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FORMAL ADJOINTS AND A CANONICAL FORM FOR LINEAR OPERATORS

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ABSTRACT. We describe a canonical form for linear differential operators that are formally self-adjoint or formally skew-adjoint.

Suppose E and F are smooth vector bundles on an oriented smooth manifold M. Let vol denote the bundle of volume forms on M. The *formal adjoint* of a linear differential operator $L : E \to F$ is the differential operator $L^* : F^* \otimes vol \to E^* \otimes vol$ characterised by the equation

$$\int_M \langle L^*\sigma, \tau\rangle = \int_M \langle \sigma, L\tau\rangle \quad \text{for } \sigma \in \Gamma(M, F^* \otimes vol) \text{ and } \tau \in \Gamma_*(M, E).$$

Here all sections are assumed sufficiently smooth and Γ_* indicates the space of compactly supported sections. If $F = E^* \otimes vol$, then $L^* : E \to F$ and there is a canonical decomposition

$$L = L_+ + L_- = \tfrac{1}{2}[L + L^*] + \tfrac{1}{2}[L - L^*]$$

into self-adjoint and skew-adjoint parts. Henceforth let us assume that M is equipped with a preferred volume form and a compatible torsion-free connection ∇ (e.g. M is Riemannian).

Suppose that E is trivial. Then the preferred volume form trivialises F, and so L may be viewed as taking functions to functions and written in terms of the given connection. A formula for its adjoint is determined by integration by parts. Suppose, for example, that L is second order. Then we may write

$$L = S^{ab}\nabla_a\nabla_b + T^b\nabla_b + R,$$

where the tensor S^{ab} is symmetric. Adopting the convention that ∇_a acts on everything to its right, we can re-express L in the form

$$L = \nabla_a S^{ab}\nabla_b + (\tilde{T}^b\nabla_b + \nabla_b\tilde{T}^b) + \tilde{R}$$

where the tensor $\tilde{T}$ and scalar field $\tilde{R}$ are given by $\tilde{T}^b = \frac{1}{2}(T^b - (\nabla_a S^{ab}))$ and $\tilde{R} = R - (\nabla_b\tilde{T}^b)$. This is congenial since clearly

$$L^* = \nabla_a S^{ab}\nabla_b - (\tilde{T}^b\nabla_b + \nabla_b\tilde{T}^b) + \tilde{R}.$$

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In particular,

$$L_+ = \nabla_a S^{ab} \nabla_b + \tilde{R} \quad \text{and} \quad L_- = \tilde{T}^b \nabla_b + \nabla_b \tilde{T}^b.$$

By an obvious inductive argument this generalises immediately to give the following result.

Theorem 1. *A self-adjoint kth order linear differential operator taking functions to functions on M has even order and may be canonically written in the form:*

$$\sum_{i=0}^{k/2} \underbrace{\nabla_a \cdots \nabla_b}_{i} S_{(i)}^{a\cdots bc\cdots d} \underbrace{\nabla_c \cdots \nabla_d}_{i},$$

for suitable symmetric tensors $S_{(i)}^{a\cdots d}$. *A skew-adjoint kth order linear differential operator taking functions to functions on M has odd order and may be canonically written in the form:*

$$\sum_{i=0}^{(k-1)/2} \underbrace{\nabla_a \cdots \nabla_b}_{i} (\nabla_c A_{(i)}^{a\cdots bcd\cdots e} + A_{(i)}^{a\cdots bcd\cdots e} \nabla_c) \underbrace{\nabla_d \cdots \nabla_e}_{i},$$

for suitable symmetric tensors $A_{(i)}^{a\cdots e}$.

Suppose now that E is a vector bundle (possibly a tensor bundle) with connection on M and write ∇ to indicate the coupled tensor-vector bundle connection. Let us use upper case Greek indices as abstract indices for the bundle E and its dual. Then, for example, operators $L : E \to E^*$ may be written $L_{\Psi\Phi} : E^\Phi \to E_\Psi$. Since the tensor product $E^* \otimes E^*$ decomposes canonically into symmetric and skew-symmetric parts, it follows easily that the above generalises as follows. We write $[\ell]$ to indicate the integer part of a real number ℓ.

Theorem 2. *A self-adjoint (respectively skew-adjoint) kth order linear differential operator* $L_{\Psi\Phi} : E^\Phi \to E_\Psi$ *on M may be canonically written in the form:*

$$\begin{aligned}&\textstyle\sum_{i=0}^{[k/2]} \underbrace{\nabla_a \cdots \nabla_b}_{i} S_{\Psi\Phi\ (i)}^{a\cdots bc\cdots d} \underbrace{\nabla_c \cdots \nabla_d}_{i} \\ &\quad + \textstyle\sum_{i=0}^{[(k-1)/2]} \underbrace{\nabla_a \cdots \nabla_b}_{i} (\nabla_c A_{\Psi\Phi\ (i)}^{a\cdots bcd\cdots e} + A_{\Psi\Phi\ (i)}^{a\cdots bcd\cdots e} \nabla_c) \underbrace{\nabla_d \cdots \nabla_e}_{i},\end{aligned}$$

where the sections $S_{\Psi\Phi\ (i)}^{a\cdots d}$ *are symmetric over the tensor indices and symmetric (respectively skew-symmetric) over the pair* $\Psi\Phi$ *of vector bundle indices; the sections* $A_{\Psi\Phi\ (i)}^{a\cdots e}$ *are symmetric over the tensor indices and skew-symmetric (respectively symmetric) over the pair* $\Psi\Phi$ *of vector bundle indices.*

Here is an example application. Suppose that P is a self-adjoint linear differential operator P of order $k > 0$ that takes functions to functions on M. Write $\tilde{P}$ for the modified operator obtained by subtracting from P the scalar part obtained by applying P to $f \equiv 1$. Then $\tilde{P}$ is formally self-adjoint and takes the form $f \mapsto \nabla_a(Q^{ab}(\nabla_b f))$ for a suitable differential operator $Q : T^*M \to TM$. Since the construction of $\tilde{P}$ is canonical, $\tilde{P}$ enjoys the same naturality and/or invariance properties as does P.

A good example arises in conformal geometry. Suppose M is an oriented conformal manifold of even dimension $2m$. Let L denote any conformally invariant operator on functions that, with respect to any Riemannian metric in the conformal class, takes the form

$$L = \Delta^m + \text{lower order terms.} \tag{1}$$

Since L takes functions to volume forms, so does its formal adjoint. The self-adjoint part L_+ of L is therefore also conformally invariant. As a conformal analogue of Δ^m (in the sense of [3]), we may as well replace L by L_+. Then by Theorem 1 this, in turn, may be replaced by a self-adjoint conformally invariant modification of the form $f \mapsto \nabla_a(Q^{ab}(\nabla_b f))$ for a suitable $(n-2)$nd order differential operator $Q : \Lambda^1 \to \Lambda^{n-1}$. Thus, we obtain the following result conjectured to us by Tom Branson.

Theorem 3. *Any conformally invariant linear differential operator of the form (1) admits a self-adjoint conformally invariant modification of the form $f \mapsto \nabla_a(Q^{ab}(\nabla_b f))$ for a suitable $(n-2)$nd linear order differential operator $Q : \Lambda^1 \to \Lambda^{n-1}$.*

The motivation for Branson's conjecture came from the case of the sphere where the form of the operator may be verified directly. On the sphere, the operator controls the embedding $L^2_m \hookrightarrow e^L$ (Orlitz class) as a limiting case of the sharp Sobolev embeddings $L^2_r \hookrightarrow L^{\frac{2m}{m-r}}$ for $r < n/2$ (equivalently, comparing an L^q norm with the complementary series norm); see [1] for further discussion. In [4], Graham, Jenne, Mason, and Sparling established the existence of operators of the form (1). More recently it has been shown, using scattering theory [5] (and alternatively via the Fefferman–Graham ambient metric [2]) that the particular operators constructed in [4] are, in fact, already formally self-adjoint and have the form given in Theorem 3.

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