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THE LOCATION OF CRITICAL POINTS OF FINITE BLASCHKE PRODUCTS

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Abstract. A theorem of Bôcher and Grace states that the critical points of a cubic polynomial are the foci of an ellipse tangent to the sides of the triangle joining the zeros. A more general result of Siebert and others states that the critical points of a polynomial of degree N are the algebraic foci of a curve of class $N-1$ which is tangent to the lines joining pairs of zeroes. We prove the analogous results for hyperbolic polynomials, that is, for Blaschke products with N roots in the unit disc.

1. Introduction

The well-known theorem of Lucas [5] states that the critical points of a polynomial in the complex plane lie within or on the convex hull of the zeros. But more is known about the critical points. A remarkable theorem of Bôcher [1] and Grace [3] relates the position of the zeroes of a cubic polynomial in the plane to the position of the critical points. It states:

Theorem 1.1. *The critical points of a cubic polynomial $P(Z)$ are the foci of an ellipse E which is tangent to the midpoints of the three line segments joining the roots of $P(Z)$. More generally, the zeroes of the function $F(Z) = \sum_1^3 m_i(Z-Z_i)^{-1}$ are the foci of the conic that touches the line segments (Z_1, Z_2), (Z_2, Z_3), and (Z_3, Z_1) in points which divide these segments in the ratios $m_1 : m_2$, $m_2 : m_3$, and $m_3 : m_1$, respectively.*

The first assertion follows from the second when $F(Z)$ is chosen to be the logarithmic derivative $\frac{P'(Z)}{P(Z)}$, in which case $m_i = 1,\ 1 \le i \le 3$. The theorem is actually a special case of a theorem first proved by Siebeck [11] which states:

Theorem 1.2. *The zeros of the function $F(Z) = \sum_1^p \frac{m_i}{Z-Z_i}$ are the foci of the curve of class $p-1$ which touches each line segment (Z_i, Z_j) in a point dividing the line segment in the ratio $m_i : m_j$. In particular, the critical points of a polynomial $P(Z)$ of degree p are the foci of a curve of class $p-1$ which is tangent to the lines joining pairs of roots of P.*

For the proof of these theorems, see Marden [6, pp. 7–11].

We will prove the analogous result for ‘polynomials’ in the hyperbolic plane. The notion of a *non-Euclidean polynomial* seems to be due to Walsh [16]. He uses this

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term for a function of the form

$$B(Z) = \lambda \prod_{1}^{n} \frac{Z - A_k}{1 - \overline{A}_k Z}, \quad |\lambda| = 1, \quad |A_k| < 1. \tag{1.1}$$

This is the general form for a rational function which takes the closed unit disc $\overline{D}$ to itself, and it is usually referred to as a *finite Blaschke product.* $B(Z)$ is an n-to-one map of $\overline{D}$ onto itself, has precisely n zeros in the interior D, and has modulus unity on $C\colon |Z| = 1$. We may regard D as the hyperbolic plane and the unit circle C as the set of ideal points, using the Poincaré model of the hyperbolic plane. Then it is reasonable to think of $B(Z)$ as a polynomial, and indeed it has exactly $n-1$ critical points (counting multiplicity) in D, with the remaining critical points in the exterior of the disc, symmetric with respect to inversion in the circle. There is an analogue to Lucas's theorem:

Theorem 1.3 (Walsh [16, p. 157])**.** *Let $B(Z)$ be defined by* (1.1)*. The critical points of $B(Z)$ in the interior of the disc lie within or on the (non-Euclidean) convex hull of the zeroes of $B(Z)$, with respect to the Poincaré metric.*

We will show:

Theorem 1.4 (Main theorem)**.** *Let $B(Z) = \beta \prod_1^n \frac{Z-A_k}{1-\overline{A}_k Z}$, $|\beta| = 1$, $|A_k| < 1, n > 2$ be a non-Euclidean polynomial, and let γ be the curve in D which is the envelope of the non-Euclidean geodesics (with respect to the Poincaré metric) joining pairs of points W_i, W_j on C satisfying $B(W_i) = B(W_j)$. Then γ is (part of) an algebraic curve whose real foci are the critical points of $B(Z)$ in D together with their inverses with respect to D.*

Remark 1.5. Beginning with the curve γ one can construct an interesting dynamical system, the "dual billiard" system in the hyperbolic plane, by following the tangent geodesics a fixed distance from the point of tangency. (See [14].) For the curves of Theorem 1.4, the associated diffeomorphisms on the ideal boundary are conjugate to *rational* rotations.

Remark 1.6. In the case $n = 3$ the author proved [12] that the curve is a non-Euclidean ellipse (together with its reflected image) whose geometric foci are indeed the two algebraic foci in the disc. Thus, Theorem 1.4 generalizes the theorem of Bôcher and Grace in this case. This is discussed in section 5.

2. Algebraic curves and their foci

We will be studying the real projective plane, the complex projective plane, and the complex projective line. For the convenience of the reader we present a brief review of some classical information about algebraic curves. More details may be found, for example, in [13]. Real variables will be denoted by small letters, complex coordinates by capital letters. The complexification of a variable will be represented by capitalization.

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) | x_j \in \mathbb{R}\} \subset \mathbb{C}^3 = \{(X_1, X_2, X_3) | X_j \in \mathbb{C}\},$$
$$X_j = x_j + i x_j', \text{ where } x_i = \Re(X_i), x_i' = \Im(X_i).$$

Points in complex projective space $\mathbb{C}P^2$ are complex lines through the origin in $\mathbb{C}^3$; they have homogeneous coordinates $[X_1, X_2, X_3]$, where (X_1, X_2, X_3) are the coordinates of any non-zero point on the line. We view real projective space

$\mathbb{R}P^2$ as a subset of $\mathbb{C}P^2$. A line in $\mathbb{C}P^2$ is given by an equation of the form $AX_1 + BX_2 + CX_3 = 0$; the coordinates of this line will be denoted $[A\ B\ C]$. The space of lines in $\mathbb{C}P^2$ is the dual space $\mathbb{C}P^{2*}$.

An algebraic curve γ in $\mathbb{C}P^2$ has a dual curve $\gamma*$ in $\mathbb{C}P^{2*}$ consisting of the lines tangent to the curve. As is well known, the dual of $\gamma*$ is γ. The *order* n of γ is degree of the polynomial defining it in $\mathbb{C}P^2$, while the *class* c of γ is the degree of the polynomial defining $\gamma*$. If γ has class c, then in general there are exactly c lines through a given point p in $\mathbb{C}P^2$ tangent to γ (with the usual caveat about multiplicity).

Now suppose γ is a real curve of class c. Then the curve has (no more than) c real foci defined as follows: Take lines passing through the point $[1, i, 0]$ tangent to γ. Such a line has the form $X_1 + iX_2 - PX_3 = 0$. Such a line (called an *isotropic line*) meets $\mathbb{R}P^2$ at a unique point $[x_1, x_2, 1]$, given by the equation $P = x_1 + ix_2$. These are the *real foci* of γ. Another way of defining the foci is to take the c isotropic tangent lines through the "circular point" $[1, i, 0]$ and the c lines (also called isotropic) through the other "circular point" $[1, -i, 0]$ and taking the c^2 intersections of pairs of such lines. The real foci are the real intersections.

In the case of an ellipse, for example, there are four foci, of which two are real and two imaginary. Quadratic curves have both order 2 and class 2. In general, a curve of degree n has $c \leq n(n-1)$. As the example in section 5 shows, the relation between the class and the number of real foci can be complicated. The hyperbolic ellipse has class $c = 8$ and four real foci. The definition of foci can be traced back to Plücker; see [9, pp. 119–128] for details.

3. Blaschke products

We consider a finite Blaschke product

$$B(Z) = \prod_1^n \frac{Z - A_k}{1 - \overline{A_k} Z}, \qquad |A_k| < 1. \tag{3.1}$$

This defines a relation between pairs of points $(Z, W), Z \neq W$ on the unit circle which have the same preimage:

$$\frac{B(Z) - B(W)}{Z - W} = 0. \tag{3.2}$$

This can be rewritten as the polynomial equation:

$$P(Z, W) = \frac{\prod_1^n (Z - A_k)(1 - \overline{A_k} W) - \prod_1^n (W - A_k)(1 - \overline{A_k} Z)}{W - Z} = 0. \tag{3.3}$$

The equation $(P(Z, W) = 0, |Z| = 1 = |W|)$ defines a curve η in D as the envelope of the straight lines connecting pairs of points on the circle. Since for each Z_0 there are precisely $n-1$ solutions to $P(Z_0, W) = 0$, this is a curve of class $n-1$.

If (Z, W) is a zero of P, then Z and W are the roots of the polynomial $X^2 - \nu X + \tau$, where $\tau = ZW$ and $\nu = Z + W$. As Mirman and Shukla observe [8, p. 88], the equation $P(Z, W) = 0$ may be written in terms of τ and ν in the form $P_1(\tau, \nu) = 0$, where P_1 is a polynomial of degree $n-1$. Given a pair (τ, ν) the corresponding line has coordinates $[u \quad v\ -1]$ with

$$u = \frac{\tau + 1}{\nu} = \frac{1 + ZW}{Z + W}, \quad v = \frac{i(1 - \tau)}{\nu} = \frac{i(1 - ZW)}{Z + W}. \tag{3.4}$$

The curve η obviously satisfies the *Poncelet property*: For each point x on η there is an $(n-1)$-gon tangent to η at x with each side tangent to η and vertices lying on C. As Mirman and Shukla observe, the usual definition of Poncelet curve uses the ordering of the vertices on the circle to define a convex polygon and convex curve. However, the algebraic definition yields a curve tangent to the diagonals of the polygon as well. Indeed, the curve should be thought of as living in $\mathbb{C}P^2$. The unit circle is the real part of the *unit quadric* $\mathcal{C} = \{[X_1, X_2, X_3] : X_1^2 + X_2^2 = X_3^2\}$. Points on $\mathcal{C}$ may be parametrized by complex numbers Z as follows: Given Z, let $[X_1, X_2, X_3] = [1 + Z^2, i(1 - Z^2), 2Z]$. Conversely, given $[X_1, X_2, X_3]$, let $Z = \frac{X_1+iX_2}{X_3}$. This can be extended to a correspondence between $\mathcal{C}$ and $\mathbb{C}P^1$ by mapping the point $[1, -i, 0]$ to infinity. Note that the point $[\cos\theta, \sin\theta, 1]$ corresponds to the complex number $Z = \cos\theta + i\sin\theta$, so the real unit circle is represented as the unit circle in the complex plane by this parametrization.

Now the equation $P(Z,W) = 0$ may be interpreted as defining the lines joining points in $\mathcal{C}$ with coordinates Z and W. In particular, setting $Z = 0$ and solving for W gives the equation for a real focus for the curve γ. We have:

Proposition 3.1. *The curve η which is the envelope of the lines joining pairs of unit complex numbers (Z,W) with $B(Z) = B(W), Z \neq W$ is a curve of class $n-1$ with foci F_i satisfying $B(F_i) = B(0), F_i \neq 0$.*

In the case $n = 3$ the curve is an ellipse and the foci are the geometric foci of the ellipse ([2]).

4. Proof of Theorem 1.4

If we consider the disc D as a model for hyperbolic space having the Klein metric, for which geodesics are Euclidean straight line segments, then the envelope described above can be viewed as the formulated with respect to the Poincaré metric. The relationship between the two metrics can be described as follows. If we consider the complex plane as the boundary of upper half space $\mathbb{H}^3$ with the Poincaré metric $ds^2 = (dx_1^2 + dx_2^2 + dx_3^2)/x_3^2$, then the *unit hemisphere model* for the hyperbolic plane is given by the unit hemisphere $x_3 = \sqrt{1 - x_1^2 - x_2^2}$ with the induced metric. (See [7], page 191.) The Klein model, which Milnor refers to as the *projective disc model*, is gotten by ignoring the x_3 - coordinate. On the other hand, the Poincaré model is achieved by stereographic projection of the unit hemisphere onto the disc from the south pole $(0, 0, -1)$. We define a canonical map relating these two models.

Definition 4.1. The Klein-to-Poincaré map $KP : D \longrightarrow D$ is defined by

$$KP(Z) = \frac{Z}{1 + \sqrt{1 - |Z|^2}}$$

or

$$KP(x,y) = (\frac{x}{1 + \sqrt{1 - x^2 - y^2}}, \frac{y}{1 + \sqrt{1 - x^2 - y^2}}).$$

The Klein-to-Poincaré map is the unique isometry from the disc D with the Klein metric to the disc with the Poincaré metric which keeps the ideal boundary pointwise fixed. It takes the straight line segment between points on the boundary to the NE geodesic between the same two points. It therefore takes the envelope η described in Proposition 3.1 to the non-Euclidean envelope γ.

The inverse of KP, the *Poincare-to-Klein map* PK is actually somewhat nicer;

$$PK(Z) = KP^{-1}(Z) = \frac{2Z}{1+Z\overline{Z}},$$

or

$$PK(x,y) = (\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}),$$

it can be defined on $\mathbb{C}P^2$ by

$$PK[X_1, X_2, X_3] = [2X_1X_3, 2X_2X_3, X_1^2+X_2^2+X_3^2].$$

Remark 4.2. Construct the family of circles orthogonal to the unit circle passing through pairs (Z,W) satisfying (3.2). If the circle C has radius $r = \tan\phi$ and center $p_1 + ip_2$, then the condition that C be orthogonal to the unit circle is

$$p_1^2 + p_2^2 = 1 + r^2.$$

So we define θ by

$$p_1 = \sqrt{1+r^2}\cos\theta, \qquad p_2 = \sqrt{1+r^2}\sin\theta.$$

We have $Z = e^{\theta-\phi}$ and $W = e^{\theta+\phi}$. So

$$\tau = ZW = e^{2i\theta}, \qquad \nu = Z+W = 2\cos\phi e^{i\theta}, \qquad Z-W = 2i\sin\phi e^{i\theta},$$

$$p = p_1 + ip_2 = \tan\phi e^{i\theta} = \frac{2ZW}{Z+W}, \qquad r = \|\frac{Z-W}{Z+W}\|. \tag{4.1}$$

Thus the curve γ is (part of) the envelope of circles whose centers lie at points $\frac{2ZW}{Z+W}$ for $P(Z,W) = 0$.

The relationship between γ and η is simplest to describe when γ is represented by an equation of the form $\Gamma(Z) = 0$; then η is represented by the equation $\Gamma((KP(Z)) = 0$. Of course, we do not actually have an equation for η; rather, we have an equation for its dual. The strategy for determining the foci of γ is therefore indirect. Suppose $\ell : ux+vy = h$ is a line in the complex plane. (If $h \neq 0$, ℓ will not be geometrically a line in the Poincaré disc, of course.) Pulling the line back via KP, we get the equation

$$ux + vy = h + h\sqrt{1-x^2-y^2}. \tag{4.2}$$

This defines an irreducible quadratic curve. The condition that ℓ is tangent to the curve γ translates into the condition that the pulled-back curve is tangent to η. Now we extend this construction to $\mathbb{C}P^2$. In the special case of an (isotropic) line through $[1, i, 0]$, we may assume $u = 1, v = i$; H is the complex coordinate of a focus of γ. In that case, equation (4.2) becomes

$$Z = H + H\sqrt{1-Z\overline{Z}} \tag{4.3}$$

or

$$Z^2 - 2HZ + H^2 Z\overline{Z} = Z(Z + H^2\overline{Z} - 2H) = 0. \tag{4.4}$$

Thus the isotropic line pulls back to a pair of lines, namely the isotropic line $[1 \quad i \quad 0]$ and the line $\ell_H = [1+H^2 \quad i(1-H^2) \quad 2H]$. The line ℓ_H has line coordinates $u = \frac{1+H^2}{2H}$ and $v = \frac{i(1-H^2)}{2H}$. Comparing with formula (3.4), We have $ZW = H^2$ and $Z + W = 2H$. Therefore, the focus H corresponds to a pair $(Z,W) = (H,H)$ with $P(H,H) = 0$. But from equation (3.2) it follows that the polynomial $P(Z, Z)$ vanishes precisely at the critical points of $B(Z)$. This proves the main theorem.

5. Example: The hyperbolic ellipse

The ellipse in the hyperbolic plane can be represented in the Klein model of the hyperbolic plane $x^2 + y^2 < 1$ by a compact quadratic curve given in orthogonal coordinates as $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1, 0 < \beta < \alpha < 1$ (see [15]). Schilling [10] proved that the ellipse defined in this manner is the locus of a point the sum of whose hyperbolic distances from the two geometric foci are constant. As noted in [11], the coordinates $\pm\delta$ of these geometric foci are farther from the origin than the Euclidean foci $\pm c$. The precise relationship is

$$\delta^2 = \frac{c^2}{1-\beta^2}. \tag{5.1}$$

Pulling back the equation of the ellipse by PK, the hyperbolic ellipse (in the Poincaré disc) is given by

$$\Gamma : \frac{4x^2}{\alpha^2} + \frac{4y^2}{\beta^2} - (x^2 + y^2 + 1)^2 = 0. \tag{5.2}$$

The geometric foci are located at $\pm\epsilon$, where $\delta = \frac{2\epsilon}{1+\epsilon^2}$ or

$$\epsilon = \frac{\delta}{1 + \sqrt{1-\delta^2}} = \frac{\sqrt{1-\beta^2} - \sqrt{1-\alpha^2}}{c}. \tag{5.3}$$

The algebraic curve (5.2) is an example of a *bicircular quartic* (see, e.g., [4, p. 304]). This curve has two real components, one interior to and the other exterior to the unit disc; the two components are related by reflection through the unit circle. As an algebraic curve it is of class 8 and deficiency (genus) 1, having four real (algebraic) foci.

Proposition 5.1. *The real foci of the bicircular quartic* Γ *given by equation* (5.2) *coincide with the geometric foci of the hyperbolic ellipse and their reflections in the unit circle.*

Proof. The algebraic foci are the points (x, y) in the plane for which the isotropic lines are tangent to the curve Γ. To find these points, we look at the general equation of an isotropic line and see when the equation for its intersection with the curve has a double root. Thus we look at the pair of equations:

$$\frac{4x^2}{\alpha^2} + \frac{4y^2}{\beta^2} - (x^2 + y^2 + 1)^2 = 0, \quad x + iy = U.$$

Eliminating x leads to the equation

$$(\frac{4}{\beta^2} - \frac{4}{\alpha^2})y^2 - \frac{8iU}{\alpha^2}y + \frac{4U^2}{a^2} = (1 - 2iUy + U^2)^2. \tag{5.4}$$

This is quadratic in y, because the curve Γ is bicircular, meaning that it has double points at both circular points. Therefore, a line through a circular point meets the curve twice at the circular point and therefore has only two other intersections. Rewrite this equation as $Ay^2 + By + C = 0$; then

$$A = \frac{4}{b^2} - \frac{4}{a^2} + 4U^2, \quad B = 4iU(1 + U^2 - \frac{2}{a^2}), \quad C = \frac{4U^2}{a^2} - (1 + U^2)^2. \tag{5.5}$$

The discriminant equation $B^2 - 4AC = 0$ would appear to be sixth order in U but there is a cancellation. The equation is actually biquadratic and reduces to

$$U^2 = \frac{1}{c^2}[2 - \alpha^2 - \beta^2 \pm \sqrt{(1-\alpha^2)(1-\beta^2)}]. \tag{5.6}$$

Taking square roots, we get the four algebraic foci:

$$U = \pm\frac{1}{c}[\sqrt{1-\beta^2} \pm \sqrt{1-\alpha^2}]. \tag{5.7}$$

These four foci are all real, and they correspond to the geometric foci and their reciprocals. □

In addition to the four foci determined above, there are two additional *singular foci*, which are given by the real intersections of the lines through a circular point and tangent to the curve *at the circular point itself.* By parametrizing the general isotropic line we can find these points:

First rewrite equation (5.2) in homogeneous form:

$$\Gamma : \frac{4x^2z^2}{\alpha^2} + \frac{4y^2z^2}{\beta^2} - (x^2 + y^2 + z^2)^2 = 0. \tag{5.8}$$

Let $x = 1 + tX$, $y = i + tY$, and $z = t$ where X and Y are the (unknown) coordinates of the focus. Plugging these into equation (5.8) gives

$$t^2(2X + 2iY + t(X^2 + Y^2 + 1))^2 - \frac{4t^2}{a^2}(1 + 2tX + t^2X^2) - \frac{4t^2}{b^2}(-1 + 2itY + t^2Y^2) = 0. \tag{5.9}$$

The factor of t^2 corresponds to the line passing through the circular point twice, once for each branch. When the line is tangent to the curve at the circular point, there will be an extra factor of t. This occurs when

$$4(x + iY)^2 - \frac{4}{\alpha^2} - \frac{4}{\beta^2} = 0.$$

Thus the singular foci occur at

$$X = 0,\ Y = \pm\frac{c}{\alpha\beta}.$$

A consequence of the general theory of bicircular quartics [4, p. 305] is that there is an ellipse with foci at these singular foci such that the curve is the envelope of the circles centered at points on the ellipse and orthogonal to the unit circle. It is not hard to verify that the envelope of Remark 4.2 is an ellipse with foci equal to the singular foci in this example. □

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