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# NON-PERSISTENTLY RECURRENT POINTS, QC-SURGERY AND INSTABILITY OF RATIONAL MAPS WITH TOTALLY DISCONNECTED JULIA SETS

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Abstract. Let $R$ be a rational map with a totally disconnected Julia set $J(R)$. If the postcritical set on $J(R)$ contains a non-persistently recurrent (or conical) point, then we show that the map $R$ cannot be a structurally stable map.

## Introduction and statements

Fatou's problem of the density of hyperbolic maps in the space of rational maps is one of the principal problems in the field of holomorphic dynamics. Due to Mané, Sad and Sullivan [MSS], we can reformulate this problem in the following way:

*If the Julia set $J(R)$ contains a critical point, then the rational map $R$ is a structurally unstable map.*

For convenience we give the definition of the structural stability of a rational map. For other basic notations and definitions we refer to the book of Milnor [M].

**Definition 1.** Let $Rat_d$ be the space of all rational maps of degree $d$ with the topology of coefficient convergence. A map $R \in Rat_d$ is called structurally stable if there exists a neighborhood $U \subset Rat_d$ of $R$ such that:

For any map $R_1 \in U$ there exists a quasiconformal map $f : \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}}$ conjugating $R$ to $R_1$.

We give a condition, Assumptions "G" (see below), on a rational map with totally disconnected Julia set and with a critical point on $J(R)$ to be unstable. In the pioneer paper [BH], Branner and Hubbard prove that the Lebesgue measure of the Julia set is zero if there exists only one critical point on $J(R)$. Our result (see Theorem A below) restricted to the Branner–Hubbard case is weaker, but it can be applied for maps with two or more critical points on $J(R)$.

Let $R$ be a rational map with a totally disconnected Julia set. Let us normalize $R$ so that the point $z = \infty$ becomes the attractive fixed point. Let $Pc(R)$ be a postcritical set of the map $R$ and $P(R) = Pc(R) \cap J(R)$ be a postcritical set on the Julia set. Let $S = \mathbb{C} \backslash \overline{\bigcup_n R^{-n}(Pc(R))}$, then $R : S \to S$ is an unbranched autocovering.

**Definition 2.** We say that a closed simple geodesic $\gamma \subset S$ is linked with $P(R)$ if the interior $I(\gamma)$ of $\gamma$ intersects $P(R)$.

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**Assumptions "G".** Let $R$ be a rational map with totally disconnected Julia set. Assume that there exists a simple closed geodesic $\gamma \in S$ such that:

(1) there exists an infinite subsequence of simple closed geodesics $\gamma_i \in \bigcup_n R^{-n}(\gamma)$ linked with the postcritical set $P(R)$, and
(2) for all $i = 1, ...$ the lengths $L(\gamma_i)$ are bounded uniformly away from $\infty$, in the hyperbolic metric on $S$.

The aim of this paper is to prove the following theorem.

**Theorem A.** *Let $R$ be a rational map with a totally disconnected Julia set satisfying Assumptions* "G"*. Then the map $R$ is not a structurally stable map (that is to say, it is an unstable map).*

A priori it is not clear when Assumptions "G" holds. We give a series of sufficient conditions on $R$ that imply Assumptions "G". The next proposition translates Assumptions "G" into the language of "non-persistently recurrent points" on $P(R)$.

**Definition 3.** A point $x \in P(R)$ is called persistently recurrent if any backward orbit $U_0, U_{-1}, ...$ of any neighborhood $U_0$ of $x$ along $P(R)$ hits a critical point infinitely many times.

**Lemma** (Sufficient condition)**.** *Let $R$ be a rational map with a totally disconnected Julia set and assume that there exists a non-persistently recurrent point $x \in P(R)$. Then $R$ satisfies Assumptions* "G"*.*

*Proof.* Follows immediately from Definition 3. □

Another sufficient condition is connected with the conical points of $P(R)$.

**Definition 4.** Let $R$ be a rational map, and denote by $U(x_0, R^k, \delta)$ the component of $R^{-k}(\mathbb{D}(R^k(x_0), \delta))$ that contains $x_0$, where $\mathbb{D}(R^k(x_0), \delta)$ is the disk centered in $R^k(x_0)$ with radius $\delta$. A point $x_0$ is conical if and only if there is a constant $\delta > 0, d \in \mathbb{N}$, and a sequence $k_j \to \infty$ such that

$$R^{k_j} : U(x_0, R^k, \delta) \mapsto \mathbb{D}(R^k(x_0), \delta)$$

has degree no more than $d$.

Several other notions of conical points appear in the literature. One can see that the definition of a conical point is somehow in the spirit of the notion of the conical set of Lyubich and Minsky [LM]. Definition 4 above appears in [P], where Przytycki compares different notions of conical points. McMullen [MM] and, independently, Urbanski [DMNU] call a point conical if the mappings in Definition 4 can be chosen to be conformal.

**Theorem 1.** *Let $R$ be a rational map with a totally disconnected Julia set and assume that there exists a conical point $x \in P(R)$. Then $R$ is an unstable map.*

The following two results are immediate corollaries of Theorem 1.

**Corollary 1.** *Let $R$ be a rational map with a totally disconnected Julia set and assume that the postcritical set $P(R)$ contains a periodic point $x$. Then $R$ is an unstable map.*

*Proof.* By assumption, the periodic point $x \in J(R)$. Hence $x$ is either parabolic or repelling. Now assume that $R$ is a structurally stable map, then $x$ should be repelling and hence conical. Applying Theorem 1 we are done. □

**Corollary 2.** *Let $R$ be a rational map with a totally disconnected Julia set. Assume $J(R) = P(R)$, then $R$ is an unstable map.*

*Proof.* In this case $P(R)$ contains all repelling periodic points and by Corollary 1 we are done. □

## Proof of Theorem A

To prove Theorem A, we use quasiconformal surgery in the spirit of Shishikura [Sh].

Let $\Delta(r)$ be a disk of radius $r$ centered at $z = 0$ and $\Delta = \Delta(1)$. Let $A(p, q) = \{z : p < |z| < q\}$ be a ring. Let $A(p) = A(p, 1), p < 1$ and $a = \frac{1+3p}{4} \in A(p, \frac{1+p}{2}) \subset A(p)$ be a point. Now we define a quasiconformal homeomorphism $f_p : \Delta \mapsto \Delta$ as follows:

(1) $f_p(z) = \frac{z+a}{1+az}$ on $\Delta(p)$ and
(2) $f_p$ is a quasiconformal mollifier on $A(p)$, that is
  (i) $f_p = id$ on $\partial\Delta$ and
  (ii) $f_p = \frac{z+a}{1+az}$ on the other boundary component of $A(p)$ and
  (iii) the $L_\infty$-norm of the dilatation $\mu = \frac{\overline{\partial} f_p}{\partial f_p}$ is minimal among all dilatations of the quasiconformal homeomorphisms satisfying (i)–(ii).

*Remark* 1. Note that the $L_\infty$-norm of $\mu$ depends only on the modulus of the ring $A(p)$, or in other words, if $p_i \in \Delta$ converges to $p_0 \in \Delta$, then the $L_\infty$-norms of the dilatations $\mu_i$ are uniformly bounded away from 1.

According to the results of D. Sullivan [S], C. McMullen and D. Sullivan [MS], the space of full orbits of the points on $S$ forms a Riemann surface $S(R)$ which is (conformally) the torus with a finite number of punctures: The punctures correspond to the full orbits of the critical points belonging to $F(R)$. Hence there exists a fundamental domain $F \subset F(R)$ for the action $R$ on $F(R)$. We can choose the fundamental domain as follows:

(1) $F \subset F(R)$ is a closed topological ring;
(2) the boundary components $\alpha_1 \cup \alpha_2 = \partial F \subset S$ are smooth closed Jordan curves;
(3) $R(\alpha_1) = \alpha_2$ and restrictions $R^n_{|F}$ are univalent for all $n$.

Let $O(F)$ be the full orbit of the fundamental domain $F$. Let $\alpha \subset S$ be any geodesic, then $\alpha$ intersects a finite number, say $n(\alpha)$, of elements of $O(F)$, say $F_1(\alpha), ..., F_{n(\alpha)}(\alpha)$.

*Remark* 2. By the properties of the fundamental domain we can always assume that there exists $i_0$ so that the forward orbit $O_+(F_{i_0}(\alpha)) = \bigcup_{i\geq 1} R^i(F_{i_0}(\alpha))$ never intersects the interior of the geodesic $\alpha$. For convenience we redefine $F_1(\alpha) = F_{i_0}(\alpha)$.

Let $B(\alpha) \subset S \cap \{\bigcup_{i=1}^{n(\alpha)} F_i(\alpha)\}$ be an annulus containing $\alpha$ as a non-trivial curve with modulus $m(\alpha)$ of $B(\alpha)$ as large as possible. Note that $B(\alpha)$ is not unique. Now let $\beta \subset S$ be an iterated preimage of $\alpha$ (that is, there exists an integer $k$ such that $R^k(\beta) = \alpha$). If $d(\beta)$ is the degree of the covering $R^k : \beta \mapsto \alpha$, then the hyperbolic length $l(\beta) = d(\beta)l(\alpha)$, and $m(\beta) \geq \frac{m(\alpha)}{d(\beta)}$, as well as $n(\beta) \leq d(\beta)n(\alpha)$.

Let us start with any closed simple geodesic $\gamma \in S$ linked with $P(R)$. Now we associate a qc-homeomorphism $f(\gamma, p) : \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}}$ as follows:

Let $I(\gamma)$ be the interior of $\gamma$ and the point $b$ be the first hit in $I(\gamma)$ of the forward orbit of a critical point $c \notin I(\gamma)$. Let $h : I(\gamma) \mapsto \Delta$ be the Riemann map with $h(b) = 0, h'(b) = 1$. Now, let $p > 0$ be a number so that $A(p) \subset h(B(\gamma))$. Adjusting $h$ by a rotation we can construct a conformal map $\phi(\gamma) : I(\gamma) \mapsto \Delta$ so that the point $a = \frac{1+3p}{4} \in \phi(F_1(\gamma))$. Then we set

$$f(\gamma, p) = \begin{cases} \phi(\gamma)^{-1} \circ f_p \circ \phi(\gamma) \text{ on } I(\gamma), \\ id \text{ off of } I(\gamma). \end{cases}$$

Hence, for any simple closed geodesic $\gamma \subset S$ and a suitable number $0 < p < 1$ we can define a quasi-regular map $P(\gamma, p) = f(\gamma, p) \circ R : \overline{\mathbb{C}} \mapsto \overline{\mathbb{C}}$.

**Lemma 1.** *Let $\gamma \subset S$ be a closed simple geodesic linked with $P(R)$ and $1 > p > 0$ be a suitable number. Then*

1. *there exists an invariant conformal structure $\sigma$ on $\overline{\mathbb{C}}$ so that $P(\gamma, p) : (\overline{\mathbb{C}}, \sigma) \mapsto (\overline{\mathbb{C}}, \sigma)$ is a holomorphic map,*
2. *the norm of the dilatation of $\sigma$ (that is, $L_\infty$-norm of the corresponding Beltrami differential) depends only on the numbers $p$ and $n(\gamma)$.*

*Proof.* Follows immediately from the definition of $f(\gamma, p)$. □

By the Riemann Mapping Theorem there exists a quasi-conformal homeomorphism $f_\sigma$ fixing the points $0, 1$ and $\infty$ so that $R(\gamma, p) = f_\sigma \circ R \circ f_\sigma^{-1}$ is a rational map.

Let $s(R)$ be the number of critical points whose forward orbits converge to $\infty$.

**Corollary 1.** *Let $\gamma$ and $p$ be as in Lemma 1 above. Assume that $\gamma$ has a sufficiently small spherical diameter. Then $s(R(\gamma, p)) \geq s(R) + 1$.*

*Proof.* Let the spherical diameter of $\gamma$ be so small that the interior $I(\gamma)$ does not contain any critical point of the Fatou set $F(R)$. Hence, if the critical point $c \in F(R)$, then $P^n(\gamma, p)(c) = R^n(c) \to \infty$. Now let $c \in J(R)$ be the critical point coming from the definition of $f(\gamma, p)$. Then again by the construction, $P^n(\gamma, p)(c) \to \infty$. □

Now we are ready to prove Theorem A. Let $\gamma_i \in \bigcup_n R^{-n}(\gamma)$ be the geodesics of the assumption. Let us redefine:

1. $\phi_i = \phi(\gamma_i)$,
2. $P_i = P(\gamma_i, p_i)$ and $R_i = R(\gamma_i, p_i)$,
3. $f_i = f_{\sigma_i}$ and hence $R_i = f_i \circ P_i \circ f_i^{-1}$.
4. Let $\nu_i$ be the Beltrami differentials of the structures of $\sigma_i$ respectively.

Now our aim is to show that there exists a subsequence $\{\nu_{i_j}\}$ with norms uniformly bounded away from 1. By Remark 1 and Lemma 1 it is enough to show that we can choose a subsequence $p_{i_j}$ uniformly bounded away from 1.

Let $k_i$ be integers so that $R^{k_i}(\gamma_i) = \gamma$. Let $B_i$ be a component of $R^{-k_i}(B(\gamma))$ containing the geodesic $\gamma_i$. Then by our assumptions there exists a constant $C$ so that the moduli $m(B_i) = \frac{m(B(\gamma)}{d(\gamma_i)} \geq C > 0$ are uniformly bounded. Let $A_i = \phi_i(B_i)$.

**Lemma 2.** *There exists a subsequence $\{i_j\}$ and a number $p < 1$ so that $A(p) \subset A_{i_j}$ for any $j$.*

*Proof.* The argument is simple. Let $A_{i_0}$ be any annulus of minimal modulus. Then there exist conformal injections $h_i : A_{i_0} \mapsto A_i$ such that $h_i(\partial\Delta) = \partial\Delta$ and $h_i(1) = 1$. The family $\{h_i\}$ is normal so let $\{h_{i_j}\}$ be a convergent subsequence. Then the limit map $h_\infty \neq const$. Now, let $q < 1$ be so that $A(q) \subset A_{i_0}$. Then by the reflection principle $\{h_{i_j}\}$ converges to $h_\infty$ uniformly on $A(\frac{q+1}{2})$. Hence, $h_\infty(A(\frac{q+1}{2})) \subset h_{i_j}(A_{i_0})$ for all large enough $j$. Let $p < 1$ be an integer so that $A(p) \subset h_\infty(A(\frac{q+1}{2}))$; then by the discussion above

$$A(p) \subset h_{i_j}(A_{i_0}) \subset A_{i_j}$$

for all large $j$. The lemma is thus proved. □

By Remark 1, Lemma 1 and Lemma 2 we have that the family of quasiconformal homeomorphisms $\{f_{i_j}\}$ is normal, and after passing to a subsequence we can assume that $\{f_{i_j}\}$ converges to a quasi-conformal homeomorphism $f_\infty$. The Julia set $J(R)$ is a Cantor set, hence, the spherical diameter $diam(\gamma_i) \to 0$. Then the homeomorphisms $f(\gamma_{i_j}, p)$ converge to the identity uniformly on $\overline{\mathbb{C}}$, and, hence, $P_{i_j} \to R$.

After passing again to a subsequence we can assume that $lim_{j\to\infty} R_{i_j} = R_\infty$, where $R_\infty$ is a rational map of degree smaller or equal to the degree of $R$. Then we can pass to the limit in the following equality:

$$f_{i_j} \circ P_{i_j} \circ f_{i_j}^{-1} = R_{i_j} \to f_\infty \circ R \circ f_\infty^{-1} = R_\infty.$$

Now, to obtain a contradiction assume that $R$ is a structurally stable map. Then $R_\infty$ is structurally stable (being a qc-deformation of $R$) and $s(R) = s(R_\infty)$. By construction, $R_\infty = lim_{j\to\infty} R_{i_j}$ and, by Corollary 1, $s(R_{i_j}) \geq s(R)+1 = s(R_\infty)+1$ which contradicts the structural stability of $R_\infty$.

## Proof of Theorem 1

Here we show that the existence of a conical point $x \in P(R)$ implies Assumptions "G".

**Lemma 3.** *Assume that $R$ satisfies the assumptions of Theorem* 1*. Then $R$ satisfies Assumptions* "G"*.*

*Proof.* Let $x_0 \in P(R)$ be a conical point. Let integer $d$ and a sequence $\{k_j\}$ be as in the definition of the conical point. Let $U(x_0, R^{k_j}, \delta)$ be the component of $R^{-k}(\mathbb{D}(R^{k_j}(x_0), \delta))$ that contains $x_0$, where $\mathbb{D}(R^k(x_0), \delta)$ is the disk centered in $R^k(x_0)$ with the radius $\delta$. After passing to a subsequence we can assume that the sequence $R^{k_j}(x_0)$ converges to a point $y \in P(R)$. The disks $\mathbb{D}(R^{k_j}(x_0), \delta)$ converge as well as to the disk $\mathbb{D}(y, \delta)$. Now, let $\gamma \subset S \cap \mathbb{D}(y, \frac{\delta}{2})$ with $y \in I(\gamma)$. Then for large $j$ we have:

(1) $R^{k_j}(x_0) \in I(\gamma)$,
(2) there exists a geodesic $\gamma_j \in R^{-k_j}(\gamma) \cap U(x_0, R^{k_j}, \delta)$, so that $x_0 \in I(\gamma_j)$, and
(3) the hyperbolic length $L(\gamma_j) \leq dL(\gamma)$.

The lemma is proved. □

An application of Theorem A completes the proof of Theorem 1.

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