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AN EXPLICIT COUNTEREXAMPLE TO THE EQUIVARIANT $K = 2$ CONJECTURE

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Abstract. We construct an explicit example of a geometrically finite Kleinian group G with invariant component Ω in the Riemann sphere $\hat{\mathbf{C}}$ such that any quasiconformal map from Ω to the boundary of the convex hull of $\hat{\mathbf{C}} - \Omega$ in $\mathbf{H^3}$ which extends to the identity map on their common boundary in $\hat{\mathbf{C}}$, and which is equivariant under the group of Möbius transformations preserving Ω, must have maximal dilatation $K > 2.002$.

1. Introduction

Let Ω be a simply connected domain in the Riemann sphere $\hat{\mathbf{C}}$ whose boundary contains more than two points. Thinking of $\hat{\mathbf{C}}$ as the boundary of hyperbolic 3-space $\mathbf{H^3}$, the *hyperbolic convex hull* of $\hat{\mathbf{C}} - \Omega$ is the smallest closed set in $\mathbf{H^3}$ which contains every hyperbolic geodesic arc with endpoints in $\hat{\mathbf{C}} - \Omega$. The boundary of the hyperbolic convex hull of $\hat{\mathbf{C}} - \Omega$ in $\mathbf{H^3}$ is called the *dome* of Ω and is denoted by $\mathrm{Dome}(\Omega)$.

On the one hand, Riemann's mapping theorem tells us that Ω is conformal to the unit disk $\mathbf{D}$ with its conformal structure. On the other hand, Thurston ([EM87]) proved that $\mathrm{Dome}(\Omega)$ with its induced metric from $\mathbf{H^3}$ is isometric to $\mathbf{D}$ with its hyperbolic structure. Since Ω and $\mathrm{Dome}(\Omega)$ share the same boundary $\partial\Omega$ in $\hat{\mathbf{C}}$, it is natural to look for quasiconformal maps $f : \Omega \to \mathrm{Dome}(\Omega)$ such that the continuous extension of f to $\partial\Omega$ acts as the identity map on $\partial\Omega$. We cannot expect, in general, the existence of a conformal map from Ω to $\mathrm{Dome}(\Omega)$ with this boundary condition; as far as we know, it is unknown whether the existence of a conformal map from Ω to $\mathrm{Dome}(\Omega)$ with this boundary condition would imply that Ω is a round disk in $\hat{\mathbf{C}}$. In the study of these quasiconformal maps from Ω to $\mathrm{Dome}(\Omega)$, Sullivan ([Sul81]) proved the existence of a universal constant K such that for any Ω there is a K-quasiconformal map from Ω to $\mathrm{Dome}(\Omega)$ whose extension to $\partial\Omega$ is the identity map. Later, Epstein and Marden ([EM87]) gave a more detailed proof of this result and included an upper bound on K in the case where the quasiconformal map must be equivariant under the group of Möbius transformations preserving Ω.

To fix some notation, let $\mathrm{M\ddot{o}b}(\Omega)$ denote the group of Möbius transformations which preserve Ω. Then $\mathrm{M\ddot{o}b}(\Omega)$ also acts on $\mathrm{Dome}(\Omega)$ as a group of hyperbolic isometries by means of the Poincaré extension. Let $K(\Omega)$ be the infimum of the maximal dilatations of quasiconformal maps from Ω to $\mathrm{Dome}(\Omega)$ that extend to the identity map on $\partial\Omega = \partial\mathrm{Dome}(\Omega)$. Also, let $K = \sup_\Omega K(\Omega)$. For the equivariant

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case, let $K_{eq}(\Omega)$ be the infimum of the maximal dilatations of Möb(Ω)-equivariant quasiconformal maps from Ω to Dome(Ω) with this boundary condition. Finally, let $K_{eq} = \sup_{\Omega} K_{eq}(\Omega)$. Thurston's $K = 2$ Conjecture, which appears in [Thu98], is that $K = K_{eq} = 2$.

It follows immediately from the definitions that $K \leq K_{eq}$, but it is unknown whether $K < K_{eq}$, or even whether there is a domain Ω satisfying $K(\Omega) < K_{eq}(\Omega)$. In [EM87], Epstein and Marden obtained the upper bound $K_{eq} < 82.7$, and Bishop ([Bis04]) improved this result to $K < 7.82$. Bishop ([Bis02]) also showed that Thurston's $K = 2$ Conjecture relates to another important conjecture, called Brennan's Conjecture. Bishop showed that if $K(\Omega) \leq 2$, then Brennan's Conjecture holds for Ω; and that, in particular, Thurston's $K = 2$ Conjecture implies Brennan's Conjecture. (See [Bis02] for details.)

In [EMM04], Epstein, Marden, and Markovic showed the existence of punctured torus groups whose invariant components Ω satisfy $K_{eq}(\Omega) > 2$, thus showing Thurston's $K = 2$ Conjecture to be false in the equivariant case. Then, in [EM05], Epstein and Markovic showed that the complement Ω of the logarithmic spiral $z = e^{(1+i)s}$ ($s \in \mathbf{R}$) satisfies $K(\Omega) > 2.1$, and that there is a quasi-Fuchsian group of a compact surface whose invariant component Ω satisfies $K(\Omega) > 2.1$.

The punctured torus groups found by Epstein, Marden, and Markovic as counterexamples to the equivariant $K = 2$ conjecture are located close to a point on the boundary of the trace $A = 2\sqrt{2}$ slice of the space of punctured torus groups (see Figure 2 in [EMM04]); however, it is not clear exactly how close to this boundary point we must get before we can find an explicit counterexample to the conjecture. The boundary point itself has an invariant component which is a counterexample to the conjecture, but this boundary group is not geometrically finite, so explicit information on this group (including matrices of generators) is difficult to obtain. In this paper, we construct an explicit geometrically finite group whose invariant component is a counterexample to the equivariant $K = 2$ conjecture; it is located in the trace $A = 2$ slice of the space of punctured torus groups, known as the Maskit slice (see [KS93]). Our group is generated by very simple Möbius transformations and we can also estimate $K_{eq}(\Omega)$ not only from below but also from above:

Theorem 1.1. *The group $G = \langle S, T \rangle$ generated by $S(z) = z + 2$ and $T(z) = (1/z) + (1 + 2i)$ is a regular b-group of type $(1, 1)$ whose invariant component Ω satisfies $2.002 < K_{eq}(\Omega) < 2.0156$.*

The idea of the proof is that the surfaces Ω/G and Dome(Ω)$/G$ are marked once-punctured tori, hence they determine points in the upper half plane $\mathbf{H^2}$ which can be naturally identified with the Teichmüller space of once-punctured tori. The hyperbolic (Poincaré) distance from Ω/G to Dome(Ω)$/G$ is equal to the Teichmüller distance $\log K(G)$ where $K(G)$ is the infimum of the maximal dilatations of quasiconformal maps between Ω/G and Dome(Ω)$/G$. It follows that $K(G)$ is equal to the infimum of maximal dilatations of G-equivariant quasiconformal maps between Ω and Dome(Ω) that extend to the identity map on $\partial\Omega$; thus, $K(G) \leq K_{eq}(\Omega)$. Moreover, we have the following:

Lemma 1.2. *For the group G given in Theorem 1.1 with its invariant component Ω, $K(G)$ is equal to $K_{eq}(\Omega)$.*

This lemma will be proved in Section 4. Hence to show Theorem 1.1, it is enough to estimate the Teichmüller distance $\log K(G)$ between Ω/G and Dome(Ω)$/G$. In

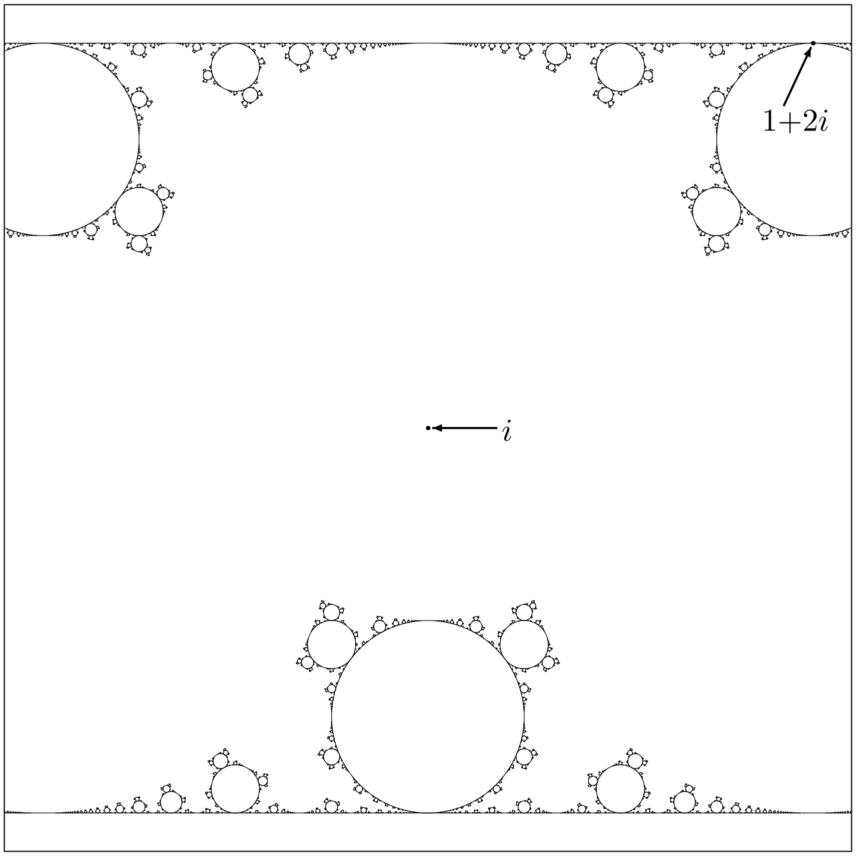

FIGURE 1. This fractal curve is a part of the limit set of the group G from Theorem 1.1. The invariant component lies between the top and bottom portions of the limit set. This picture was drawn using the limit set programs written by David J. Wright, available at www.math.okstate.edu/∼wrightd.

Section 2 we will show that the Teichmüller parameter of Dome$(\Omega)/G$ is $(1/2)+(\sqrt{3}/6)\,i = M^2(\omega)$ where $\omega = (-1+\sqrt{3}i)/2$ and M is the element of the elliptic modular group $PSL(2,\mathbf{Z})$ given by $M(\tau) = \tau/(\tau+1)$. On the other hand in Section 3, the Teichmüller parameter of Ω/G will be computed as $(1/2)+y\,i$ where $0.143223349 < y < 0.144192163$ by means of the approximation of the period of the torus Ω/G studied by the second-named author [Mat01]. Since both Teichmüller parameters lie on the same vertical line $\Re(z) = 1/2$ in $\mathbf{H^2}$, we can easily work out their hyperbolic distance: it is $\log(\,(\sqrt{3}/6)/y\,)$, and so $K_{eq}(\Omega) = (\sqrt{3}/6)/y$ and Theorem 1.1 follows immediately.

2. The Teichmüller parameter of Dome$(\Omega)/G$

Any (marked) once-punctured torus is uniformized by a (marked) Fuchsian once-punctured torus group $\Gamma = \langle A, B\rangle$ which is unique up to conjugation in $PSL(2,\mathbf{R})$; such a torus is also uniquely determined by the triple $(x,y,z) = (\mathrm{Tr}\,A, \mathrm{Tr}\,B, \mathrm{Tr}\,AB) \in \mathbf{R^3}$. The modular group $PSL(2,\mathbf{Z})$ acts on the upper half plane $\mathbf{H^2}$ and on the trace parameter space (Tr A, Tr B, Tr AB) $\in \mathbf{R^3}$ as models of the Teichmüller space of once-punctured tori (see Section 1 of [KRV79] for an example). The modular group is generated by the elements $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The transformation J changes the marking $\langle A, B\rangle$ to $\langle B^{-1}, A\rangle$, and thus changes the trace parameters (x, y, z) to $(y, x, xy - z)$. The transformation N changes the marking $\langle A, B\rangle$ to $\langle B^{-1}, AB\rangle$, and thus changes the trace parameters (x, y, z) to (y, z, x). We note here that the element $M = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ changes the marking $\langle A, B\rangle$ to $\langle AB, B\rangle$ and the trace parameters (x, y, z) to $(z, y, yz - x)$. The following result is well known (see p. 203 of [KRV79]):

Proposition 2.1. *Let $\Gamma = \langle A, B\rangle$ be a Fuchsian once-punctured torus group. Then the Teichmüller parameter of Γ is equal to $\omega = (-1 + \sqrt{3}i)/2$ if and only if* (Tr A, Tr B, Tr AB) $= (3, 3, 3)$.

Proof. The point ω is a fixed point of the element $N = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in $PSL(2, \mathbf{Z})$. Since N changes the trace parameters (x, y, z) to (y, z, x), we get $x = y = z$. Since Γ is a once-punctured torus group, Markoff's equation $x^2 + y^2 + z^2 = xyz$ guarantees that $x = y = z = 3$. □

Next we consider the Fenchel–Nielsen coordinates of the Teichmüller space of once-punctured tori (see [PP98] for an example). Let X be a once-punctured torus uniformized by $\Gamma = \langle A, B\rangle$, and γ be a simple closed geodesic on X representing $A \in \Gamma$. We denote its hyperbolic length by λ. Then $X \setminus \gamma$ is a once-punctured cylinder with geodesic boundaries having the same length λ. It is uniformized by $\langle A, A'\rangle$ where

$$A = \begin{pmatrix} \cosh\lambda/2 & \cosh\lambda/2 + 1 \\ \cosh\lambda/2 - 1 & \cosh\lambda/2 \end{pmatrix}, \tag{1}$$

$$A' = \begin{pmatrix} \cosh\lambda/2 & \cosh\lambda/2 - 1 \\ \cosh\lambda/2 + 1 & \cosh\lambda/2 \end{pmatrix}. \tag{2}$$

The original surface X can be reconstructed by gluing together the geodesic boundaries of $X \setminus \gamma$. In terms of Fuchsian groups, this operation can be realized by forming the HNN extension of $\langle A, A'\rangle$ by an element $B \in PSL(2, \mathbf{R})$ satisfying

$$B^{-1}AB = A'.$$

From this condition, B can be written as

$$B = \begin{pmatrix} \cosh\tau/2\coth\lambda/4 & -\sinh\tau/2 \\ -\sinh\tau/2 & \cosh\tau/2\tanh\lambda/4 \end{pmatrix} \tag{3}$$

where τ is a free real parameter which has the following geometric interpretation: If the common perpendicular δ to the hyperbolic axis of A and that of $B^{-1}AB = A'$ meets these axes in points X and Y, then τ is the signed distance from X to $B(Y)$ where the axis of A is oriented from the attracting fixed point $\coth\lambda/4$, to the repelling fixed point $-\coth\lambda/4$. The map $\Gamma \mapsto (\lambda, \tau)$ gives the Fenchel–Nielsen coordinates of the Teichmüller space of once-punctured tori and λ and τ are called the length and twist parameters, respectively. Let us compute the Fenchel–Nielsen coordinates of the marked torus stated in Proposition 2.1.

Proposition 2.2. *The Fenchel–Nielsen coordinates of the once-punctured torus whose Teichmüller parameter is $\omega = (-1 + \sqrt{3}i)/2$ are given by*

$$\lambda = 2\cosh^{-1}(3/2),\ \tau = \lambda/2.$$

Proof. Setting Tr $A = 3$ and solving for λ yields the solution $\lambda = 2\cosh^{-1}(3/2)$. Setting Tr $B = 3$ yields $\tau = \pm\lambda/2$. If $\tau = \lambda/2$, then Tr $AB = 3$; if $\tau = -\lambda/2$, then Tr $AB = 6$. □

Let $\Gamma_0 = \langle A_0, B_0\rangle$ be the Fuchsian group satisfying the trace condition $(\text{Tr } A_0, \text{Tr } B_0, \text{Tr } A_0B_0) = (3,3,3)$ stated in Proposition 2.1 and let λ_0 and τ_0 be the Fenchel–Nielsen coordinates of it given in Proposition 2.2. Now we start to deform Γ_0 in $PSL(2,\mathbf{C})$ to get the Kleinian group we are looking for. Keeping λ_0 fixed, let us release the twist parameter τ as a free complex parameter. Then we can still consider the subgroup $\Gamma(\tau) = \langle A, B\rangle$ of $PSL(2,\mathbf{C})$ defined by equations (1) and (3) which acts on $\mathbf{H^3}$ as a group of hyperbolic isometries by means of the Poincaré extension. In particular, there is a natural homomorphism f_τ from $\Gamma_0 = \Gamma(\tau_0)$ to $\Gamma(\tau)$ defined by $f_\tau(A_0) = A$ and $f_\tau(B_0) = B$. Here we should remark that $A = A_0$ and $f_\tau(A') = A'$ whereas $f_\tau(B_0) \neq B_0$ in general. Hence Γ_0 and $\Gamma(\tau)$ share the same subgroup $\langle A, A'\rangle$ on which f_τ is identity.

Associated with $\tau \in \mathbf{C}$, we can consider the map $\psi_\tau : \mathbf{H^2} \to \mathbf{H^3}$, called the pleated surface for f_τ, defined as follows (see [KS97]). Consider $\mathbf{H^2}$ as the totally geodesic surface in $\mathbf{H^3}$ defined by Im $z = 0$, and remove the Γ_0-orbit of the axis of A_0 from $\mathbf{H^2}$. Take the connected component whose boundary contains the axis of A and that of A', and consider its closure in $\mathbf{H^3}$. We call it the flat piece and denote it by F. By means of this flat piece F, we can define the pleated surface ψ_τ as follows:

$$\psi_\tau(z) = \begin{cases} z, & \text{if } z \in F, \\ f_\tau(g)g^{-1}(z), & \text{if } z \in g(F) \text{ for } g \in \Gamma_0. \end{cases}$$

We remark that the stabilizer of F in Γ_0 is $\langle A, A'\rangle$, the subgroup on which f_τ is the identity; hence, ψ_τ is well defined. The pleated surface ψ_τ satisfies the f_τ-equivariant property

$$\psi_\tau(g(z)) = f_\tau(g)\psi_\tau(z)$$

for $z \in \mathbf{H^2}$ and $g \in \Gamma_0$. We denote $\psi_\tau(\mathbf{H^2})$ by $\mathbf{H^2}(\tau)$ which is the union of the $\Gamma(\tau)$-orbit of F. In particular, when $\tau = \tau_0 + i\theta$, adjacent flat pieces $g_1(F)$ and $g_2(F)$ of $\mathbf{H^2}(\tau)$ are bent along the axis of some conjugate of A in $\Gamma(\tau)$ with bending angle θ. This process is called pure bending. We denote $\Gamma(\tau_0 + i\theta)$ and $\mathbf{H^2}(\tau_0 + i\theta)$ by $\Gamma(\theta)$ and $\mathbf{H^2}(\theta)$. Assuming that $|\theta|$ is sufficiently small, it is shown in [KS97] that $\Gamma(\theta)$ is a quasi-Fuchsian group and $\mathbf{H^2}(\theta)$ is a boundary component ∂C_0 of the convex core, the hyperbolic convex hull of the limit set of $\Gamma(\theta)$. The quotient surface $\partial C_0/\Gamma(\theta)$ is a pleated surface bent along the simple closed geodesic represented by A. Also $(\partial C_0/\Gamma(\theta); A, B)$ is conformal to $(\mathbf{H^2}/\Gamma_0; A, B_0)$ as a marked surface.

Proposition 2.3. *Define the real number* θ_0 *by* $-\pi < \theta_0 < 0$ *and* $\cos\theta_0 = -1/9$.

1. $AB^2 \in \Gamma(\theta)$ *is purely hyperbolic for* $\theta_0 < \theta < 0$ *while it is parabolic when* $\theta = \theta_0$.
2. *For* $\theta_0 < \theta < 0$, $\Gamma(\theta)$ *is a quasi-Fuchsian punctured torus group whose convex core has two boundary components which are bent along simple closed geodesics represented by* A *and* AB^2*, respectively.*
3. $\Gamma(\theta_0)$ *is a regular b-group of type* $(1,1)$ *in which* AB^2 *is accidentally parabolic.*

Proof. Claim (1) follows immediately from computing the trace of AB^2.

To prove (2), we first we apply the local pleating theorem (Theorem 26 in [KS04]) to Γ_0. Then there is some $\theta_1 < 0$ such that for $\theta_1 < \theta < 0$, $\Gamma(\theta)$ is a quasi-Fuchsian punctured torus group whose convex core has two boundary components whose bending loci are simple closed geodesics represented by A and AB^2, respectively. Next we apply the local pleating theorem to $\Gamma(\theta_1)$. Then there is some $\theta_2 < \theta_1$ such that $\Gamma(\theta)$ satisfies the same property for $\theta_2 < \theta < 0$. We continue this procedure and obtain a decreasing sequence $\{\theta_n\}$. Now assume that the sequence $\{\theta_n\}$ does not converge to θ_0. Then $\{\theta_n\}$ must converge to some θ_∞ which is strictly greater than θ_0. Then the limit pleating theorem (Theorem 15 in [KS04]) implies that $\Gamma(\theta_\infty)$ is also a quasi-Fuchsian punctured torus group satisfying the previous property. Hence we can still apply the local pleating theorem to $\Gamma(\theta_\infty)$ and go further, which contradicts the assumption. Therefore for any θ satisfying $\theta_0 < \theta < 0$, $\Gamma(\theta)$ is a quasi-Fuchsian punctured torus group whose convex core boundaries are bent along simple closed geodesics represented by A and AB^2, respectively.

Claim (3) is an immediate consequence of (1) and (2). □

Let ∂C_0 be the convex core boundary facing to the invariant component of $\Gamma(\theta_0)$. From the previous bending construction, the marked surface $(\partial C_0/\Gamma(\theta_0); AB^2, B)$ is conformal to $(\mathbf{H^2}/\Gamma_0; AB^2, B)$. Hence considering the action of the modular transformation $M(\tau) = \frac{\tau}{\tau+1} \in PSL(2, \mathbf{Z})$ mentioned in the beginning of this section, the Teichmüller parameter of $(\mathbf{H^2}/\Gamma_0; AB^2, B)$ is $M^2(\omega) = \frac{1}{2} + \frac{\sqrt{3}}{6}i$. Now we know the Teichmüller parameter of $(\mathrm{Dome}(\Omega)/G; S, T)$ for the group G defined in Theorem 1.1:

Theorem 2.4. *The marked group $G = \langle S, T\rangle$ stated in Theorem 1.1 is conjugate to $\Gamma(\theta_0) = \langle AB^2, B\rangle$ in $PSL(2, \mathbf{C})$. Therefore the Teichmüller parameter of $(\mathrm{Dome}(\Omega)/G; S, T)$ is $\frac{1}{2} + \frac{\sqrt{3}}{6}i$.*

Proof. Both triples $(\mathrm{Tr}\ AB^2, \mathrm{Tr}\ B, \mathrm{Tr}\ AB^3)$ and $(\mathrm{Tr}\ S, \mathrm{Tr}\ T, \mathrm{Tr}\ ST)$ are equal to $(2, -2+i, -2+3i)$. The claim follows from the fact that the triple $(x, y, z) = (\mathrm{Tr}\, A, \mathrm{Tr}\, B, \mathrm{Tr}\, AB)$ determines the marked group $\langle A, B\rangle$ unique up to conjugation in $PSL(2, \mathbf{C})$. □

3. The Teichmüller parameter of Ω/G

The Teichmüller parameter for Ω/G is given by the period

$$\frac{\int_b \zeta}{\int_a \zeta},$$

where $\{\zeta\}$ is a basis for the space of holomorphic abelian differentials on the one-point compactification of Ω/G, and where $\{a, b\}$ forms a canonical homology basis on Ω/G. A method for approximating the period of certain once-punctured tori $\Omega(G_\mu)/G_\mu$ is developed in [Mat01]. Our group G is the group G_μ from [Mat01] for $\mu = 1 + 2i$, but some of the results presented in that paper do not apply unless $\Im(\mu) > 2$. In this section we develop some new results needed in the approximation of the period of Ω/G, and we use those results of [Mat01] that do apply in our case.

After choosing base points $Q_1, Q_2 \in \Omega$, the projection of any pair of paths in Ω from Q_1 to $S(Q_1)$ and from Q_2 to $T(Q_2)$ forms a canonical homology basis $\{a, b\}$ on Ω/G. The use of relative Poincaré series in the construction of holomorphic

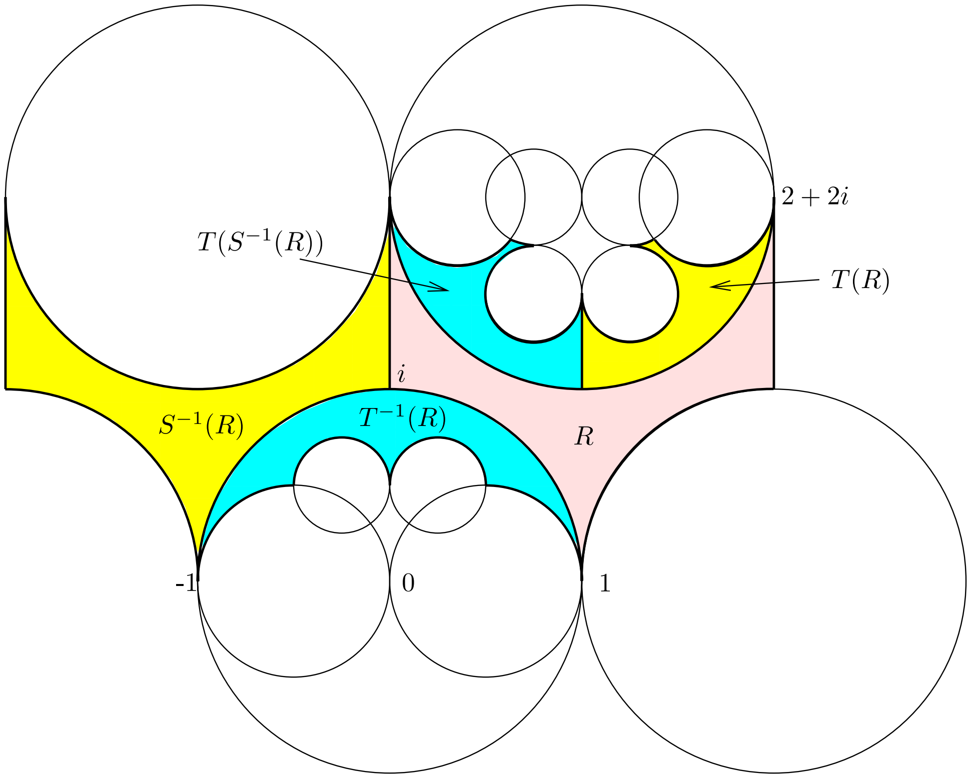

FIGURE 2. Five fundamental domains for G.

forms is well known, and on page 252 of [Mat01] it is shown that the square root of the relative Poincaré series

$$P_2(z) = \sum_{g \in \langle S \rangle \backslash G} g'(z)^2$$

projects to a holomorphic 1-form on the compactification of Ω/G; hence, the period of Ω/G is given by

$$\frac{\int_{Q_2}^{T(Q_2)} \sqrt{P_2(z)}\, dz}{\int_{Q_1}^{S(Q_1)} \sqrt{P_2(z)}\, dz}.$$

Figure 2 shows a fundamental domain R for the action of G in Ω. The line segment from $Q_1 = -1+i$ to $S(Q_1) = 1+i$ is contained in Ω, as is the line segment from $Q_2 = i$ to $T(Q_2) = 1+i$. We will integrate over these line segments in the computation of the period of Ω/G.

There are symmetries in the series $P_2(z)$ which can make this computation a little shorter. First, for all $x \in \mathbf{R}$, $\overline{P_2(i+x)} = P_2(i-x)$ (see Lemma 3.6 and the proof of Proposition 3.9 in [Mat01]). Hence, $\int_{Q_1}^{S(Q_1)} \sqrt{P_2(z)}\, dz$ must be real. Second, the real part of the period of Ω/G must equal 1/2 (see Corollary 3.10 of

[Mat01], for an example). Thus,

$$\int_{Q_1}^{S(Q_1)} \sqrt{P_2(z)}\, dz = 2 \cdot \Re \int_{Q_2}^{T(Q_2)} \sqrt{P_2(z)}\, dz.$$

Also, since the transformation $R(z) = 1 + 2i - z$ conjugates S to S^{-1} and T to T^{-1}, we have the symmetry $P_2(z) = P_2(1 + 2i - z)$, and it follows that

$$\int_{i}^{i+1} \sqrt{P_2(z)}\, dz = 2 \int_{i}^{i+1/2} \sqrt{P_2(z)}\, dz.$$

Therefore the period of Ω/G is $1/2 + y\, i$, where

$$y = \frac{\Im \int_i^{i+1/2} \sqrt{P_2(z)}\, dz}{2 \cdot \Re \int_i^{i+1/2} \sqrt{P_2(z)}\, dz}.$$

In order to obtain an approximation for y, we approximate the infinite series $P_2(z)$ by a finite sum $P_2(z, \varepsilon)$ with an error bound coming from an area argument. Once we have this approximation, we can use the trapezoid rule to obtain approximations for the line integrals. The results in Section 5 of [Mat01] can be applied to compute error bounds for the trapezoid rule approximations to the integrals of $\sqrt{P_2(z)}$, and this gives us an error bound for our approximation to y.

The group G has a natural tree structure obtained by establishing a vertex for each element of G and forming an edge between two vertices g, h if and only if $g^{-1}h \in \{S, S^{-1}, T, T^{-1}\}$. The tree structure of $\langle S \rangle \setminus G$ is then obtained from this tree for G by deleting the branches containing words of which the letters S or S^{-1} is a prefix. The sum $P_2(z, \varepsilon)$ is then constructed by traversing the tree for $\langle S \rangle \setminus G$, adding the terms $g'(z)^2$ for each vertex traversed, and truncating infinite branches at a depth determined by ε. For two words g_1 and g_2, let $g_1 \sqsubseteq g_2$ denote that g_1 is a prefix of g_2. Let $I(g)$ denote the isometric circle of g. Then

$$P_2(z, \varepsilon) = \sum_{g \in H_\varepsilon} g'(z)^2,$$

where H_ε is the subset of $\langle S \rangle \setminus G$ consisting of those g for which:

1. if $g_1 T^{-1} \sqsubseteq g$ and $g_1 T^{-1} \neq g$, then the area inside $g_1(I(T))$ is at least ε;
2. if $g_1 T \sqsubseteq g$ and $g_1 T \neq g$, then the area inside $g_1(I(T^{-1}))$ is at least ε;
3. if $g_1 S^{-1} S^{-1} \sqsubseteq g$, then the area inside $g_1(\{z : \Re z < -2\})$ is at least ε; and
4. if $g_1 SS \sqsubseteq g$, then the area inside $g_1(\{z : \Re z > 3\})$ is at least ε.

Each infinite branch in $\langle S \rangle \setminus G$ that is truncated in the computation of $P_2(z, \varepsilon)$ is one of the branches with prefixes given above such that the area inside the corresponding disk is less than ε. Indeed, any infinite branch $\{g \in \langle S \rangle \setminus G : g_0 \sqsubseteq g\}$ is contained in the union of a finite number of branches of the types listed above. We have not proven that $P_2(z, \varepsilon)$ is a finite sum, but the sum was always finite in our computer experiments.

The area argument used to find an error bound for this approximation to $P_2(z)$ comes from the mean value property for holomorphic functions: If $D(z, r)$ is a disk contained in some fundamental domain for G in Ω and $g \in \langle S \rangle \setminus G$, then $|g'(z)^2|$ is less than or equal to the area of the disk $g(D(z, r))$ divided by πr^2. Before we use the mean value property to get our error bound, we must first prove a few preliminary results.

Proposition 3.1. *Let $g = g_1 g_2 \cdots g_n \in \langle S \rangle \setminus G$, $g_1 = T^{\pm 1}$, $g_i g_{i+1} \neq 1$ for $1 \leq i \leq n-1$, and $g_i \in \{S, S^{-1}, T, T^{-1}\}$ for $2 \leq i \leq n$. Then,*

(i) *if $g_n = T$, then $g^{-1}(\infty)$ is inside $I(T)$;*
(ii) *if $g_n = T^{-1}$, then $g^{-1}(\infty)$ is inside $I(T^{-1})$;*
(iii) *if $g_n = S$, then $g^{-1}(\infty)$ is to the left of $\Re(z) = 0$; and*
(iv) *if $g_n = S^{-1}$, then $g^{-1}(\infty)$ is to the right of $\Re(z) = 1$.*

Proof. We use induction on the length of the word n. If $n = 1$, then $g = T^{\pm 1}$ and the proof is clear. Assume the proposition is true for all words of length $\leq n$; we want to show its truth for words g of length $n+1$. Write $g = g_1 g_2 \cdots g_{n+1}$.

First suppose $g_{n+1} = S$. Then if $g_n = S$, $g^{-1}(\infty)$ is to the left of $\Re(z) = -2$ by the induction hypothesis. If $g_n = T$, then $g_n^{-1} \cdots g_1^{-1}(\infty)$ is in $I(T)$, and so $g^{-1}(\infty)$ is to the left of $\Re(z) = -1$. If $g_n = T^{-1}$, then $g_n^{-1} \cdots g_1^{-1}(\infty)$ is in $I(T^{-1})$, so $g^{-1}(\infty)$ is to the left of $\Re(z) = 0$.

Next suppose $g_{n+1} = T$. Then if $g_n = T$ also, then $g_n^{-1} \cdots g_1^{-1}(\infty)$ is inside $I(T)$ by hypothesis. Since the isometric circles of T and T^{-1} do not intersect, $g^{-1}(\infty)$ is inside $I(T)$.

If $g_n = S$, then $g_n^{-1} \cdots g_1^{-1}(\infty)$ is to the left of $\Re(z) = 0$, which is outside $I(T^{-1})$, and so $g^{-1}(\infty)$ is inside $I(T)$.

If $g_n = S^{-1}$ we must consider the previous letter. If $g_{n-1} g_n = S^{-1} S^{-1}$, then $g_n^{-1} \cdots g_1^{-1}(\infty)$ is to the right of $\Re(z) = 3$. Hence, $g_n^{-1} \cdots g_1^{-1}(\infty)$ is outside $I(T^{-1})$ and $g^{-1}(\infty)$ is inside $I(T)$. If $g_{n-1} g_n = T S^{-1}$, then $g_{n-1}^{-1} \cdots g_1^{-1}(\infty)$ is inside $I(T)$, and $g_n^{-1} \cdots g_1^{-1}(\infty)$ is below the line $\Im(z) = 1$; so $g_n^{-1} \cdots g_1^{-1}(\infty)$ is outside $I(T^{-1})$ and thus $g^{-1}(\infty)$ is inside $I(T)$. If $g_{n-1} g_n = T^{-1} S^{-1}$, then $g_{n-1}^{-1} \cdots g_1^{-1}(\infty)$ is inside $I(T^{-1})$, $g_n^{-1} \cdots g_1^{-1}(\infty)$ is outside $I(T^{-1})$, and $g^{-1}(\infty)$ is inside $I(T)$.

The cases where $g_{n+1} = S_1^{-1}$ and $g_{n+1} = T^{-1}$ can be proven using the symmetry of the transformation $R(z) = 1 + 2i - z$, which fixes ∞ and conjugates S^{-1} to S and T^{-1} to T. First note that

$$R g^{-1}(\infty) = R g_{n+1}^{-1} \cdots g_1^{-1} R(\infty) = g_{n+1} \cdots g_1(\infty).$$

By the preceding arguments, if $g_{n+1} = S^{-1}$, then $R g^{-1}(\infty)$ is to the left of $\Re(z) = 0$. Thus $g^{-1}(\infty)$ is to the right of $\Re(z) = 1$. If $g_{n+1} = T^{-1}$, then $R g^{-1}(\infty)$ is inside $I(T)$, so $g^{-1}(\infty)$ is inside $I(T^{-1})$. □

Corollary 3.2. *Under the hypotheses of Proposition* 3.1*, if $g_n = T$, S, or S^{-1}, then $g^{-1}(\infty)$ is outside $I(T^{-1})$; if $g_n = T^{-1}$, S, or S^{-1}, then $g^{-1}(\infty)$ is outside $I(T)$.*

Proof. Suppose $g_n = T$. Then by Proposition 3.1, $g^{-1}(\infty)$ is inside $I(T)$, which is disjoint from $I(T^{-1})$. By similar reasoning, if $g_n = T^{-1}$, then $g^{-1}(\infty)$ is outside $I(T)$.

If $g_n = S$, we consider the previous letter. If $g_{n-1} g_n = SS$, then by Proposition 3.1, $g^{-1}(\infty)$ is to the left of $\Re(z) = -2$. Hence $g^{-1}(\infty)$ is outside $I(T)$ and $I(T^{-1})$. If $g_{n-1} g_n = TS$, then by Proposition 3.1, $g^{-1}(\infty)$ is to the left of $\Re(z) = -1$; so $g^{-1}(\infty)$ is outside $I(T)$ and $I(T^{-1})$. If $g_{n-1} g_n = T^{-1} S$, then $g^{-1}(\infty)$ is above the horizontal line $\Im(z) = 1$ and to the left of the line $\Re(z) = 0$; so $g^{-1}(\infty)$ is outside both isometric circles.

For the final case $g_n = S^{-1}$ we apply the transformation $R(z) = 1 + 2i - z$ which conjugates S^{-1} to S and T^{-1} to T and which fixes ∞. Now $R g^{-1}(\infty) = g_n \cdots g_1(\infty)$, and by the preceding argument, $g_n \cdots g_1(\infty)$ is outside $I(T)$ and $I(T^{-1})$. Hence, $g^{-1}(\infty)$ is outside $I(T)$ and $I(T^{-1})$ also. □

Let z be any point on the line segment from i to $i + 1/2$, and let D denote the disk centered at z with radius 0.25.

Proposition 3.3. *Let $g = g_1 g_2 \cdots g_n$, where $n \geq 2$ and $g_i \in \{S, S^{-1}, T, T^{-1}\}$ for all i, and where $g_i g_{i+1} \neq 1$ for $1 \leq i \leq n-1$. Then,*

(i) *if $T^{-1} \sqsubseteq g$, then $g(D)$ is contained in $I(T)$;*
(ii) *if $T \sqsubseteq g$, then $g(D)$ is contained in $I(T^{-1})$; and*
(iii) *if $S \sqsubseteq g$ or $S^{-1} \sqsubseteq g$, then $g(D)$ is outside $I(T)$ and $I(T^{-1})$.*

Proof. We use induction on n. The basis step when $n = 2$ is straightforward to check computationally. Assume the statement is true for all words of length $\leq n$, and let $g = g_1 g_2 \cdots g_{n+1}$.

If $g_1 = T^{-1}$, then by hypothesis $g_2 \cdots g_{n+1}(D)$ is outside $I(T^{-1})$, so $g(D)$ is inside $I(T)$. Likewise, if $g_1 = T$, then by hypothesis $g_2 \cdots g_{n+1}(D)$ is outside $I(T)$, so $g(D)$ is inside $I(T^{-1})$.

Let $g_1 = S$. Then $g_j = S$ for $2 \leq j \leq n+1$, or $g_j = S$ for $2 \leq j \leq n$ and $g_{n+1} = T^{\pm 1}$, or there is an index $j_0 < n+1$ such that $g_j = S$ for $1 \leq j < j_0$ and $g_{j_0} = T^{\pm 1}$. In the first two cases it is easy to see that $g(D)$ is outside both $I(T)$ and $I(T^{-1})$. In the third case it follows from the induction hypothesis that $g_{j_0} \cdots g_{n+1}(D)$ is inside $I(T)$ or $I(T^{-1})$, and so $g(D)$ is outside $I(T)$ and $I(T^{-1})$.

The proof for the case $g_1 = S^{-1}$ is similar. □

Theorem 3.4. *Let z be any point on the line segment from i to $i + 1/2$. Then:*

(1) *If $H = \{g \in \langle S \rangle \setminus G : g_1 T^{-1} \sqsubseteq g$ and $g_1 T^{-1} \neq g\}$, then*

$$\sum_{g \in H} |g'(z)^2| < \frac{1}{\pi(.25)^2} \cdot \mathit{Area}(g_1(I(T))).$$

(2) *If $H = \{g \in \langle S \rangle \setminus G : g_1 T \sqsubseteq g$ and $g_1 T \neq g\}$, then*

$$\sum_{g \in H} |g'(z)^2| < \frac{1}{\pi(.25)^2} \cdot \mathit{Area}(g_1(I(T^{-1}))).$$

(3) *If $H = \{g \in \langle S \rangle \setminus G : g_1 S^{-1} S^{-1} \sqsubseteq g\}$, then*

$$\sum_{g \in H} |g'(z)^2| < \frac{1}{\pi(.25)^2} \cdot \mathit{Area}(g_1(\{z : \Re z < -2\})).$$

(4) *If $H = \{g \in \langle S \rangle \setminus G : g_1 S S \sqsubseteq g\}$, then*

$$\sum_{g \in H} |g'(z)^2| < \frac{1}{\pi(.25)^2} \cdot \mathit{Area}(g_1(\{z : \Re z > 3\})).$$

Proof. Let z be any point on the line segment from i to $i + 1/2$, and let D denote the disk centered at z with radius 0.25.

First consider $H = \{g \in \langle S \rangle \setminus G : g_1 T^{-1} \sqsubseteq g$ and $g_1 T^{-1} \neq g\}$, where g_1 is fixed. For $g \in H$, write $g = g_1 T^{-1} h$. By Proposition 3.3, $T^{-1} h(D)$ lies inside $I(T)$. By Corollary 3.2, $g_1^{-1}(\infty)$ lies outside $I(T)$; so g_1 takes the inside of $I(T)$ to the inside of $g_1(I(T))$. Thus, for all $g \in H$, $g(D)$ is inside $g_1(I(T))$. The radius of the disk D was chosen small enough that D is contained in a fundamental domain for G, so the images $g(D)$ are disjoint. It follows from the mean value property for holomorphic functions that $\sum_H |g'(z)^2|$ is less than $1/(\pi(0.25)^2)$ times the area inside $g_1(I(T))$.

Next suppose $H = \{g \in \langle S \rangle \setminus G : g_1 T \sqsubseteq g$ and $g_1 T \neq g\}$ where g_1 is fixed. For $g \in H$, write $g = g_1 T h$. By Proposition 3.3, $Th(D)$ lies inside $I(T^{-1})$. By

Corollary 3.2, $g_1^{-1}(\infty)$ lies outside $I(T^{-1})$; so g_1 takes the inside of $I(T^{-1})$ to the inside of $g_1(I(T^{-1}))$. Thus, for all $g \in H$, $g(D)$ is inside $g_1(I(T^{-1}))$. By the mean value property for holomorphic functions, $\sum_H |g'(z)^2|$ is less than $1/(\pi(0.25)^2)$ times the area inside $g_1(I(T^{-1}))$.

Now let $H = \{g \in \langle S\rangle \setminus G : g_1 S^{-1} S^{-1} \sqsubseteq g\}$. For $g = g_1 S^{-1} S^{-1} h \in H$, the set $h(D)$ is to the left of the vertical line $\Re(z) = 2$ by Proposition 3.3. Hence the set $S^{-1}S^{-1}h(D)$ is to the left of the line $\Re(z) = -2$. Since the word g_1 does not end in the letter S, Proposition 3.1 guarantees that $g_1^{-1}(\infty)$ is inside $I(T)$, inside $I(T^{-1})$, or to the right of $\Re(z) = 1$; so g_1 takes the set

$$D_1 = \{z : \Re(z) < -2\}$$

to the inside of $g_1(D_1)$. Thus $\sum_H |g'(z)^2|$ is less than $1/(\pi(0.25)^2)$ times the area inside the disk $g_1(D_1)$.

As a final case, consider $H = \{g \in \langle S\rangle \setminus G : g_1 SS \sqsubseteq g\}$. For $g = g_1 SSh \in H$, the disk $h(D)$ is to the right of the vertical line $\Re(z) = -1$ by Proposition 3.3. Hence, the set $SSh(D)$ is inside the half space

$$D_2 = \{z : \Re(z) > 3\}.$$

Since the word g_1 does not end in the letter S^{-1}, Proposition 3.1 guarantees that $g_1^{-1}(\infty)$ is not inside D_2; so g_1 takes D_2 to the inside of the disk $g_1(D_2)$. Hence $\sum_H |g'(z)^2|$ is less than $1/(\pi(0.25)^2)$ times the area inside the disk $g_1(D_2)$. □

The error bound for the approximation $P_2(z, \varepsilon)$ to $P_2(z)$ is the sum of all of the bounds on $\displaystyle\sum_{g\in H} |g'(z)^2|$ over all of the infinite branches H that are truncated when computing $P_2(z, \varepsilon)$; these error bounds are of the four types specified in Theorem 3.4.

A computer program was used to do these approximations with the truncation level $\varepsilon = 2 \times 10^{-12}$ at 12000 points along the line segment from i to $i + 1/2$ for use with the trapezoid rule. The program was written in FORTRAN and ran on a Linux machine with a 1.7 GHz, Pentium 4 processor for about 9 days. The maximum error bound for the approximation $P_2(z, 2 \times 10^{-12})$ to $P_2(z)$ was less than 0.001505, and there were about 217,000,000 terms in the finite sum. After applying the results of Section 5 in [Mat01] (which give a formula for the error bound on the approximation of the period of Ω/G as a function of the error bound on the approximation $P_2(z, \varepsilon)$ of $P_2(z)$), the final output of the program was that the value of y is approximately 0.143707391, and with error bound analysis, y must be between 0.143223349 and 0.144192163.

4. Proof of Lemma 1.2

In order to prove that $K(G) = K_{eq}(\Omega)$, we start by finding Möb(Ω). Let $R(z) = 1 + 2i - z$. Then we have the following:

Proposition 4.1. *G is a subgroup of index 2 in Möb(Ω) and R represents the non-trivial coset.*

Our proof of Proposition 4.1 will follow from several lemmas.

Lemma 4.2. *$G = \langle S, T\rangle$ and R are contained in Möb(Ω).*

Proof. Since Ω is the invariant component of G, $G \subset$ Möb(Ω). Since $R = R^{-1}$ and $RGR = G$, $G(R(\Omega)) = R(\Omega)$. Since Ω is the unique invariant component of G, $R(\Omega) = \Omega$ and so $R \in$ Möb(Ω). □

Let $\Omega(G)$ denote the region of discontinuity of G.

Lemma 4.3. *For any $g \in$ Möb(Ω) and any component Δ of $\Omega(G)$, $g(\Delta)$ is also a component of $\Omega(G)$.*

Proof. Since g preserves the invariant component Ω of G, g also preserves the boundary $\partial\Omega$ which is the limit set of G. Hence g preserves components of $\Omega(G)$. □

Consider the following non-invariant components of $\Omega(G)$:

$$\mathbf{L} := \{z \in \mathbf{C} | \mathrm{Im}\, z < 0\},\ \mathbf{U} := \{z \in \mathbf{C} | \mathrm{Im}\, z > 2\}.$$

We remark that $\mathbf{L}$ and $\mathbf{U}$ are tangent at ∞ which is the parabolic fixed point of S, and the involution R interchanges $\mathbf{L}$ and $\mathbf{U}$.

Lemma 4.4. *For any $g_1 \in$ Möb(Ω), there exists $g_2 \in G$ such that $g_2^{-1} g_1(\mathbf{L}) = \mathbf{L}$ and $g_2^{-1} g_1(\mathbf{U}) = \mathbf{U}$, or $g_2^{-1} g_1(\mathbf{L}) = \mathbf{U}$ and $g_2^{-1} g_1(\mathbf{U}) = \mathbf{L}$.*

Proof. Fix a Fuchsian once-punctured torus group $\Gamma = \langle A, B \rangle$. It is well known that G is a cusp boundary point of the quasiconformal deformation space of Γ, occurring at the point at which the generator A becomes accidentally parabolic (see [KS93], [KS04]). This implies that the thrice-punctured sphere $(\Omega(G) - \Omega)/G$ is obtained from the once-punctured torus $\mathbf{H^2}/\Gamma$ by pinching the simple closed geodesic corresponding to A. From this point of view, there is a natural correspondence between non-invariant components of the region of discontinuity of G and connected components of the complement of the Γ-orbit of the hyperbolic axis of A in $\mathbf{H^2}$. Any adjacent pair of components of the complement of the Γ-orbit of the hyperbolic axis of A in $\mathbf{H^2}$ is the image of the adjacent pair of components sharing the axis of A as their common boundary, by some element of Γ. Hence any pair of non-invariant components of the region of discontinuity of G which are tangent to each other is an image of the pair $\mathbf{L}$ and $\mathbf{U}$ sharing the point ∞ on their boundaries, under some element of G. Now the claim follows from Lemma 4.3 and the remarks following that lemma. □

First we consider the case that $g_2^{-1} g_1(\mathbf{L}) = \mathbf{L}$ and $g_2^{-1} g_1(\mathbf{U}) = \mathbf{U}$. Since $g_2^{-1} g_1$ preserves $\mathbf{L}$, $g_2^{-1} g_1$ is an element of $PSL(2, \mathbf{R})$ and fixes ∞, the tangent point of $\mathbf{L}$ and $\mathbf{U}$. Moreover $g_2^{-1} g_1$ preserves the boundary of $\mathbf{U}$, which implies that $g_2^{-1} g_1$ is a translation along the real line. Therefore there is a constant $c \in \mathbf{R}$ such that $g_2^{-1} g_1(z) = z + c$. In practice, c is a multiple of 2:

Lemma 4.5. *$g_2^{-1} g_1 = S^n$ for some $n \in \mathbf{Z}$.*

Proof. Since $\mu = 1 + 2i$ is on the 1/2 pleating ray in the Maskit slice, there is the 1/2-circle chain in Ω (see [KS93]). Since $g_2^{-1} g_1(z) = z + c$ preserves Ω, it also preserves the 1/2-circle chain in Ω, which forces c to be $2n$ ($n \in \mathbf{Z}$). □

Therefore $g_1 = g_2 S^n \in G$ in this case. For the other case that $g_2^{-1} g_1(\mathbf{L}) = \mathbf{U}$ and $g_2^{-1} g_1(\mathbf{U}) = \mathbf{L}$, considering the action of R, we can show that $g_1 = g_2 S^n R \in GR$. This proves Proposition 4.1.

Proof of Lemma 1.2. Suppose that f is the extremal G-equivariant quasiconformal map from Ω to Dome(Ω). Then f induces the Teichmüller map from Ω/G to Dome(Ω)$/G$. The map RfR induces a map from Ω/G to Dome(Ω)$/G$ which is homotopic to the Teichmüller map and whose maximal dilatation is the same as that of the Teichmüller map induced by f. Because the Teichmüller map is uniquely extremal, it follows that $RfR = f$. Therefore, by Proposition 4.1, we conclude that $K(G) = K_{eq}(\Omega)$. □

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