Im} z - 2)^2 = a_j (\operatorname{Re} z)^3, \quad j = 1, 2. \tag{4.1}$$

This is a more precise statement of (3) for the cusp $2i \in \partial\mathcal{M}$. In Sections 6 and 7 we obtain an asymptotic description of $\mathcal{M}$ close to $2i$ which enables us to prove this statement using geometric limits (Theorems 6.2 and 7.3) in a slightly different form with explicit bounds on the constants which determine the shape of the cusp (see (1.2) and (1.3)). Our estimate is equivalent to (4.1): The curve

$$\left\{ 2i \cosh\left(\frac{\pi i}{t + Ci}\right) : t \in \mathbb{R} \right\} \cup \{2i\} \tag{4.2}$$

is the image under $z \mapsto 2i \cosh z$ of the circle of radius $\pi / \operatorname{Im} \mu$ tangent to the origin in the right half plane. This map has a simple zero at the origin, which corresponds to $t = \infty$. This implies that the curve (4.2) satisfies (4.1) (see [17], Section 2).

5. Conformal Dehn surgery and geometric limits

Let $\mu \in \mathbb{H}$. Consider a sequence of parameters

$$\tau_n(\mu) = 2i \cosh(\pi i / (\mu + n)). \tag{5.1}$$

Jørgensen and Marden [6] showed that the geometric limit of the sequence $\langle h_{\tau_n}(\mu) \rangle$ as $n \to \infty$ is the rank-two parabolic group

$$\Gamma_\mu = \langle h_{2i}, f_\mu \rangle \tag{5.2}$$

where

$$f_\mu = \begin{pmatrix} 1 - \mu & \mu i \\ \mu i & 1 + \mu \end{pmatrix}. \tag{5.3}$$

This follows from the calculations in Section 5 of [6] by considering the modulus of the rank two parabolic group generated by $h_{2i} = f_{-1}$ and f_μ. The sequences $\langle h_{\tau_n}(\mu) \rangle$ are called conformal Dehn surgery sequences.

Let us denote

$$H_\mu = \langle \Gamma_\mu, g \rangle. \tag{5.4}$$

In general, geometric limits of Kleinian groups can be quite complicated. Our restriction to conformal Dehn surgery sequences simplifies the situation considerably. Our inside estimate of the cusp $2i \in \partial\mathcal{M}$ is based on the following Proposition, which is an immediate consequence of Proposition 4.7 (i) of [6].

Proposition 5.1. *Let $\mu \in \mathbb{H}$ such that*

(1) *H_μ is discrete,*
(2) *$H_\mu = \Gamma_\mu * \langle g \rangle$, and*
(3) *if $h \in H_\mu$ is parabolic, then h is conjugate in H_μ to an element of Γ_μ or $\langle g \rangle$.*

If there is an increasing sequence $k(n) \in \mathbb{N}$ such that $\tau_{k(n)}(\mu) \in \mathcal{M}$ for all n, then $G_{\tau_{k(n)}(\mu)}$ converges geometrically to H_μ

It is fairly easy to find a subset of $\mathbb{H}$ where Proposition 5.1 can be applied. Lemma 5.2 provides the largest possible half-space for our application, and Proposition 5.3 shows how $\mathcal{M}$ can be used as a parameter space for deformations of H_μ.

Lemma 5.2. *μ satisfies conditions* (1)–(3) *of Proposition 5.1 if* $\operatorname{Im} \mu > 2$.

Proof. If $\operatorname{Im} \mu > 2$, then the isometric circles of all transformations $\gamma \in \Gamma_\mu \setminus \langle g \rangle$ are contained in the disk of radius $1/|\mu|$ centered at i. Klein's combination theorem implies the result (see Figure 4). □

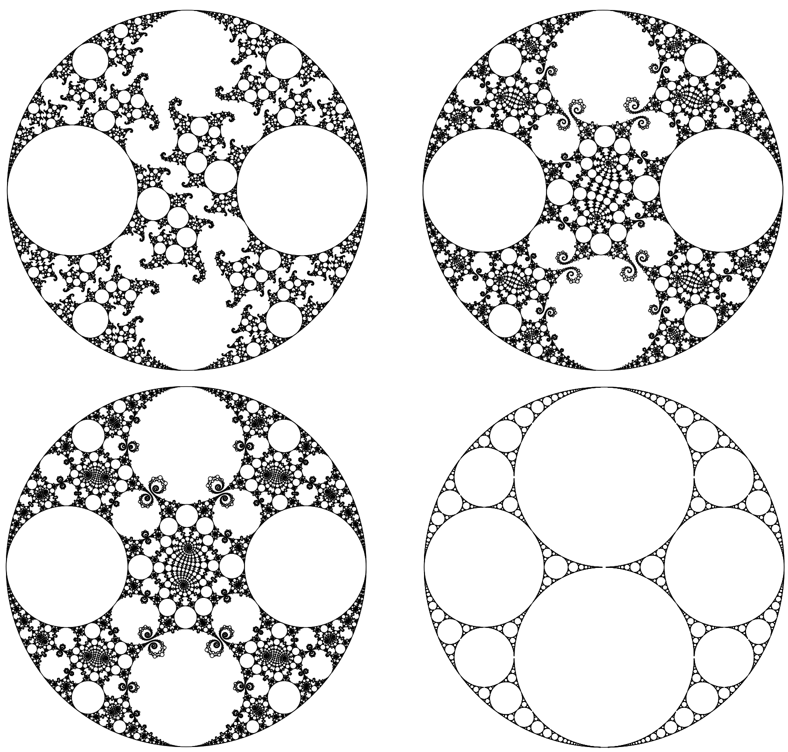

FIGURE 3. The limit sets of the groups $G_{\tau_n(2i)}$, for parameters $\tau_5(2i) \approx -0.225153 + 1.754094\ i$, $\tau_{10}(2i) \approx -0.035969 + 1.912927\ i$ and $\tau_{15}(2i) \approx -0.011214 + 1.958540\, i$ in the conformal Dehn surgery sequence $\tau_n(2i) = 2i \cosh(\pi i/(2i+n))$. The fourth image shows the limit set of the algebraic limit G_{2i} . This figure was produced using *lim* by Curtis T. McMullen.

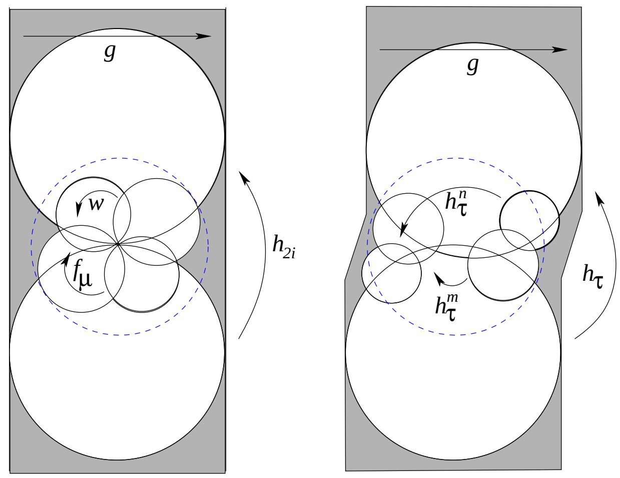

FIGURE 4. (a) The fundamental domain D_μ (the shaded region) of H_μ for a typical value μ with $\operatorname{Im}\mu > 2$. H_μ is generated by h_{2i} and f_μ, and $w \in H_\mu$ depends on μ. (b) A fundamental domain of $G_{\tau_n(\mu)}$ for $n \in \mathbb{N}$ 'big enough'.

Figure 3 illustrates the difference between the geometric limit of a conformal Dehn surgery sequence and the group G_{2i}. We will use the deformation space of H_μ to study the shape of $\mathcal{M}$.

Let $\operatorname{Im}\mu > 2$. Lemma 5.2 implies that H_μ is discrete, $\Omega(H_\mu) \neq \emptyset$, and that

$$D_\mu = \operatorname{Ford}\Gamma_\mu \cap \{z \in \mathbb{C} : -1 < \operatorname{Im} z < 1\} \tag{5.5}$$

is a fundamental domain of H_μ. An inspection of D_μ shows that $\Omega(H_\mu)/H_\mu$ consists of a punctured torus and a thrice-punctured sphere. Maskit's second combination theorem ([12], Theorem VII.E.5) implies that the stabilizer of any component of $\Omega(H_\mu)$ which projects to the punctured torus is conjugate to the subgroup

$$H^0_\mu = \langle h_{2i}, g h_{2i} g^{-1}, g f_\mu\rangle = \langle h_{2i}, g h_{2i} g^{-1}\rangle *_{g f_\mu^{-1}} \tag{5.6}$$

(see Figure 5). The subgroup generated by h_{2i} and $gh_{2i}g^{-1}$ is a torsion-free triangle group which acts discontinuously in the complement of the circle of radius 1 centered at $1 + i$, and H^0_μ is a torsion-free terminal regular b-group of signature (1,1).

Maskit's second combination theorem also implies that $H_\mu = H^0_\mu *_g$: The half planes $B_1 = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ and $B_2 = \{z \in \mathbb{C} : \operatorname{Re} z > 2\}$ are precisely invariant under $\langle h_{2i}\rangle$ and $\langle g h_{2i} g^{-1}\rangle$, and g maps the interior of B_1 onto the exterior of B_2. In particular, if H^0_μ is a torsion-free terminal regular b-group of signature (1,1), then H_μ is as in Proposition 5.1.

The parameter

$$-i \operatorname{tr} g f_\mu = -i \operatorname{tr} \begin{pmatrix} 1 - \mu + 2\mu i & 1 + 2\mu + \mu i \\ \mu i & 1 + \mu i \end{pmatrix} = 2\mu - 2i \tag{5.7}$$

is the horocyclic coordinate of the Teichmüller space of once-punctured tori, that is, H^0_μ is conjugate to the group $G_{2\mu-2i} = \langle g, h_{2\mu-2i}\rangle$. This implies the following proposition.

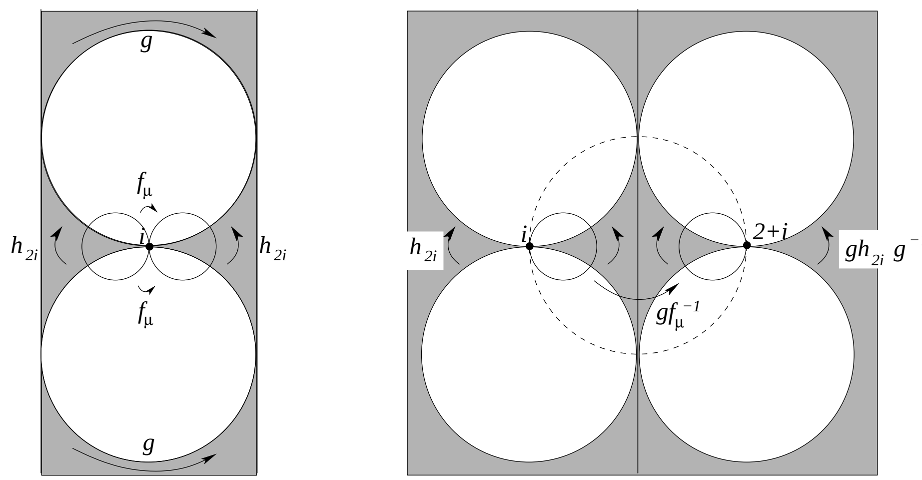

FIGURE 5. (a) A fundamental domain of H_μ. (b) Fundamental domains of the triangle group $\langle h_{2i}, gh_{2i}g^{-1}\rangle$ (the shaded region) and the component stabilizer $H_\mu^0 < H_\mu$ (the shaded region with the two disks contained in the invariant disk of $\langle h_{2i}, gh_{2i}g^{-1}\rangle$ and tangent to i and $2+i$ deleted).

Proposition 5.3. *H_μ is discrete if and only if $G_{2\mu-2i}$ is discrete. The deformation space of H_{μ_0} for any μ_0 such that $2\mu_0 - 2i \in \mathcal{M}$ is complex analytically isomorphic with the set*

$$\mathcal{M}'' = \mathcal{M}/2 + i = \mathcal{M}' + i. \tag{5.8}$$

6. THE INSIDE ESTIMATE

Geometric convergence implies (see [6], Section 4.4) that the isometric fundamental polygons of the loxodromic cyclic groups $\langle h_{\tau_n}(\mu)\rangle$ converge to the isometric fundamental polygon of the rank 2 parabolic group $\Gamma_\mu = \langle h_{2i}, f_\mu\rangle$ in the sense of Hausdorff distance. We use this observation and Klein's combination theorem to prove the inside estimate (1.2).

Proposition 6.1. *Let $\mu \in \mathbb{H}$ such that $\operatorname{Im}\mu > 2$. There is some $N \in \mathbb{N}$ such that $\tau_n(\mu) \in \mathcal{M}$ for all $n \geq N$.*

Proof. Let D_μ be the fundamental domain of H_μ given by (5.5). Let $\tau_n = \tau_{n(\mu)}$. We can assume $0 \leq \operatorname{Re}\tau \leq 1$ and $\sqrt{3} < \operatorname{Im}\tau$ because $\tau_n \to 2i$. The isometric circle of h_{τ_n} is the unit circle and the isometric circle of $h_{\tau_n}^{-1}$ is the circle of radius 1 centered at τ_n. Their union has width 2 for all τ_n in this range.

Assume that there is an increasing sequence $k(j)$ of integers such that the width of the isometric fundamental polygon of $\langle h_{\tau_{k(j)}}\rangle$ is greater than 2 for all j. Geometric convergence of the cyclic groups generated by $h_{\tau_{k(j)}}$ to H_μ implies that the width of

$$\operatorname{Ford}\left(\langle h_{\tau_{k(j)}}\rangle\right) \cap \{z \in \mathbb{C} : |\operatorname{Im} z| > \delta \text{ and } |\operatorname{Im} z - 2| > \delta\} \tag{6.1}$$

is eventually less than 2. Thus, the widest parts of Ford $\left(\langle h_{\tau_{k(j)}}\rangle\right)$ must be in the strips

$$\{z \in \mathbb{C} : |\operatorname{Im} z| < \delta \text{ or } |\operatorname{Im} z - 2| < \delta\} \tag{6.2}$$

for large values of j. This implies that there is a sequence of integers $m(j)$ such that

$$\operatorname{Isom}\left(h_{\tau_{k(j)}}^{m(j)}\right) \cup \operatorname{Isom}\left(h_{\tau_{k(j)}}\right) \tag{6.3}$$

is wider than 2 for all j and that $\operatorname{Isom}\left(h_{\tau_{k(j)}}^{m(j)}\right)$ must converge to $\operatorname{Isom}(h_{2i})$ and $\operatorname{Isom}\left(h_{\tau_{k(j)}}^{-m(j)}\right)$ must converge to $\operatorname{Isom}\left(h_{2i}^{-1}\right)$. Furthermore, the fixed points of h_{τ_n} converge to i. An elementary calculation shows that these facts imply $h_{\tau_{k(j)}}^{m(j)} \to h_{2i}$. But this implies $h_{\tau_{k(j)}}^{m(j)-1}$ converges to the identity. The proof of Proposition 5.3 in [6] shows that this cannot happen.

Thus, there is an integer $N(\mu) \in \mathbb{N}$ such that for all $n \geq N(\mu)$ there is a fundamental domain A_n of $\langle g\rangle$ such that $A_n \cap \operatorname{Ford}\langle h_{\tau_{k(j)}}(\mu)\rangle$ is a fundamental domain of $H_{\tau_n(\mu)}$. This proves the claim. □

Theorem 6.2. *Let $y > 2$. There is $M \geq 0$ such that*

$$\left\{2i\cosh\left(\frac{\pi i}{t+iy}\right) : |t| \geq M\right\} \subset \mathcal{M}. \tag{6.4}$$

Proof. The arc

$$\left\{2i\cosh\left(\frac{\pi i}{t+iy}\right) : t \in \mathbb{R}\right\} \subset \mathcal{M} \tag{6.5}$$

is the union of the images under Dehn surgery of the compact interval

$$\left\{2i\cosh\left(\frac{\pi i}{t+iy}\right) : t \in [0,1]\right\} \subset \mathcal{M}. \tag{6.6}$$

Let $t \in \mathbb{R}$. Proposition 6.1 implies that there is an integer $N(t)$ such that $\tau_n(t+iy) \in \mathbb{M}$ for all $n \geq N(t)$. The isometric circles vary continuously on the parameter t. This implies that the sets

$$U_N = \{t \in \mathbb{R} : \tau_n(t+iy) \in \mathcal{M} \text{ for all } n \geq N\} \tag{6.7}$$

are open. The compactness of the interval $[0,1]$ implies the result (6.4) for positive t. The symmetry of $\mathcal{M}$ under the map $\tau \mapsto -\overline{\tau}$ ([7], Proposition 2.5) gives (6.4). □

7. The outside estimate

In this section we study conformal Dehn surgery sequences which correspond to parameters $\mu \in \mathbb{H} \setminus \overline{\mathcal{M}''}$. We use Minsky's results on the classification of punctured torus groups [16] to obtain a pointwise estimate, Theorem 7.1, which is better than the analogous inside estimate, Proposition 6.1, in the sense that it applies to all parameters $\mu \in \mathbb{H} \setminus \overline{\mathcal{M}''}$. The outside estimate by the cardioid curve, Theorem 7.3, is then deduced using a proof similar to that of Theorem 6.2. A numerical estimate of the shape of the cardioid, Corollary 7.5 is proved using an elementary discreteness argument based on the Shimizu–Leutbecher lemma.

Theorem 7.1. *Let $\mu \in \mathbb{H} \setminus \overline{\mathcal{M}''}$. There is $M \in \mathbb{N}$ such that*

$$\tau_n(\mu) = 2i \cosh\left(\frac{\pi i}{n+\mu}\right) \in \mathbb{H} \setminus \overline{\mathcal{M}} \tag{7.1}$$

for all $n \geq M$.

Proof. Assume there is a subsequence $\tau_{n(k)}(\mu) \in \overline{\mathcal{M}}$. Proposition 3.8 of [6] implies that there is a geometrically convergent subsequence, which we denote $G_{\tau_{n(k)}(\mu)}$. The geometric limit G of the sequence $G_{\tau_{n(k)}(\mu)}$ is discrete and torsion-free by [6], Lemma 3.2 and [13], Theorem 7.1.

The normalization (1.1) of the groups G_τ implies that the lower half-plane $\mathbb{H}^*$ is contained in $\Omega(G_{\tau_{n(k}})$ for all $k \in \mathbb{N}$. Proposition 4.7 (ii) of [6] implies that $\mathbb{H}^* \subset \Omega(G)$. Thus, the set of discontinuity of the subgroup $H^0_\mu = \langle h_{2i}, g f_\mu^{-1}\rangle < G$ is nonempty. Furthermore, H^0_μ is a two-generator group, and thus it is free by [4]. This implies that H_μ is a punctured torus group in the sense of [16]. The classification of punctured torus groups implies (see [16], pp. 621–622) that H_μ is in the closure of the Maskit embedding. Thus, $\mu \in \overline{\mathcal{M}''}$. □

Remark 7.2. The results of [4] and [16] used in the proof of Theorem 7.1 are specific to groups with two generators.

Let

$$b = \inf_{\mu \in \mathcal{M}} \operatorname{Im} \mu. \tag{7.2}$$

Theorem 7.3. *Let $y < b$. There is $M \geq 0$ such that*

$$\left\{ 2i \cosh\left(\frac{\pi i}{t+iy}\right) : |t| \geq M \right\} \subset \mathbb{C} \setminus \overline{\mathcal{M}}. \tag{7.3}$$

Proof. Let $t \in \mathbb{R}$. Theorem 7.1 implies that here is an integer $N(t)$ such that $\tau_n(t+iy) \in \mathbb{H} \setminus \overline{\mathcal{M}}$ for all $n \geq N(t)$. Furthermore, for any $t \in \mathbb{R}$ there is an open neighborhood U of t such that there is an integer M such that $G_{\tau_n(s+iy)} \in \mathbb{C} \setminus \overline{\mathcal{M}}$ for all $n \geq M$ and all $s \in U$. For if no such U exists, then there is a geometrically convergent sequence of discrete groups $G_{\tau_k(s_k+iy)}$, $s_k \to t$, with nondiscrete geometric limit H_{t+iy}. We can assume $t \in [0,1]$ as in the proof of Theorem 6.2. The compactness of $[0,1]$ and the symmetry under $\tau \mapsto -\overline{\tau}$ of $\mathcal{M}$ imply the result. □

It is conjectured in [21], Conjecture 6.1, that $b \approx 1.617$. In Lemma 7.4 we give an elementary proof of the estimate $b \geq 1 + \sqrt{3}/4 \approx 1.433$. It appears to be most convenient to do the calculations in $\mathcal{M}'$, the Maskit embedding of the Teichmüller space of four times punctured spheres, and use the isomorphism of (5.8) to translate the estimate to our situation. We will also use the fact that a group is discrete if and only if its normalizer in $\mathrm{PSL}_2\,\mathbb{C}$ is discrete ([12], Proposition V.E.10).

Lemma 7.4. *G'_α is not discrete if $0 < \operatorname{Im} \alpha < \sqrt{3}/4$.*

Proof. Let

$$k_2 = \begin{pmatrix} 0 & -1/\sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}. \tag{7.4}$$

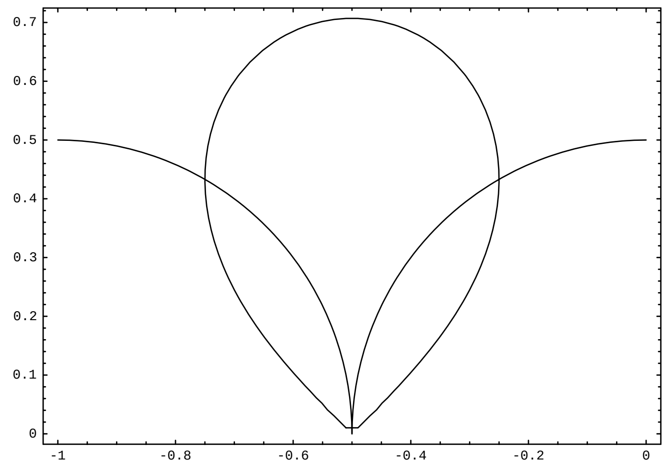

FIGURE 6. Lemma 7.4 implies that the group G_α is not discrete for parameters $\alpha \in \mathbb{H}$ which are contained below the curves in the figure or its translates by integers.

Now

$$\widetilde{b} = kbk^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{7.5}$$

and

$$\widetilde{b}_\alpha = kb_\alpha k^{-1} = \begin{pmatrix} 1+2\alpha & 1 \\ -4\alpha^2 & 1-2\alpha \end{pmatrix}. \tag{7.6}$$

The Shimizu-Leutbecher Lemma ([12], II.C.5) applied to the subgroup generated by $\widetilde{b}$ and $\widetilde{b}_\alpha$ shows that G'_α is not discrete if $0 < |\alpha| < 1/2$.

The transformation

$$w = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} b_\alpha b^{-1} = \begin{pmatrix} 1+2\alpha+4\alpha^2 & 1+2\alpha+2\alpha^2 \\ 4\alpha & 1+2\alpha \end{pmatrix} \tag{7.7}$$

is in the normalizer of G'_α (see Sections 6.1 and 8.1 of [10]). Now,

$$\widetilde{w} = kwk^{-1} = \begin{pmatrix} 1+2\alpha & -2\alpha \\ -2(1+2\alpha+2\alpha^2) & 1+2\alpha+4\alpha^2 \end{pmatrix}. \tag{7.8}$$

The Shimizu-Leutbecher Lemma applied to the subgroup generated by $\widetilde{b}$ and $\widetilde{w}$ implies that G'_α is not discrete if

$$0 < |2(1+2\alpha+2\alpha^2)| < 1. \tag{7.9}$$

An elementary calculation, combined with the fact that $G'_{\alpha+1} = G'_\alpha$ for all α proves the claim. See Figure 6. □

Corollary 7.5. *Let $y < 1 + \sqrt{3}/4$. There is $M \geq 0$ such that*

$$\left\{ 2i \cosh\left(\frac{\pi i}{t+iy}\right) : |t| \geq M \right\} \subset \mathbb{C} \setminus \overline{\mathcal{M}}. \tag{7.10}$$

Proof. This follows from Lemma 7.4 and Proposition 5.3. □

8. The fine structure of the cusp

In this section we present a conjectural picture of the asymptotic structure of the cusp $2i \in \partial\mathcal{M}$ based on the results of the previous sections. The shape of any other cusp of $\mathcal{M}$ should have a similar description.

Proposition 6.1 and Theorem 7.1 lead to the conjecture that the shape of the cusp at $2i$ is 'asymptotically determined by the image of $\mathcal{M}$ under the map $z \mapsto 2i\cosh(\pi i/z)$'. A more precise formulation of this conjecture is:

Conjecture 8.1. *$2\mu - 2i \in \mathcal{M}$ if and only if there is $N(\mu) \in \mathbb{N}$ such that $\tau_n(\mu) \in \mathcal{M}$ for all $n \geq N(\mu)$.*

One would like to prove the remaining parts of Conjecture 8.1 in the same manner as Proposition 6.1. However, we cannot use isometric fundamental polygons for all $\mu \in \mathcal{M}$ with $\operatorname{Im}\mu \leq 2$. This means that in order to prove Conjecture 8.1 using the same strategy we would need to have control over the convergence of more general fundamental domains of cyclic groups in a geometrically convergent sequence. The subset of $\mathcal{M}$ where the method of Proposition 6.1 can be applied is somewhat larger than in Proposition 6.1:

Proposition 8.2. *Let $\mu \in \mathbb{H}$. If $\operatorname{Ford}(\Gamma_\mu) \setminus \{-1, 1, -1+2i, 1+2i\}$ is contained in an open strip of width 2, then there is $N(\mu) \in \mathbb{N}$ such that $\tau_n(\mu) \in \mathcal{M}$ for all $n \geq N(\mu)$.*

Corollary 8.3. *If*

$$\operatorname{Im}\mu + |\mu + k| > |\mu + k|^2 \text{ for all } k \in \mathbb{Z}, \tag{8.1}$$

then there is $N(\mu) \in \mathbb{N}$ such that $\tau_n(\mu) \in \mathcal{M}$ for all $n \geq N(\mu)$.

Proof. The condition (8.1) implies that the boundary of the isometric fundamental domain of H_μ is contained in the strip $\{z \in \mathbb{C} : -1 < \operatorname{Re} z < 1\}$ with the exception of the points ± 1 and $\pm 1 + 2i$. □

It does not seem likely that there should be a uniform constant $N(\mu) = N$ which would work for all μ in Conjecture 8.1. Compactness and continuity arguments as in Theorem 6.2 lead to the following, more reasonable conjecture.

Conjecture 8.4. *Let $f(\sigma) = 2i\cosh(\pi i/(\sigma/2 + i))$. For any $K \subset \mathcal{M}$ such that all points of K are at least a distance $\delta > 0$ from $\partial\mathcal{M}$ there is $M > 0$ such that $f(K \cap \{z \in \mathbb{C} : |\operatorname{Re} z| > M\}) \subset \mathcal{M}$.*

Let G be a Kleinian group and F a subgroup of G. The limit set $\Lambda(G)$ of G equals the closure in $\mathbb{C} \cup \{\infty\}$ of the union of the translates of $\Lambda(F)$ by elements of G. This observation is used to produce Figure 3 which shows parts of the orbits of the torsion-free triangle group $\langle g, hgh^{-1}\rangle$. Figure 7 shows parts of the orbits of the limit set of a Fuchsian subgroup for groups in a conformal Dehn surgery sequence which corresponds to a parameter $\mu \in \mathcal{M}''$ which is not contained in the subset covered by Proposition 8.3. The fact that none of the circles in the images overlap provides 'experimental evidence' that Conjecture 8.1 should hold.

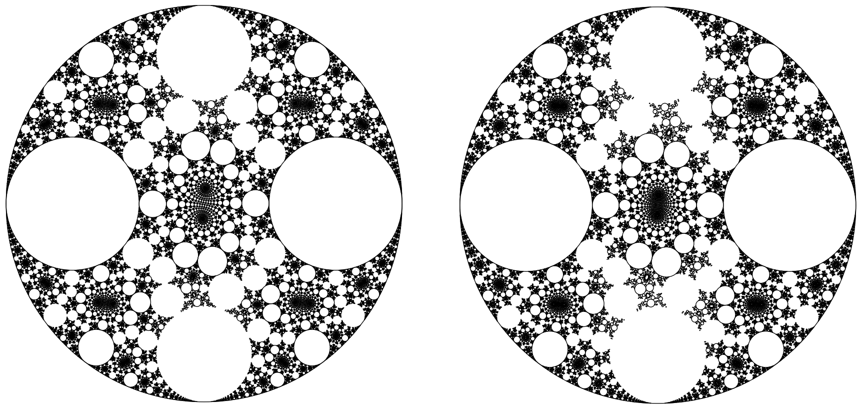

Figure 7. The limit sets of the groups $G_{\tau_n(2i)}$, which correspond to parameters $\tau_{20}(\mu) \approx -0.004291 + 1.976666\ i$ and $\tau_{30}(\mu) \approx -0.001305 + 1.989373\ i$, where $\mu = \tau/2 + i$, and $\tau \approx 0.582943 + 1.707985\ i$ is the unique parameter in $\mathcal{M}$ for which $\operatorname{tr} g^{-1}h_\tau^3 = 2.1$. Clearly, μ does not satisfy (8.1) but the computation of the orbit of $\mathbb{R}\cup\{\infty\} = \Lambda(\langle h_{2i}, gh_{2i}g^{-1}\rangle)$ up to word lengths of about 300 for $n = 20$ and 500 for $n = 30$ suggests that the groups are discrete. This figure was produced using *lim* by Curtis T. McMullen.

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