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EXPANSION COMPLEXES FOR FINITE SUBDIVISION RULES. II

J. W. CANNON, W. J. FLOYD, AND W. R. PARRY

Abstract. This paper gives applications of earlier work of the authors on the use of expansion complexes for studying conformality of finite subdivision rules. The first application is that a one-tile rotationally invariant finite subdivision rule (with bounded valence and mesh approaching 0) has an invariant partial conformal structure, and hence is conformal. The paper next considers one-tile single valence finite subdivision rules. It is shown that an expansion map for such a finite subdivision rule can be conjugated to a linear map, and that the finite subdivision rule is conformal exactly when this linear map is either a dilation or has eigenvalues that are not real. Finally, an example is given of an irreducible finite subdivision rule that has a parabolic expansion complex and a hyperbolic expansion complex.

1. Introduction

We continue here the study of expansion complexes for finite subdivision rules that was begun in [9]. As explained more deeply there, our study of expansion complexes arose out of our ongoing effort to resolve Cannon's Conjecture. Our interest in Cannon's Conjecture led us to finite subdivision rules and the problem of determining when a finite subdivision rule is conformal in the sense of Cannon [4]. The basic theory of finite subdivision rules and conformality for them is developed in [7].

A finite subdivision rule $\mathcal{R}$ consists of

(i) a finite 2-dimensional CW complex $S_{\mathcal{R}}$ (called the *model subdivision complex*),
(ii) a subdivision $\mathcal{R}(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$, and
(iii) a continuous cellular map $\sigma_{\mathcal{R}}\colon \mathcal{R}(S_{\mathcal{R}}) \to S_{\mathcal{R}}$ (called the *subdivision map*) which restricts to a homeomorphism on each open cell of $\mathcal{R}(S_{\mathcal{R}})$.

The model subdivision complex is the union of its closed 2-cells, and each 2-cell is the image of an n-gon (called a *tile type*), $n \geq 3$, under an attaching map which takes each open cell homeomorphically onto an open cell. An $\mathcal{R}$-complex is a 2-dimensional CW complex X together with a continuous cellular map $f\colon X \to S_{\mathcal{R}}$ (called the *structure map*) which takes each open cell homeomorphically onto an open 2-cell. If X is an $\mathcal{R}$-complex, the subdivision $\mathcal{R}(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$ pulls back to a

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subdivision $\mathcal{R}(X)$ of X. Furthermore, $\mathcal{R}(X)$ is an $\mathcal{R}$-complex, so we can continue the process and define subdivisions $\mathcal{R}^n(X)$ for $n \geq 1$.

In [9] we lay foundations for a new technique for proving conformality of finite subdivision rules. We define partial conformal structures on subdivision complexes, and prove in Theorem 4.7 that a finite subdivision rule $\mathcal{R}$ with bounded valence and mesh approaching 0 is conformal if the model subdivision complex $S_{\mathcal{R}}$ has an invariant partial conformal structure. The proofs make central use of special $\mathcal{R}$-complexes called expansion complexes. An expansion $\mathcal{R}$-complex is a planar $\mathcal{R}$-complex X together with a continuous map $\varphi\colon X \to X$ such that $\sigma_{\mathcal{R}} \circ f = f \circ \varphi$, where $f\colon X \to S_{\mathcal{R}}$ is the structure map for X.

Here we give the first applications of this technique. In Section 3 we consider one-tile rotationally invariant finite subdivision rules, for which the subdivisions of the tile types are cellularly isomorphic to each other (by orientation-preserving isomorphisms) and the subdivision of a tile type is invariant under an orientation-preserving cellular isomorphism of order the number of edges of the tile type. We prove (Theorem 3.5) that a one-tile rotationally invariant finite subdivision rule with bounded valence and mesh approaching 0 has an invariant partial conformal structure. From Theorem 3.5 and [9, Theorem 4.7] we immediately get Theorem 3.6, that every one-tile rotationally invariant finite subdivision rule is conformal.

The search for a proof of Theorem 3.6 motivated much of the work in [9]. Theorem 3.6 can also be proved using [9] and our paper [8] with Kenyon. Suppose that $\mathcal{R}$ is a one-tile rotationally invariant finite subdivision rule. We prove in [8, Theorem 3.1] that $\mathcal{R}$ has "bounded overlap" with a finite subdivision rule $\mathcal{Q}$ that can be realized by a rational map. Then $\mathcal{Q}$ has an invariant partial conformal structure, and so by [9, Theorem 4.7] $\mathcal{Q}$ is conformal. By the bounded overlap theorem [11, Theorem 4.3.1], $\mathcal{R}$ is conformal.

We next consider in Section 4 one-tile single valence finite subdivision rules. As above, the expression "one-tile" signifies that the subdivisions of any two tile types are cellularly isomorphic by an orientation-preserving isomorphism. The single valence property is that there is a constant r such that any interior vertex of an arbitrary subdivision of one of the tile types has valence r. We first use an Euler characteristic argument to prove that either

(i) each tile type is a triangle and $r = 6$,
(ii) each tile type is a quadrilateral and $r = 4$, or
(iii) each tile type is a hexagon and $r = 3$.

We then consider an expansion complex for such a finite subdivision rule $\mathcal{R}$, and show that the expansion map is conjugate to a linear map. It then follows from [9, Theorem 6.10] that $\mathcal{R}$ is conformal exactly when either this linear map is a dilation or its eigenvalues are not real.

The statement above is reminiscent of a special case of Thurston's topological characterization of critically finite branched maps. Suppose $f\colon S^2 \to S^2$ is a critically finite branched map with associated orbifold the rectangular pillowcase $(2, 2, 2, 2)$. Then f lifts to a covering map of the torus T^2 and there is a 2×2 matrix A_f which represents the induced map on $H_1(T^2, \mathbb{Z})$. Thurston's theorem states that f is equivalent to a rational map exactly if either A_f is a scalar matrix or its eigenvalues are not real. For a discussion of connections between critically finite branched maps and finite subdivision rules, see [8].

The result above concerning one-tile single valence finite subdivision rules is based on [10]. There we prove the following elementary theorem concerning polyomino tilings of the plane. Let $\mathcal{T}$ be a regular tiling of $\mathbb{R}^2$: the tiles of $\mathcal{T}$ are either equilateral triangles with six meeting at every vertex, squares with four meeting at every vertex, or regular hexagons with three meeting at every vertex. Suppose that the origin 0 is a vertex of a tile of $\mathcal{T}$. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism such that

(1) $\varphi(0) = 0$;
(2) if t is a tile of $\mathcal{T}$, then $\varphi(t)$ is a union of tiles of $\mathcal{T}$;
(3) if s and t are tiles of $\mathcal{T}$, then there exists an orientation-preserving isometry $\tau\colon \varphi(s) \to \varphi(t)$ such that $\varphi^{-1} \circ \tau \circ \varphi$ maps the vertices of s to the vertices of t.

Then there exists a subset Λ of the set of vertices of tiles of $\mathcal{T}$ such that Λ is a lattice in $\mathbb{R}^2$ and $\varphi\big|_\Lambda$ is a group homomorphism.

In Section 5 we give an interesting example that indicates some of the subtleties of expansion complexes. This is an irreducible finite subdivision rule $\mathcal{R}$ with bounded valence and mesh approaching 0 which has a hyperbolic expansion complex X and a parabolic expansion complex Y. Furthermore, any compact subcomplex of X (respectively Y) is isomorphic to a subcomplex of Y (respectively X). We are interested in this example primarily because of what it illustrates about the difficulty of determining whether an $\mathcal{R}$-complex which is homeomorphic to the plane is parabolic or hyperbolic. Such an $\mathcal{R}$-complex is parabolic (respectively hyperbolic) exactly if its "face barycenter" triangulation is the carrier complex of a parabolic (respectively hyperbolic) circle packing. So this example is also another illustration of the difficulty of the type problem for circle packings (see, for example, the He-Schramm paper [14]); that is, the problem of determining whether the circle packing associated to a triangulation of the plane is parabolic or hyperbolic. This example can also be viewed in the context of Riemann surface laminations of mixed type, though we will not make this connection concrete here. In his survey paper [13], Ghys gave an example of Kenyon's of a Riemann surface lamination which has one leaf homeomorphic to a four-punctured sphere (and hence hyperbolic) and has all other leaves parabolic. Blanc generalized this example greatly in his Ph.D. thesis [2]. In particular, he showed that there is a minimal surface lamination $\mathcal{L}$ of a compact space X such that, if S is any noncompact orientable surface, then $\mathcal{L}$ has a leaf homeomorphic to S.

2. Regular conformal structures

In Section 3 of [9] we introduce partial conformal structures on subdivision complexes. One might ask whether any such partial conformal structures exist. The purpose of this section is to construct one. We call this partial conformal structure the regular partial conformal structure. We learned of the regular partial conformal structure in [3]. It led us to the notion of a general partial conformal structure, which is essential for our investigation of finite subdivision rules. We define the notion of regular partial conformal structure following [3, Section 2 and Section 3] and [1, Subsection 3.3]. Our definition is slightly different from that in [3] and [1] mainly because our complexes are more general. Let $\mathcal{R}$ be a finite subdivision rule.

In this paragraph we put a piecewise metric structure on $S_{\mathcal{R}}$; that is, we put a metric structure on every open cell of $S_{\mathcal{R}}$ in a compatible way. There is only one way to put a metric on a vertex. To put a metric on an open edge of $S_{\mathcal{R}}$, we identify that open edge with the open unit interval. Now let t be a tile of $S_{\mathcal{R}}$ with tile type s. The characteristic map from s to t determines a metric on every open edge of s. Maintaining compatibility with these metrics, we identify s with a regular polygon in $\mathbb{C}$ whose edges have length 1. This defines a metric on $\operatorname{int}(s)$ and thereby a metric on $\operatorname{int}(t)$. We now have a metric structure on every open cell of $S_{\mathcal{R}}$.

We choose the face barycenter triangulation $\boxtimes(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$ to be compatible with the piecewise metric structure of $S_{\mathcal{R}}$. In other words, the barycenter of every tile is chosen to be the metric central point of the tile, and every edge is chosen to be geodesic.

We put a partial conformal structure on $S_{\mathcal{R}}$ as follows. We orient the open tiles and butterflies of $S_{\mathcal{R}}$. We also orient $\mathbb{C}$. Let t be a tile of $S_{\mathcal{R}}$. Suppose that the tile type of t has n edges. Let T be a regular polygon in $\mathbb{C}$ with n edges of length 1. We let $\mu_t\colon \operatorname{int}(t) \to \operatorname{int}(T)$ be an orientation-preserving isometry from the interior of t to the interior of T. The map μ_t defines a chart for $\operatorname{int}(t)$. Now let b be a butterfly of $S_{\mathcal{R}}$. Suppose that b is the union of $\operatorname{int}(e)$, $\operatorname{int}(s_1)$, and $\operatorname{int}(s_2)$, where e is an edge of $S_{\mathcal{R}}$ and where s_1 and s_2 are distinct tiles of $\boxtimes(S_{\mathcal{R}})$ which contain e. Suppose that the tile type of the tile t_i of $S_{\mathcal{R}}$ containing s_i has n_i edges for $i \in \{1, 2\}$. In $\mathbb{C}$ we form the union of a regular n_1-gon T_1 and a regular n_2-gon T_2 which have disjoint interiors and have an edge E of length 1 in common. We triangulate $T_1 \cup T_2$ by introducing the central points of T_1 and T_2 as new vertices and line segments joining them to old vertices in the straightforward way. Let B be the union of $\operatorname{int}(E)$, $\operatorname{int}(S_1)$ and $\operatorname{int}(S_2)$, where $S_1 \subseteq T_1$ and $S_2 \subseteq T_2$ are the triangles of $T_1 \cup T_2$ which contain E. We define $\mu_b\colon b \to B$ to be the unique orientation-preserving homeomorphism such that $\mu_b(\operatorname{int}(s_i)) = \operatorname{int}(S_i)$ for $i \in \{1, 2\}$ and the restrictions of μ_b to $\operatorname{int}(e)$, $\operatorname{int}(s_1)$ and $\operatorname{int}(s_2)$ are isometries. The map μ_b is a chart for b. This gives us an atlas $\mathcal{A}$ for $S_{\mathcal{R}}$, and it is a straightforward matter to verify that $\mathcal{A}$ is a partial conformal structure on $S_{\mathcal{R}}$.

In this paragraph we show that $\mathcal{A}$ is a nonsingular partial conformal structure on $S_{\mathcal{R}}$. By the definitions in Section 3 of [9], it suffices to show that if X is any $\mathcal{R}$-complex which is an oriented surface, then $\mathcal{A}$ determines a conformal structure on $\operatorname{int}(X)$. So let X be an $\mathcal{R}$-complex which is an oriented surface. It is shown in Section 3 of [9] that the pullback of $\mathcal{A}$ determines a conformal structure at every point of $\operatorname{int}(X)$ which is not a vertex. So let v be a vertex of X. The piecewise metric on X determines a shortest path metric on X, and we let $B \subseteq X$ be the open ball of radius $1/3$ centered at v for this shortest path metric. The open ball B is a topological disk. Intersecting B with the tiles of X which contain v yields sectors $B_1, \dots, B_d$, where d is the valence of v. The shortest path metric on B_i determines an angle θ_i at v for $i \in \{1, \dots, d\}$. Let $\alpha = 2\pi / \sum_{i=1}^{d} \theta_i$. For $i \in \{1, \dots, d\}$, we map B_i isometrically and in an orientation-preserving way to

$$\{z \in \mathbb{C} : |z| < \frac{1}{3}, \sum_{j=1}^{i-1} \theta_j \leq \arg(z) \leq \sum_{j=1}^{i} \theta_j\},$$

and then we follow this map with an appropriate branch of the map $z \mapsto z^{\alpha}$. These composition maps assemble to yield a homeomorphism

$$\mu_v : B \to \{z \in \mathbb{C} : |z| < (1/3)^{\alpha}\}.$$

These maps μ_v are compatible with the pullback of $\mathcal{A}$ to X, and so the pullback of $\mathcal{A}$ defines a conformal structure on $\mathrm{int}(X)$. We call this conformal structure the **regular conformal structure** on $\mathrm{int}(X)$, and call $\mathcal{A}$ the **regular partial conformal structure** on $S_{\mathcal{R}}$.

3. One-tile rotationally invariant finite subdivision rules

Recall that a finite subdivision rule $\mathcal{R}$ is **orientation-preserving** if there is an orientation on the union of the open tiles of $S_{\mathcal{R}}$ such that the restriction of $\sigma_{\mathcal{R}}$ to each open tile of $\mathcal{R}(S_{\mathcal{R}})$ preserves orientation. Note that an orientation on the union of the open tiles of $S_{\mathcal{R}}$ determines an orientation on the union of the tile types of $\mathcal{R}$.

We define a **one-tile rotationally invariant finite subdivision rule** $\mathcal{R}$ as follows. We assume that $\mathcal{R}$ has bounded valence, that $\mathcal{R}$ is orientation preserving, and that the mesh of $\mathcal{R}$ approaches 0. We fix an orientation on the union of the open tiles of $S_{\mathcal{R}}$ such that the restriction of $\sigma_{\mathcal{R}}$ to each open tile of $\mathcal{R}(S_{\mathcal{R}})$ preserves orientation. We make the following two further assumptions.

(1) If s and t are tile types of $\mathcal{R}$, then there exists an orientation-preserving cellular isomorphism from s to t which takes $\mathcal{R}(s)$ to $\mathcal{R}(t)$.
(2) If t is a tile type of $\mathcal{R}$ with q sides, then there exists an orientation-preserving cellular automorphism of t of order q which is also a cellular automorphism of $\mathcal{R}(t)$.

Even though $\mathcal{R}$ may have more than one tile type, we still call $\mathcal{R}$ a one-tile rotationally invariant finite subdivision rule because the subdivisions of the tile types of $\mathcal{R}$ look the same. While the combinatorics of the subdivisions of the tile types are the same, because of the constraints of the edge orientations it may not be possible to realize $\mathcal{R}$ with a single tile type.

Constructing finite subdivision rules with specified properties can be challenging. The following lemma can be used to construct many one-tile rotationally invariant finite subdivision rules.

Lemma 3.1. *Let X be a closed topological disk with the cell structure of a polygon with $q \geq 3$ sides. Let Y be a CW complex subdivision of X with the following properties. Every edge of X properly subdivides in Y. Only one tile of Y contains a given vertex of X. Every tile of Y has q sides. There exists an orientation-preserving cellular automorphism of Y of order q which is also a cellular automorphism of X. Then there exists a one-tile rotationally invariant finite subdivision rule $\mathcal{R}$ such that if t is a tile type of $\mathcal{R}$, then there exists a cellular isomorphism from t to X which maps $\mathcal{R}(t)$ to Y. We may furthermore choose $\mathcal{R}$ so that there exists an expansion $\mathcal{R}$-complex whose structure map preserves orientation.*

Proof. We construct $\mathcal{R}$ so that $S_{\mathcal{R}}$ has one vertex and one edge e. The orientation-preserving cellular automorphism of Y acts transitively on the edges of X, and so they subdivide into the same number of subedges. We subdivide e into that number of edges, and we denote the resulting complex by $\mathcal{R}(e)$. We orient e, which induces orientations on the edges of $\mathcal{R}(e)$.

Now we fix an orientation of X, which induces an orientation of Y. We orient the edges of X arbitrarily. We orient the edges of Y in any way so that if d is an edge of X, then the orientations of the subedges of d agree with the orientation of d. For every orientation of the edges of X and every such orientation of the edges of Y we construct a tile type for $\mathcal{R}$ as follows. Let t be a closed topological disk. We choose a homeomorphism $f\colon t \to X$, which gives t the structure of an oriented CW complex. The map f also induces orientations on the edges of t, and Y pulls back to a subdivision of t, which we denote by $\mathcal{R}(t)$. Hence we have a subdivision of ∂t, which we denote by $\mathcal{R}(\partial t)$. We construct a continuous cellular map from ∂t to e such that the vertices of t map to the vertex of $S_{\mathcal{R}}$ and the restriction of this map to every open edge of $\mathcal{R}(\partial t)$ is an orientation-preserving homeomorphism onto an open edge of $\mathcal{R}(e)$. We attach t to e by means of this map. Attaching all the tile types of $\mathcal{R}$ to e yields $S_{\mathcal{R}}$.

The subdivisions of e and the tile types of $\mathcal{R}$ are compatible, giving a subdivision $\mathcal{R}(S_{\mathcal{R}})$ of $S_{\mathcal{R}}$.

We define the subdivision map $\sigma_{\mathcal{R}}$ in this paragraph. We define $\sigma_{\mathcal{R}}$ on the 1-skeleton of $\mathcal{R}(S_{\mathcal{R}})$ so that $\sigma_{\mathcal{R}}$ is a continuous function which maps the open edges of $\mathcal{R}(S_{\mathcal{R}})$ to the open 1-cell of e by means of orientation-preserving homeomorphisms. Next let t be a tile of $\mathcal{R}(S_{\mathcal{R}})$. We arbitrarily choose one tile t' of $S_{\mathcal{R}}$ for which there exists a cellular isomorphism from t to t' which preserves orientations of faces and edges. We continuously extend the definition of $\sigma_{\mathcal{R}}$ from ∂t to t so that the restriction of $\sigma_{\mathcal{R}}$ to the open 2-cell of t is an orientation-preserving homeomorphism onto the open 2-cell of t'. This completes the definition of $\sigma_{\mathcal{R}}$.

We now have a finite subdivision rule $\mathcal{R}$. For every tile type t of $\mathcal{R}$ there exists a cellular isomorphism from t to X which maps $\mathcal{R}(t)$ to Y. Because only one tile of Y contains a given vertex of X, the finite subdivision rule $\mathcal{R}$ has bounded valence. Using the facts that every edge of X properly subdivides in Y and there exists a cellular automorphism of Y of order q, one checks that the mesh of $\mathcal{R}$ approaches 0 combinatorially. (See Section 1.1 of [7] for a definition of approaching 0 combinatorially.) Theorem 2.3 of [7] implies that $\mathcal{R}$ can be defined so that its mesh approaches 0. Now it is clear that $\mathcal{R}$ is a one-tile rotationally invariant finite subdivision rule.

Finally we turn to the construction of an expansion $\mathcal{R}$-complex. By construction, if t is a tile type of $\mathcal{R}$ and v is a vertex of t, then only one tile of $\mathcal{R}(t)$ contains v. Let t be a tile type of $\mathcal{R}$ with vertex v such that the edges of t which contain v are oriented away from v. Let $f\colon t \to S_{\mathcal{R}}$ be the structure map for t. Let s be the tile of $\mathcal{R}(t)$ which contains v, and let $g\colon s \to S_{\mathcal{R}}$ be the structure map for s. It is possible to choose t and v so that there exists a continuous cellular map $h\colon t \to s$ such that $h(v) = v$, the restriction of h to every open cell of t is a homeomorphism onto an open cell of s, and h preserves orientations of tiles and edges. We redefine $\sigma_{\mathcal{R}}$ on $g(s)$ if necessary so that $\sigma_{\mathcal{R}}$ maps $g(s)$ to $f(t)$ and $(f\big|_{\mathrm{int}(t)})^{-1} \circ \sigma_{\mathcal{R}} \circ g\big|_{\mathrm{int}(s)}$ maps points in $\mathrm{int}(s)$ near v to points in $\mathrm{int}(t)$ near v. Now we let S be the $\mathcal{R}$-complex derived from t by identifying the two edges of t which contain v. Rotational invariance shows that the tile of $\mathcal{R}(S)$ which contains the image of v lies in $\mathrm{int}(S)$. Lemma 2.4 of [9] implies that there exists an expansion $\mathcal{R}$-complex W with a seed $\mathcal{R}$-isomorphic to S. It is clear that the structure map of W preserves orientation.

This proves Lemma 3.1. $\square$

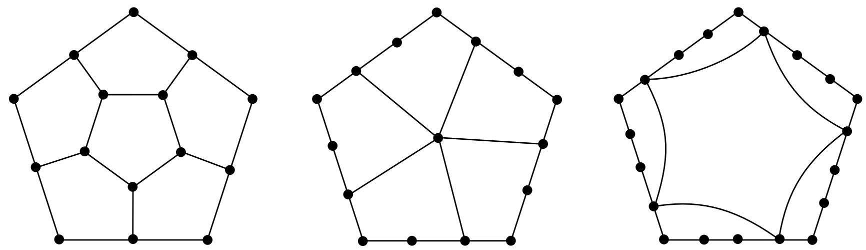

FIGURE 1. Rotationally invariant subdivisions of three pentagons

Example 3.2. Figure 1 shows rotationally invariant subdivisions of three pentagons. Lemma 3.1 shows that for each of these subdivisions of a pentagon there exists a one-tile rotationally invariant finite subdivision rule whose tile types subdivide as in Figure 1.

We next define the notion of an infinite cellular isomorphism, which plays a major role in our investigation of one-tile rotationally invariant finite subdivision rules. Let X and Y be CW complexes. Let $X = X_1, X_2, X_3, \ldots$ be a sequence of successive subdivisions of X, and let $Y = Y_1, Y_2, Y_3, \ldots$ be a sequence of successive subdivisions of Y. An **infinite cellular isomorphism** from X to Y (with respect to these subdivisions) is a homeomorphism $f\colon X \to Y$ which is a cellular isomorphism from X_n to Y_n for every positive integer n.

The following lemma gives a basic property of infinite cellular isomorphisms.

Lemma 3.3. *Let X be a closed topological disk with the structure of a CW complex. Let $X = X_1, X_2, X_3, \ldots$ be a sequence of successive subdivisions of X whose mesh approaches 0. Then every orientation-preserving infinite cellular automorphism of X is topologically conjugate to a rotation of the closed unit disk in $\mathbb{R}^2$.*

Proof. Let $f\colon X \to X$ be an orientation-preserving infinite cellular automorphism.

We first show that if f fixes a point in ∂X, then f is the identity map on X. Suppose that $x \in \partial X$ is fixed by f. Let $e \subseteq \partial X$ be an edge of X which contains x. Because f preserves orientation, f maps every cell of e into itself. Hence if t is the tile of X which contains e, then f maps every cell of t into itself. Inductively, we see that f maps every cell of X into itself. The same is true for every subdivision of X. Since the mesh of the sequence $X_1, X_2, X_3, \ldots$ approaches 0, it follows that f is the identity map on X. So if f fixes a point of ∂X, then f is the identity map on X.

Because f permutes the vertices of X in ∂X, some power of f fixes a vertex of X in ∂X, and so the previous paragraph implies that f has finite order, say, q. Since f has finite order, it follows from Kérékjartó's theorem (see, for example, the Constantin-Kolev paper [12]) that f is conjugate to a rotation.

This proves Lemma 3.3. □

Lemma 3.4. *Let $\mathcal{R}$ be a one-tile rotationally invariant finite subdivision rule. Let t_1 and t_2 be tile types of $\mathcal{R}$ with edges $e_1 \subseteq t_1$ and $e_2 \subseteq t_2$. Then there exists a unique orientation-preserving infinite cellular isomorphism $f\colon t_1 \to t_2$ such that $f(e_1) = e_2$.*

Proof. The definition of one-tile rotationally invariant finite subdivision rule implies that there exists an orientation-preserving cellular isomorphism $f_1 \colon \mathcal{R}(t_1) \to \mathcal{R}(t_2)$ such that $f(e_1) = e_2$. Inductively, there exists a cellular isomorphism $f_n \colon \mathcal{R}^n(t_1) \to \mathcal{R}^n(t_2)$ such that $f_n(e_1) = e_2$ for every positive integer n. Because the mesh of $\mathcal{R}$ approaches 0, the limit of the sequence $\{f_n\}$ exists and it is an orientation-preserving infinite cellular isomorphism taking e_1 to e_2. Uniqueness follows from Lemma 3.3.

This proves Lemma 3.4. □

Let $\mathcal{R}$ be a one-tile rotationally invariant finite subdivision rule. When constructing a regular partial conformal structure for $S_{\mathcal{R}}$, we make the following assumptions. Let t be a tile type of $\mathcal{R}$. Suppose that t has q edges. Lemma 3.4 implies that there exists an orientation-preserving infinite cellular automorphism $f \colon t \to t$ of order q. Lemma 3.3 implies that f is topologically conjugate to a rotation of order q of the closed unit disk in $\mathbb{R}^2$. When we construct a regular partial conformal structure for $S_{\mathcal{R}}$, we construct a homeomorphism $\mu_t \colon t \to T$, where T is a regular polygon in $\mathbb{R}^2$ with q sides of length 1. We now do this so that μ_t conjugates f to a rotation of T. If s is a tile type of $\mathcal{R}$ other than t, then Lemma 3.4 provides an infinite cellular isomorphism $\gamma_s \colon s \to t$. We set $\mu_s = \mu_t \circ \gamma_s$. Now let X and Y be $\mathcal{R}$-complexes which are oriented surfaces. These assumptions on the regular partial conformal structure on $S_{\mathcal{R}}$ imply that if $g \colon X \to Y$ is an orientation-preserving cellular isomorphism, then g is conformal in $\operatorname{int}(X)$ if and only if g is an infinite cellular isomorphism.

We now come to the main result of this section.

Theorem 3.5. *Every one-tile rotationally invariant finite subdivision rule has an invariant partial conformal structure.*

Proof. To clarify our argument, in Example 3.7 we present one particular finite subdivision rule with figures showing some of the subdivision complexes which occur in this proof.

Let $\mathcal{R}$ be a one-tile rotationally invariant finite subdivision rule. Then by Lemma 3.1, there exists a one-tile rotationally invariant finite subdivision rule $\mathcal{Q}$ whose tile types subdivide the same way that the tile types of $\mathcal{R}$ subdivide such that there exists an expansion $\mathcal{Q}$-complex whose structure map preserves orientation. We define a new finite subdivision rule $\mathcal{R}'$ so that $S_{\mathcal{R}'}$ is the disjoint union of $S_{\mathcal{R}}$ and $S_{\mathcal{Q}}$ with the obvious subdivision map. Then $\mathcal{R}'$ is a one-tile rotationally invariant finite subdivision rule for which there exists an expansion $\mathcal{R}'$-complex whose structure map preserves orientation. Furthermore if $\mathcal{R}'$ has an invariant partial conformal structure, then so does $\mathcal{R}$. Hence we may assume that there exists an expansion $\mathcal{R}$-complex whose structure map preserves orientation. So let X be an expansion $\mathcal{R}$-complex whose structure map preserves orientation.

Because X has bounded valence, there exists a positive integer r such that the valence of every vertex of X divides r. We put one more condition on r in the next paragraph. Let U be a topological space homeomorphic to $\mathbb{R}^2$ with a map from U to X which is a branched cover, branching only over vertices of X, such that if v is a vertex of X with valence k, then the degree of this map over v is r/k. The $\mathcal{R}$-complex structure on X lifts to an $\mathcal{R}$-complex structure on U. We orient U so that the structure map from U to $S_{\mathcal{R}}$ preserves orientation. Every vertex of U has valence r.

We next subdivide $\boxtimes(U)$ to obtain a cell complex isomorphic to a Coxeter complex. We first insert a barycenter in the interior of every edge of U. Then for every tile t of U we join the barycenter of t with an edge to the barycenter of every edge of t. The resulting subdivision of $\boxtimes(U)$ is isomorphic to the Coxeter complex of the $(2,q,r)$-triangle group, where q is the number of edges in every tile type of $\mathcal{R}$. The $(2,q,r)$-triangle group is hyperbolic if $\frac{1}{2}+\frac{1}{q}+\frac{1}{r}<1$. Hence the $(2,q,r)$-triangle group is hyperbolic if $\frac{1}{q}+\frac{1}{r}<\frac{1}{2}$. We choose r so that $\frac{1}{q}+\frac{1}{r}<\frac{1}{2}$. We equip U with a regular conformal structure as in Section 2 while adhering to the assumptions two paragraphs before Theorem 3.5. We let $\overline{U}$ denote a uniformization of U. Then $\overline{U}$ is the open unit disk, the edges of $\overline{U}$ are hyperbolic geodesic segments, and the tiles of $\overline{U}$ are regular hyperbolic q-gons. The homogeneity of $\overline{U}$ makes it possible to define maps from $\overline{U}$ to other $\mathcal{R}$-complexes, and for this reason we view $\overline{U}$ as a kind of universal complex.

Let $V=\mathcal{R}(U)$. Let $\overline{V}$ denote a uniformization of V. Corollary 5.7 of [9] implies that $\overline{V}$ is the open unit disk. We have a canonical homeomorphism from $\mathcal{R}(\overline{U})$ to $\mathcal{R}(U)$, we have the identity map from $\mathcal{R}(U)$ to V, and we have a canonical homeomorphism from V to $\overline{V}$. Let $\eta\colon \mathcal{R}(\overline{U})\to\overline{V}$ be the composition of these maps. The map η is an $\mathcal{R}$-isomorphism, but in general it is not conformal.

If W is a planar $\mathcal{R}$-complex, then we let $\mathrm{Aut}(W)$ denote the group of all orientation-preserving infinite cellular automorphisms of W and we let $\mathrm{Aut}_{\mathcal{R}}(W)$ denote the subgroup of $\mathrm{Aut}(W)$ consisting of those maps which are $\mathcal{R}$-isomorphisms. The map η induces an isomorphism $\omega\colon \mathrm{Aut}(\mathcal{R}(\overline{U}))\to\mathrm{Aut}(\overline{V})$ such that if $\alpha\in\mathrm{Aut}(\mathcal{R}(\overline{U}))$ and if $\beta=\omega(\alpha)$, then $\eta\circ\alpha=\beta\circ\eta$. (Note that the natural conformal structure on $\mathcal{R}(\overline{U})$ need not be the regular conformal structure, so the elements of $\mathrm{Aut}(\mathcal{R}(\overline{U}))$ need not be conformal. However, the elements of $\mathrm{Aut}(\overline{V})$ are conformal.) We see that $\mathrm{Aut}(\overline{U})\subseteq\mathrm{Aut}(\mathcal{R}(\overline{U}))$ and that ω maps $\mathrm{Aut}(\overline{U})$ injectively to $\mathrm{Aut}(\overline{V})$.

The fact that $\mathrm{Aut}(\overline{U})$ embeds in $\mathrm{Aut}(\overline{V})$ implies that $\overline{V}$ is regular in the following sense. Let t be a tile of $\overline{U}$; t is a regular hyperbolic q-gon. The set T of vertices of t is stabilized by a cyclic subgroup of $\mathrm{Aut}(\overline{U})$ of order q. Hence the set $\eta(T)\subseteq\overline{V}$ is stabilized by a cyclic subgroup of $\mathrm{Aut}(\overline{V})$ of order q. It follows that the elements of $\eta(T)$ are the vertices of a regular hyperbolic q-gon t'. Now let $v\in T$. The stabilizer of v in $\mathrm{Aut}(\overline{U})$ is a cyclic group of order r. The properties of the isomorphism ω imply that the stabilizer of $\eta(v)$ in $\mathrm{Aut}(\overline{V})$ contains a cyclic group of order r and in fact that the angle of t' at $\eta(v)$ is $2\pi/r$. Thus t and t' are congruent hyperbolic q-gons. It follows that there exists a conformal automorphism γ of the open unit disk which agrees with η on the vertices of $\overline{U}$. Since $\overline{U}$ is determined only up to conformal automorphisms of the open unit disk, we replace $\overline{U}$ by its image under γ. We therefore have that η fixes every vertex of $\overline{U}$.

Let $Y=\mathcal{R}(X)$, let $\overline{X}$ denote a uniformization of X, and let $\overline{Y}$ denote a uniformization of Y. The branched cover from U to X induces a branched cover $\pi\colon \overline{U}\to\overline{X}$, which is an analytic $\mathcal{R}$-map. A routine monodromy argument using the fact that every vertex of $\overline{U}$ has valence r shows that if $x,y\in\overline{U}$ with $\pi(x)=\pi(y)$, then there exists $\alpha\in\mathrm{Aut}_{\mathcal{R}}(\overline{U})$ such that $\alpha(x)=y$ and $\pi(z)=\pi(\alpha(z))$ for every $z\in\overline{U}$. So there exists a subgroup $\mathrm{Aut}(\pi)$ of $\mathrm{Aut}_{\mathcal{R}}(\overline{U})$ such that if $x,y\in\overline{U}$, then $\pi(x)=\pi(y)$ if and only if there exists $\alpha\in\mathrm{Aut}(\pi)$ with $\alpha(x)=y$.

Let $\alpha\in\mathrm{Aut}(\overline{U})$, and let $\beta=\omega(\alpha)$. It follows from the definition of the isomorphism $\omega\colon \mathrm{Aut}(\mathcal{R}(\overline{U}))\to\mathrm{Aut}(\overline{V})$ that $\eta\circ\alpha=\beta\circ\eta$. We now have that $\eta(v)=v$ for every vertex $v \in \overline{U}$. Hence $\alpha(v) = \beta(v)$ for every vertex $v \in \overline{U}$. Hence α and β are conformal automorphisms of the open unit disk which agree on every vertex of $\overline{U}$. So $\alpha = \beta$. Thus ω fixes every element of $\mathrm{Aut}(\overline{U})$, and so $\mathrm{Aut}(\overline{U})$ is a subgroup of $\mathrm{Aut}(\overline{V})$. We furthermore see that η commutes with α. In particular, η commutes with every element of $\mathrm{Aut}(\pi)$. This implies that η descends to an $\mathcal{R}$-isomorphism η_* from $\pi(\mathcal{R}(\overline{U})) = \mathcal{R}(\overline{X})$ to $\pi(\overline{V})$. Since $\overline{Y}$ and $\mathcal{R}(\overline{X})$ are $\mathcal{R}$-isomorphic, $\overline{Y}$ and $\pi(\overline{V})$ are $\mathcal{R}$-isomorphic. An $\mathcal{R}$-isomorphism between two $\mathcal{R}$-complexes with regular partial conformal structures is necessarily conformal, so there is a conformal $\mathcal{R}$-isomorphism from $\overline{Y}$ to $\pi(\overline{V})$. We use this conformal $\mathcal{R}$-isomorphism to identify $\overline{Y}$ with $\pi(\overline{V})$.

Now let φ be the expansion map of $\overline{X}$. The map $\eta_*^{-1}\colon \overline{Y} \to \mathcal{R}(\overline{X})$ is an $\mathcal{R}$-isomorphism and $\varphi\colon \mathcal{R}(\overline{X}) \to \overline{X}$ is an $\mathcal{R}$-isomorphism. Hence the map $\psi\colon \overline{Y} \to \overline{X}$ defined by $\psi = \varphi \circ \eta_*^{-1}$ is an $\mathcal{R}$-isomorphism. Since ψ is an $\mathcal{R}$-isomorphism between two $\mathcal{R}$-complexes with regular partial conformal structures, ψ is conformal.

We want to apply Theorem 6.6 of [9] with the present $\mathcal{R}$, φ and ψ as in Theorem 6.6 of [9] and the present $\overline{X}$ replacing the X of Theorem 6.6 of [9]. There are two more hypotheses of Theorem 6.6 of [9] to verify.

In this paragraph we verify the hypothesis that there exists a positive integer K such that $d(\varphi(x), \psi(x)) \leq K$ for every $x \in \overline{X}$. To prove this it suffices to prove that there exists a positive integer K' such that $d_{-1}(\varphi(x), \psi(x)) \leq K'$ for every $x \in \overline{X}$. The definition of ψ implies that $d_{-1}(\varphi(x), \psi(x)) = d_{-1}(\varphi(x), \varphi(\eta_*^{-1}(x)))$, and the end of the first paragraph of Section 6 of [9] implies that $d_{-1}(\varphi(x), \varphi(\eta_*^{-1}(x))) = d(x, \eta_*^{-1}(x))$. Moreover $d(x, \eta_*^{-1}(x)) = d(\eta_*(y), y)$, where $y = \eta_*^{-1}(x)$. So it suffices to show that there exists a global bound on the distance in terms of the pseudometric d from any point in a tile t of $\overline{X}$ to any point in $\eta_*(t)$. Pulling back to $\overline{U}$, it suffices to show that there exists a global bound on the distance in terms of the skinny path pseudometric for $\overline{U}$ from any point in a tile $\widetilde{t}$ of $\overline{U}$ to any point in $\eta(\widetilde{t})$. But this is true because $\mathrm{Aut}(\overline{U})$ is a subgroup of $\mathrm{Aut}(\overline{V})$ and it acts transitively on the tiles of $\overline{U}$: a bound for one tile of $\overline{U}$ gives a bound for all tiles of $\overline{U}$. Thus there exists a positive integer K such that $d(\varphi(x), \psi(x)) \leq K$ for every $x \in \overline{X}$.

Now Theorem 6.7 of [9] implies that $\overline{X}$ is parabolic and the mesh of the sequence of tilings $\{\psi^{-n}(\mathcal{S}(\overline{X}))\}$ locally approaches 0. We assume that the underlying space of $\overline{X}$ is $\mathbb{C}$. Thus ψ is a conformal automorphism of $\mathbb{C}$.

Thus we have verified the hypotheses of Theorem 6.6 of [9] with the present $\overline{X}$ instead of X. Theorem 6.6 of [9] implies that there exists an expansion $\mathcal{R}$-complex X' such that (1) the underlying space of X' is $\mathbb{C}$, (2) ψ is the expansion map of X', and (3) the sequence of functions $\{\psi^{-n} \circ \varphi^n\}$ converges to an $\mathcal{R}$-isomorphism $\rho\colon \overline{X} \to X'$ which commutes with the expansion maps.

We at last show that $S_{\mathcal{R}}$ has an invariant partial conformal structure. At the heart of the matter is that every orientation-preserving infinite cellular isomorphism between two $\mathcal{R}$-subcomplexes of X' is conformal. We prove this as follows.

Suppose that we have two $\mathcal{R}$-subcomplexes of X'. Because $\rho\colon \overline{X} \to X'$ is an $\mathcal{R}$-isomorphism, our $\mathcal{R}$-subcomplexes of X' have the form $\rho(W)$ and $\rho(Z)$, where W and Z are $\mathcal{R}$-subcomplexes of $\overline{X}$. Let $\tau\colon \rho(W) \to \rho(Z)$ be an orientation-preserving infinite cellular isomorphism. Then $\theta = \rho^{-1} \circ \tau \circ \rho$ is an orientation-preserving infinite cellular isomorphism from W to Z. We use the fact that $\rho = \lim_{n\to\infty} \psi^{-n} \circ \varphi^n$. Let n be a nonnegative integer. Then the function $\varphi^n \circ \theta \circ \varphi^{-n}\colon \varphi^n(W) \to \varphi^n(Z)$ is an orientation-preserving infinite cellular isomorphism between two $\mathcal{R}$-complexes with regular partial conformal structures, so $\varphi^n \circ \theta \circ \varphi^{-n}$ is conformal. Hence $\tau_n = \psi^{-n} \circ \varphi^n \circ \theta \circ \varphi^{-n} \circ \psi^n$ is conformal. Now let $z \in \operatorname{int}(\rho(W))$. Then there exists an open metric ball B and a nonnegative integer N such that $z \in B \subseteq \psi^{-n} \circ \varphi^n(W)$ for every integer $n \geq N$. So the functions in a tail of the sequence $\{\tau_n\}$ are defined on B, are conformal, and are uniformly bounded. As on page 143 of [16], it follows that they form a normal family there. Hence some subsequence converges to a conformal function. But $\lim_{n\to\infty} \tau_n = \rho \circ \theta \circ \rho^{-1} = \tau$. We have just proved that every orientation-preserving infinite cellular isomorphism between two $\mathcal{R}$-subcomplexes of X' is conformal. Since $\mathcal{R}^n(X') = \psi^{-n}(X')$, it follows that every orientation-preserving infinite cellular isomorphism between two $\mathcal{R}$-subcomplexes of $\mathcal{R}^n(X')$ is conformal for every nonnegative integer n.

In this paragraph we define charts for $S_\mathcal{R}$. Let s be either an open tile of $S_\mathcal{R}$ or a butterfly of $S_\mathcal{R}$ such that if s is a butterfly, then the orientations of its open tiles agree. If s is an open tile, then let t be an open tile of X', and if s is a butterfly, then let t be a butterfly of X'. For every positive integer n let s_n be the maximal $\mathcal{R}$-subcomplex of $\mathcal{R}^n(S_\mathcal{R})$ contained in s, and define t_n similarly. Then for every positive integer n there exists an orientation-preserving infinite cellular isomorphism $\mu_n \colon s_n \to t_n$ such that $\mu_{n+1}\big|_{s_n} = \mu_n$. Hence $\mu_s = \lim_{n\to\infty} \mu_n$ is a homeomorphism from s to t. We take μ_s to be the chart for s. If s is a butterfly of $S_\mathcal{R}$ such that the orientations of the open tiles of s disagree, then we define μ_s just as we defined charts for folding butterflies in Section 3 of [9]. We now have an atlas $\mathcal{A}$ of charts for $S_\mathcal{R}$. The previous paragraph shows that $\mathcal{A}$ is a partial conformal structure on $S_\mathcal{R}$.

The fact that the expansion map ψ is conformal implies that $\mathcal{A}$ satisfies the condition of $\mathcal{R}$-invariance in every open tile of $S_\mathcal{R}$ and in every butterfly whose open tiles have compatible orientations. For a butterfly b of $S_\mathcal{R}$ whose open tiles have opposite orientations, we proceed as in the proof of Theorem 4.1 of [9] using the fact that the open edge of b is an analytic arc. Thus $\mathcal{A}$ is an invariant partial conformal structure on $S_\mathcal{R}$.

This proves Theorem 3.5. □

Theorem 3.6. *Every one-tile rotationally invariant finite subdivision rule is conformal.*

Proof. This theorem follows immediately from Theorem 3.5 and [9, Theorem 4.7]. □

We conclude this section with an example with figures showing some of the subdivision complexes which occur in the proof of Theorem 3.5.

Example 3.7. We use Lemma 3.1 to construct a one-tile rotationally invariant finite subdivision rule $\mathcal{R}$ whose tile types are pentagons which subdivide as in the last subdivision of Figure 1. Figure 2 shows the second subdivision of a tile type of $\mathcal{R}$. We construct an expansion $\mathcal{R}$-complex X as in the proof of Lemma 3.1 except that there we take a seed S of X to consist of just one tile and here we use four tiles. Figure 3 shows S and $\mathcal{R}(S)$; note that the shaded portion of $\mathcal{R}(S)$ is $\mathcal{R}$-isomorphic to S. Figure 4 shows part of X. Note that each vertex of X has valence 2 or 4. In the proof of Theorem 3.5 we choose an integer r which is a multiple of all of the valences of the vertices of X, and we construct a complex U whose vertices all have valence r. We take $r = 4$. So the uniformization $\overline{U}$ of U gives

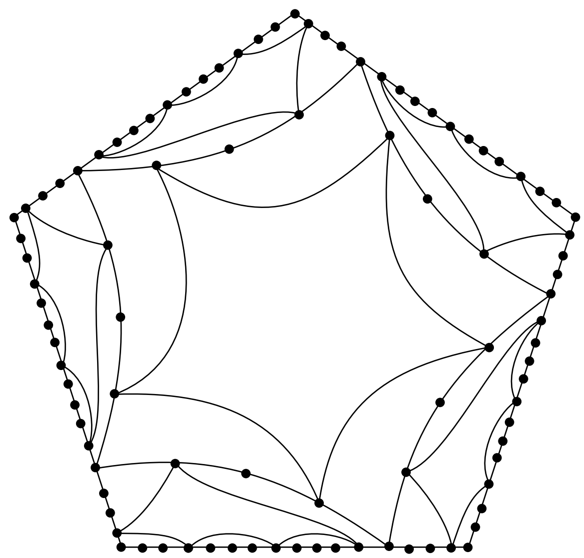

FIGURE 2. The second subdivision

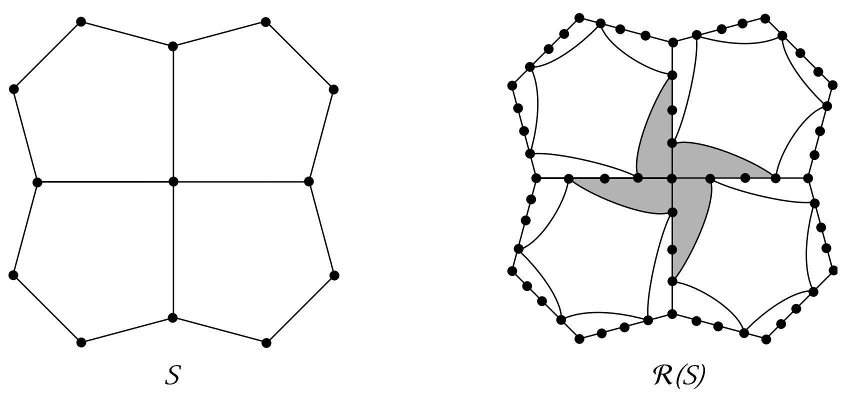

FIGURE 3. S and $\mathcal{R}(S)$

the familiar decomposition of the hyperbolic plane by right angled pentagons. A portion of $\overline{U}$ is shown in Figure 5. A portion of $\overline{V}$ is shown in Figure 6. Figure 7 shows portions of $\overline{U}$ and $\overline{V}$ simultaneously. Figure 8 shows portions of $\overline{X}$ and $\overline{Y}$ simultaneously. In Figures 7 and 8, note how the vertices of the coarser complex coincide with vertices of the finer complex. Figures 4 through 8 were drawn using Ken Stephenson's program CirclePack [17]. In large part, such circle packing figures led to Theorem 3.5.

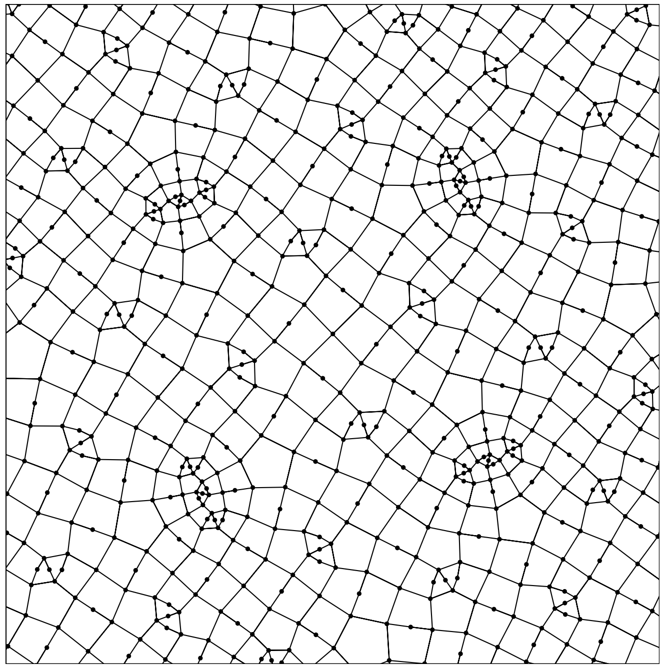

FIGURE 4. Part of X

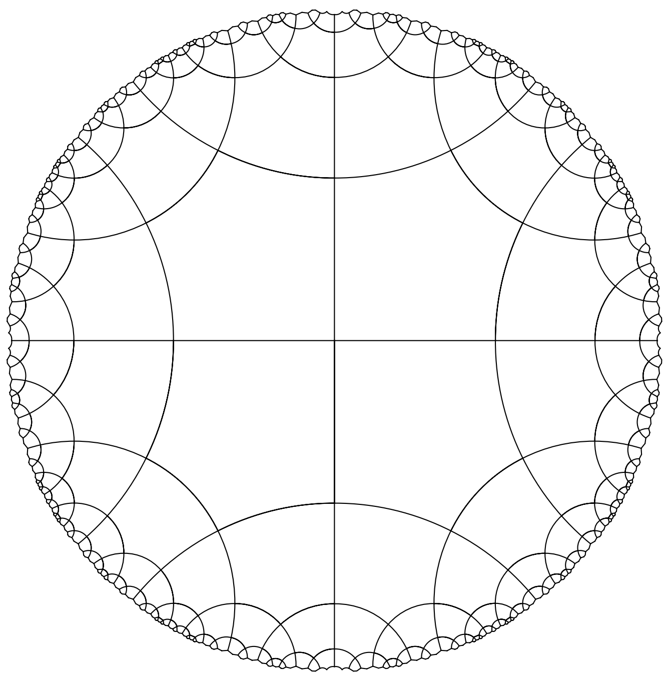

FIGURE 5. Part of $\overline{U}$

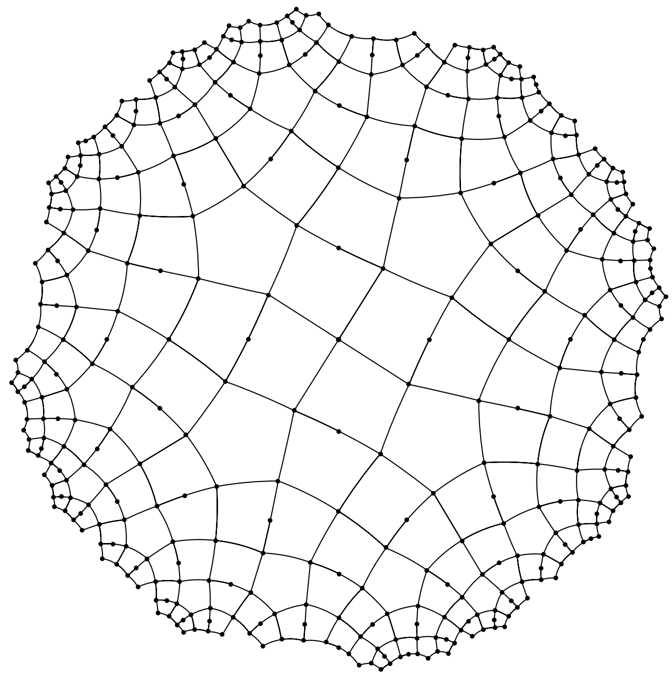

FIGURE 6. Part of $\overline{V}$

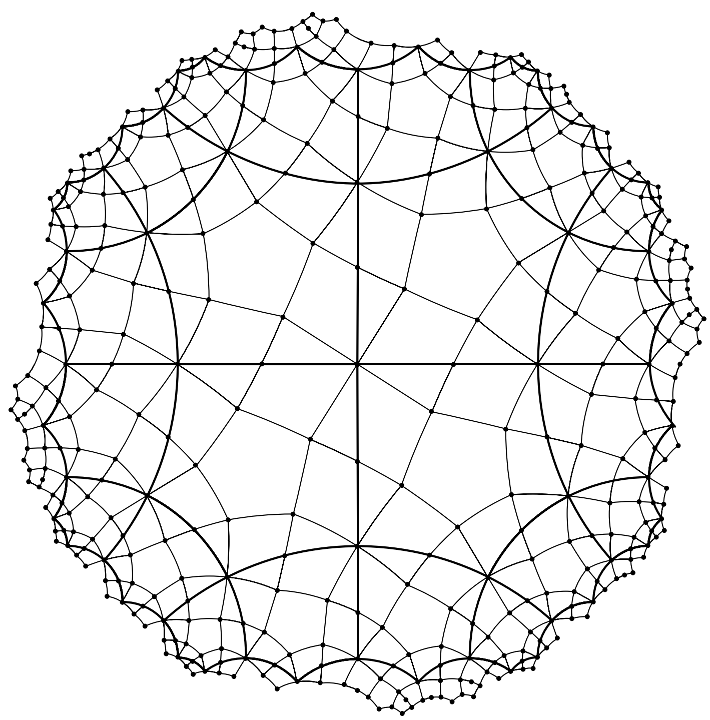

FIGURE 7. Parts of $\overline{U}$ and $\overline{V}$

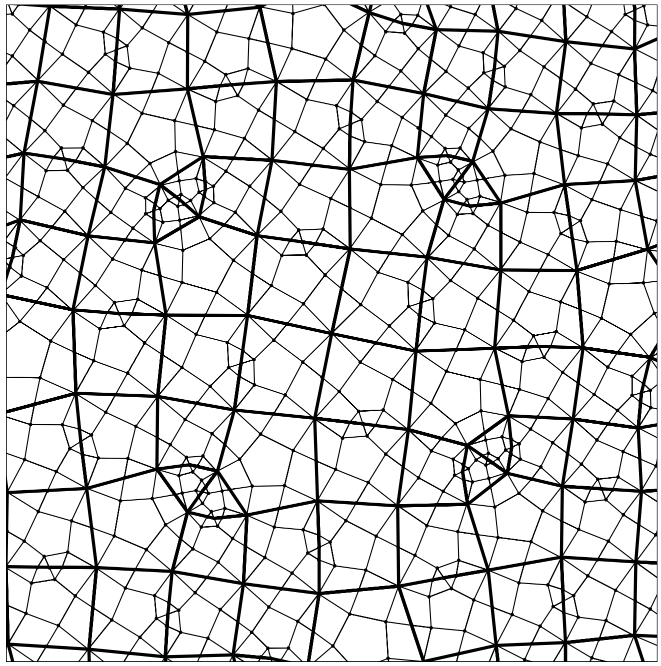

FIGURE 8. Parts of $\overline{X}$ and $\overline{Y}$

4. One-tile single valence finite subdivision rules

We define a **one-tile single valence finite subdivision rule** $\mathcal{R}$ as follows. We assume that $\mathcal{R}$ has bounded valence, that $\mathcal{R}$ is orientation preserving, and that the mesh of $\mathcal{R}$ approaches 0. We fix an orientation on the open tiles of $S_{\mathcal{R}}$ such that the restriction of $\sigma_{\mathcal{R}}$ to each open tile of $\mathcal{R}(S_{\mathcal{R}})$ preserves orientation. We make the following two further assumptions.

(1) If s and t are tile types of $\mathcal{R}$, then there exists an orientation-preserving infinite cellular isomorphism from s to t.
(2) There exists a positive integer r such that if t is a tile type of $\mathcal{R}$, then every interior vertex of $\mathcal{R}^n(t)$ has valence r for every positive integer n.

Just as for one-tile rotationally invariant finite subdivision rules, we use the expression "one-tile" to indicate that the subdivisions of the tile types of $\mathcal{R}$ look the same.

We will see that one-tile single valence finite subdivision rules provide interesting examples to which Theorem 6.10 of [9] can be applied. First we turn to the following lemma, which shows that our assumptions for one-tile single valence finite subdivision rules are very restrictive.

Lemma 4.1. *Let $\mathcal{R}$ be a one-tile single valence finite subdivision rule. Let t be a tile type of $\mathcal{R}$. Suppose that t has q edges and that r is the valence of every interior vertex of $\mathcal{R}^n(t)$ for every positive integer n. Then the ordered pair (q, r) is either* $(3, 6)$, $(4, 4)$ *or* $(6, 3)$.

Proof. If n is a positive integer, then $\mathcal{R}^n$ satisfies the assumptions of Lemma 4.1, and it suffices to prove Lemma 4.1 for $\mathcal{R}^n$. Thus since the mesh of $\mathcal{R}$ approaches 0, we may assume that some tile of $\mathcal{R}(t)$ lies in the interior of t.

In this paragraph we fix some notation and deduce some facts about the subdivisions of t. Let k be the number of tiles of $\mathcal{R}(t)$. For each nonnegative integer n, let b_n denote the number of tiles of $\mathcal{R}^n(t)$ which meet $\partial\mathcal{R}^n(t)$ and let e_n denote the number of edges of $\partial\mathcal{R}^n(t)$. The complex $\mathcal{R}^n(t)$ has k^n tiles for every nonnegative integer n. Since some tile of $\mathcal{R}(t)$ lies in the interior of t, we see that $b_{n+1} \leq (k-1)b_n$ for every n and hence $b_n \leq (k-1)^n$ for every n. Since $e_n \leq qb_n$ for every n, we see that $\lim_{n\to\infty} \frac{e_n}{k^n} = 0$.

Let n be a nonnegative integer. Choose an orientation for $\partial\mathcal{R}^n(t)$, and for each edge e in $\partial\mathcal{R}^n(t)$, let v_e be the valence of the initial vertex of e. We compute the Euler characteristic of t using $\mathcal{R}^n(t)$. By first counting vertices and edges as if they are in the interior of $\mathcal{R}^n(t)$ and then making corrections for boundary vertices and edges we obtain the following:

$$1 = \left(\frac{k^n q}{r} + \sum_{e\in\partial\mathcal{R}^n(t)} \left(1 - \frac{v_e - 1}{r}\right)\right) - \left(\frac{k^n q}{2} + \frac{e_n}{2}\right) + k^n.$$

Hence

$$1 - k^n\left(1 - \frac{q}{2} + \frac{q}{r}\right) = \sum_{e\in\partial\mathcal{R}^n(t)} \left(\frac{1}{2} - \frac{v_e - 1}{r}\right).$$

We have that $0 \leq v_e - 1 \leq r$ and so $|\frac{1}{2} - \frac{v_e-1}{r}| \leq 1$ for every boundary edge e. Hence

$$|1 - k^n(1 - \frac{q}{2} + \frac{q}{r})| \leq e_n.$$

Now we divide this inequality by k^n, let n go to ∞, and use the last result of the previous paragraph to conclude that $1 - \frac{q}{2} + \frac{q}{r} = 0$. Hence $2r - qr + 2q = 0$. Letting $Q = q - 2$ and $R = r - 2$, we obtain $QR = 4$. Hence (Q, R) is either $(1, 4)$, $(2, 2)$ or $(4, 1)$, and so (q, r) is either $(3, 6)$, $(4, 4)$ or $(6, 3)$.

This proves Lemma 4.1. □

Lemma 4.1 was inspired by Eric Swenson. It shows that if we apply Lemma 2.5 of [9] to a tile type of $\mathcal{R}$, then we obtain an expansion complex which is isomorphic as a cell complex to the standard cellular decomposition of $\mathbb{R}^2$ by either equilateral triangles, squares, or regular hexagons.

We continue with four examples.

Example 4.2. In this example there are two tile types and three edge types. The tile types are equilateral triangles. The subdivisions of the tile types are shown in Figure 9, where we label the edges of each tile by either a, b, and c or d, e, and f to indicate combinatorially the subdivision rule. In this example, edges labeled a and d have the same edge type, edges labeled b and e have the same edge type and edges labeled c and f have the same edge type. The model subdivision complex is a torus.

Example 4.3. In this example there is one tile type and it is a regular hexagon. The subdivision of the tile type is shown in Figure 10, where we label the edges of each tile by a, b, c, d, e, and f to indicate combinatorially the subdivision rule. There are three edge types; edges labeled by a and f have one edge type, edges

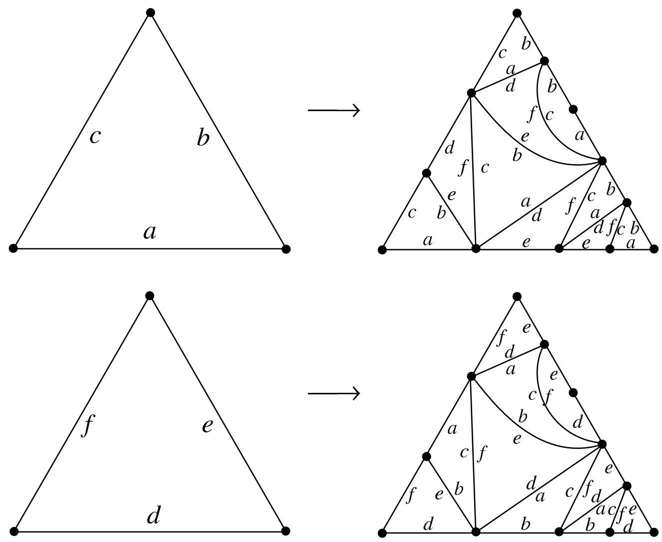

FIGURE 9. The subdivision of the tile types for Example 4.2

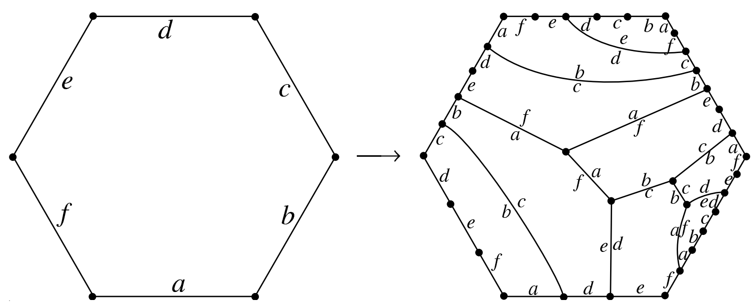

FIGURE 10. The subdivision of the tile type for Example 4.3

labeled by b and c have a second edge type, and edges labeled by d and e have the third edge type. The model subdivision complex is a sphere.

Example 4.4. In this example there is one tile type and it is a square. The subdivision of the tile type is shown in Figure 11, where we label the edges of each tile by a, b, c, and d to indicate combinatorially the subdivision rule. Edges labeled a and c have the same edge type, and edges labeled b and d have the same edge type. The model subdivision complex is a torus.

Example 4.5. In this example there is one tile type and it is a square. The subdivision of the tile type is shown in Figure 12, where we label the edges of each tile by a, b, c, and d to indicate combinatorially the subdivision rule. Edges labeled a and c have the same edge type, and edges labeled b and d have the same edge type. The model subdivision complex is a torus.

One can check that all of these examples are one-tile single valence finite subdivision rules.

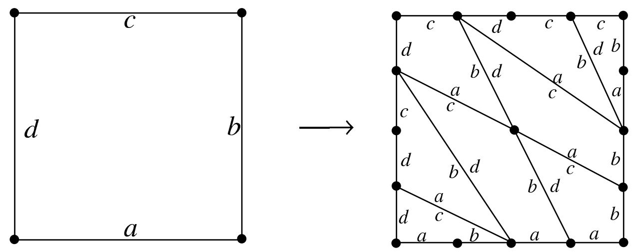

FIGURE 11. The subdivision of the tile type for Example 4.4

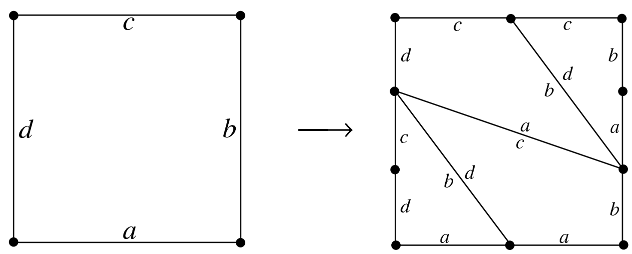

FIGURE 12. The subdivision of the tile type for Example 4.5

Let $\mathcal{R}$ be a one-tile single valence finite subdivision rule. Let X be an orientation-preserving expansion $\mathcal{R}$-complex with expansion map φ such that $X = \mathbb{R}^2$. With Lemma 4.1 in mind, we assume that X gives a regular tiling of the plane by either equilateral triangles, squares, or regular hexagons. In addition to this, we assume that 0 is a vertex of X and that $\varphi(0) = 0$. We want to determine whether or not $(X, \mathcal{R})$ is conformal. In the main theorem of [10] we prove that there exists a subset Λ of the set of vertices of X which is a lattice in $\mathbb{R}^2$ such that the restriction of φ to Λ is a group homomorphism.

Assuming the main theorem of [10], we now show how to determine whether or not $(X, \mathcal{R})$ is conformal. Because the restriction of φ to Λ is a group homomorphism and φ is injective, $\varphi(\Lambda)$ is a subgroup of $\mathbb{R}^2$ isomorphic to $\mathbb{Z}^2$. Since $\varphi(\Lambda)$ is discrete, it is a lattice in $\mathbb{R}^2$. Hence there exists an $\mathbb{R}$-linear isomorphism $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that the restriction of ψ to Λ equals the restriction of φ to Λ. We next show for every $x \in \mathbb{R}^2$ that $\varphi(x)$ and $\psi(x)$ are near one another as in the assumptions of Theorem 6.6 of [9]. As noted just before Example 2.1 in [9], there is an $\mathcal{R}$-complex W such that $X = \mathcal{R}(W)$ and $\varphi\colon X \to W$ is an $\mathcal{R}$-isomorphism. Let d_0 denote the skinny path pseudometric for X, and let d_{-1} denote the skinny path pseudometric for W. Let F be a parallelogram which is a closed fundamental domain for Λ. There exists a positive real number L with the following property. Let $x \in \mathbb{R}^2$. Then there exists $y \in \Lambda$ such that $x \in y + F$ and $d_0(x, y) \le L$. We maintain the meaning of x and y. Hence $d_{-1}(\varphi(x), \varphi(y)) \le L$. Hence there exists a positive real number M independent of x and y such that $d_0(\varphi(x), \varphi(y)) \le M$. We also have

$$\psi(x) \in \psi(y + F) = \psi(y) + \psi(F) = \varphi(y) + \psi(F).$$

Just as for L, there exists a positive real number N independent of x and y such that $d_0(\psi(x), \varphi(y)) \le N$. The triangle inequality now implies that $d_0(\varphi(x), \psi(x)) \le M + N$. This means that $\varphi(x)$ and $\psi(x)$ are near one another as in the assumptions

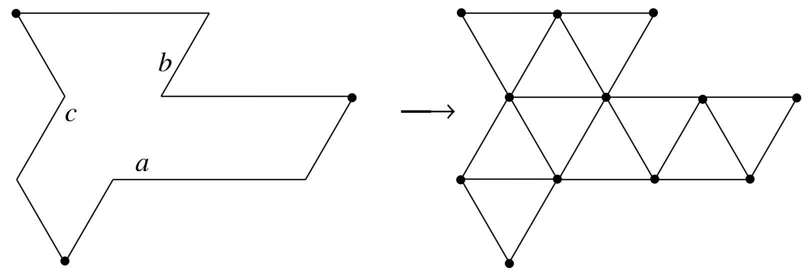

FIGURE 13. The subdivision of a tile of W for Example 4.2

of Theorem 6.6 of [9]. Because ψ agrees with φ on Λ, it is easy to see that the eigenvalues of ψ have absolute values greater than 1. Hence the eigenvalues of ψ^{-1} have absolute values less than 1, and so the mesh of the sequence of tilings $\{\psi^{-n}(\mathcal{S}(X))\}$ approaches 0. So all of the assumptions of Theorem 6.6 of [9] are satisfied in the present situation.

Hence the assumptions of Theorem 6.10 of [9] are satisfied with the X' of Theorem 6.6 of [9] replacing the X of Theorem 6.10 of [9], the present ψ replacing the φ of Theorem 6.10 of [9], and the present $\mathcal{S}(X)$ replacing the $\mathcal{T}$ of Theorem 6.10 of [9]. Theorem 6.10 of [9] now gives us a simple criterion to determine whether or not $(X, \mathcal{R})$ is conformal: $(X, \mathcal{R})$ is conformal if and only if either ψ is a dilation or the eigenvalues of ψ are not real.

We next apply this criterion to the above four examples.

For Example 4.2, Figure 13 shows the subdivision of a tile of W and Figure 14 shows part of an expansion $\mathcal{R}$-complex X (with W drawn with thick edges). The vertex in Figure 14 marked by a large dot is the origin 0. Let t denote the tile of W in Figure 13. It is the tile in Figure 14 containing 0 which is translation equivalent to the tile in Figure 13. Let s be the tile of X in t containing 0. There is an expansion map $\varphi\colon X \to W$ with $\varphi(0) = 0$ and $\varphi(s) = t$. One checks that the restriction of φ to the vertices of X is a group homomorphism; Figure 15 shows two copies of a portion of Figure 14 with a fundamental domain for the lattice Λ consisting of the vertices of X and a fundamental domain for $\varphi(\Lambda)$. Let u be the vertex of s other than 0 in the edge labeled a, and let v be the vertex of s other than 0 in the edge labeled c. With respect to the ordered basis (u, v) of $\mathbb{R}^2$, the linear operator on $\mathbb{R}^2$ which maps u to $\varphi(u)$ and v to $\varphi(v)$ has matrix $\begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix}$. Its characteristic polynomial is $x^2 - 6x + 10$, and its eigenvalues are $(6 \pm \sqrt{36-40})/2 = 3 \pm \sqrt{-1}$. Thus $(X, \mathcal{R})$ is conformal by Theorem 6.10 of [9].

Similarly, for Example 4.3, Figure 16 shows the subdivision of a tile t of W and Figure 17 shows part of an expansion $\mathcal{R}$-complex X and the complex W. The vertex in Figure 17 marked by a large dot is the origin 0. The tile t is the tile in Figure 17 containing 0 which is translation equivalent to the tile in Figure 16. Let s be the tile of X in t containing 0. There is an expansion map $\varphi\colon X \to W$ with $\varphi(0) = 0$ and $\varphi(s) = t$. Figure 18 shows two copies of a portion of Figure 17 with the fundamental domains of two lattices. One checks that the restriction of φ to the lattice Λ with the smaller fundamental domain is a group homomorphism onto the lattice with the larger fundamental domain. The corresponding linear map

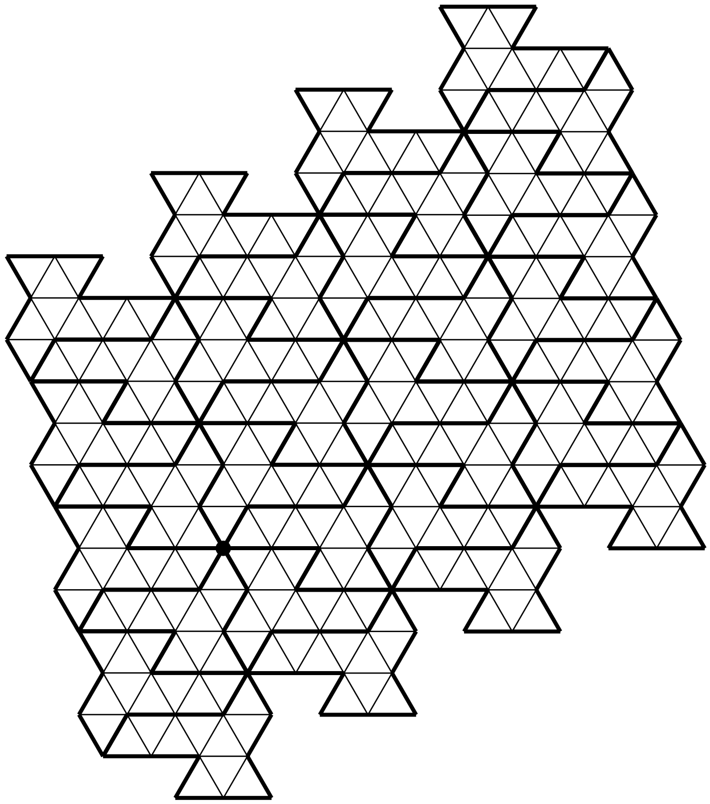

FIGURE 14. Parts of the $\mathcal{R}$-complexes X and W for Example 4.2

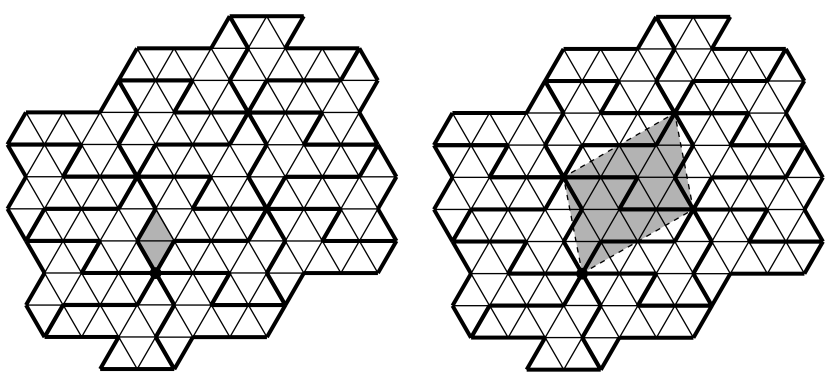

FIGURE 15. Fundamental domains for Λ and $\varphi(\Lambda)$

evidently dilates by a factor of 3 and rotates through an angle of $\pi/3$. Thus $(X, \mathcal{R})$ is conformal by Theorem 6.10 of [9].

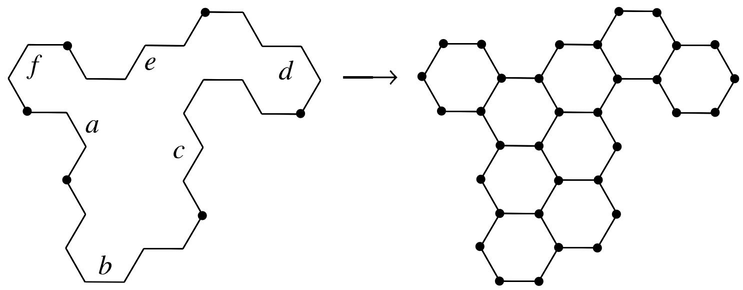

FIGURE 16. The subdivision of a tile of W for Example 4.3

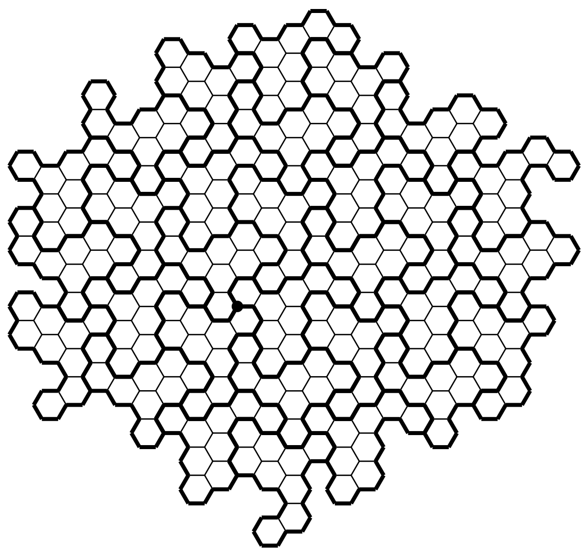

FIGURE 17. Parts of the $\mathcal{R}$-complexes X and W for Example 4.3

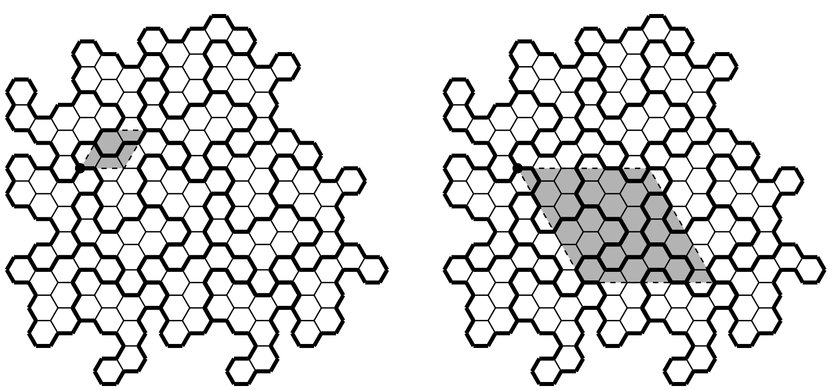

FIGURE 18. Fundamental domains for Λ and $\varphi(\Lambda)$

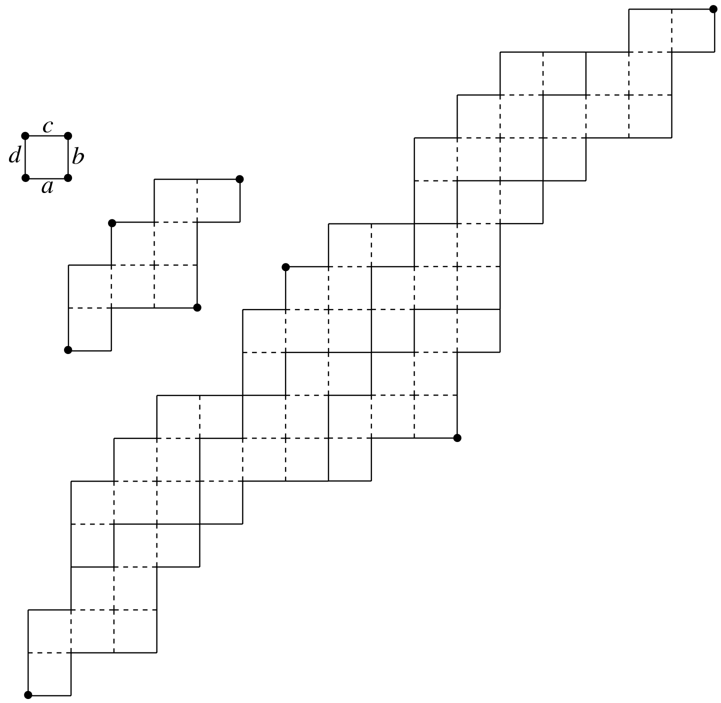

FIGURE 19. A tile t of X with $\varphi(t)$ and $\varphi^2(t)$

For Example 4.4, Figure 19 shows a tile t of X with $\varphi(t)$ and $\varphi^2(t)$. We assume that the origin is the vertex contained in the edges of t labeled a and d. In this case we obtain the matrix $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$. Its characteristic polynomial is $x^2 - 6x + 8$, and its eigenvalues are 2 and 4. Thus $(X, \mathcal{R})$ is not conformal by Theorem 6.10 of [9].

For Example 4.5, Figure 20 shows a tile t of X with $\varphi(t)$, $\varphi^2(t)$, and $\varphi^3(t)$. We assume that the origin is the vertex contained in the edges of t labeled a and d. In this case the matrix is $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Thus $(X, \mathcal{R})$ is not conformal by Theorem 6.10 of [9].

We return to our expansion complex X obtained from the finite subdivision rule $\mathcal{R}$ of Example 4.2. We saw that $(X, \mathcal{R})$ is conformal. By the proof of Theorem 6.10 of [9], there is an expansion $\mathcal{R}$-complex X' which is $\mathcal{R}$-isomorphic to X such that the expansion map for X' is given in complex coordinates by $z \mapsto (3 - i)z$. Figure 21 shows part of X'; the thick curves show the boundary of the image under the expansion map of a tile of X. The expansion complex X' has exactly two tiles up to the equivalence relation of translation. Furthermore, for any tile t of X', $(3-i)t$ is a union of tiles of X'. That is, the tiling given by X' is a self-similar tiling. We refer the reader to [15] for a discussion of self-similar tilings.

We now show that this is true in general. Let $\mathcal{R}$ be a one-tile single valence finite subdivision rule. Let X be an orientation-preserving expansion $\mathcal{R}$-complex such that X gives a regular tiling of the plane. Suppose that the expansion map

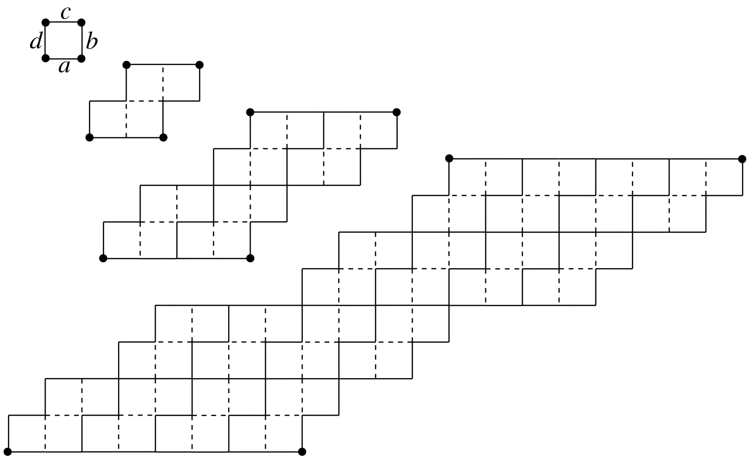

FIGURE 20. A tile t of X with $\varphi(t)$, $\varphi^2(t)$, and $\varphi^3(t)$

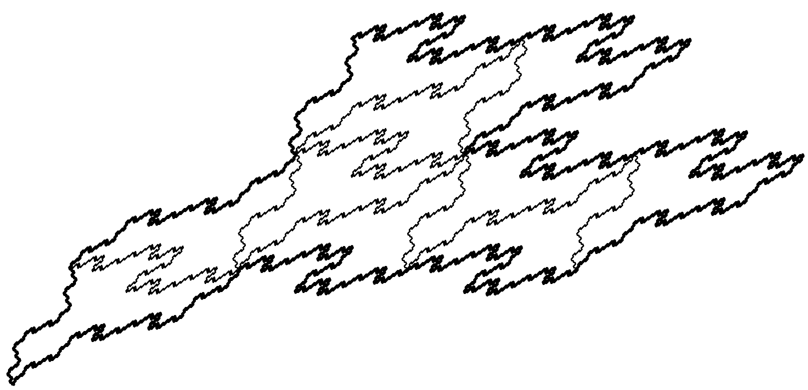

FIGURE 21. Part of a self-similar tiling for Example 4.2.

φ of X fixes a vertex of X. We saw in the discussion after Example 4.5 that there is an $\mathbb{R}$-linear isomorphism ψ such that $\mathcal{R}$, X, φ, and ψ satisfy the hypotheses of Theorem 6.6 of [9]. The conclusion of Theorem 6.6 of [9] establishes the hypotheses of Theorem 6.10 of [9]. From this we conclude that $(X, \mathcal{R})$ is conformal if and only if ψ is either a dilation or has eigenvalues which are not real numbers.

Now suppose that $(X, \mathcal{R})$ is conformal. Then there is an expansion $\mathcal{R}$-complex X' and an $\mathcal{R}$-isomorphism $\rho\colon X \to X'$ such that ρ is the limit of the sequence of functions $\{\psi^{-n} \circ \varphi^n\}$. If the tiles of X are equilateral triangles, then there are two tiles up to translation. If the tiles of X are either squares or regular hexagons, then all tiles of X are equivalent under translation. So the number of translation equivalence classes of tiles of X is either 2, 1, or 1. Let t be a tile of X. Because the tiles of the form $\varphi(t)$ need not be rotationally symmetric, the number of these translation equivalence classes is at most 6, 4, or 6. If n is a positive integer, then the same is true for tiles of the form $\varphi^n(t)$ and $(\psi^{-n} \circ \varphi^n)(t)$. It follows easily from this that X' has at most six translation equivalence classes of tiles.

Since ψ is either a dilation or has eigenvalues which are not real numbers, there is an $\mathcal{R}$-linear isomorphism $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $T^{-1} \circ \psi \circ T$ is given in complex coordinates by $z \mapsto \lambda z$ for some complex number λ. Let $\widetilde{X}$ be the tiling of $\mathbb{R}^2$ which is the image of X' under T^{-1}. Then $\widetilde{X}$ has only finitely many tiles up to translation. Furthermore, for each tile t of $\widetilde{X}$, λt is a union of tiles of $\widetilde{X}$. That is, $\widetilde{X}$ is a self-similar tiling.

5. An interesting example

In this section we describe a finite subdivision rule $\mathcal{R}$ with bounded valence and mesh approaching 0 that has a hyperbolic expansion complex X and a parabolic expansion complex Y. Furthermore, $\mathcal{R}$ is irreducible, any compact $\mathcal{R}$-subcomplex of X is $\mathcal{R}$-isomorphic to a subcomplex of Y, and any compact $\mathcal{R}$-subcomplex of Y is $\mathcal{R}$-isomorphic to a subcomplex of X.

The model subdivision complex $S_{\mathcal{R}}$ has one vertex, two edges, and six tiles. The subdivisions of the six tile types are shown in Figure 22; the tiles in the subdivisions are labeled by their tile types. A thin edge is subdivided into two thin edges, each directed away from the barycenter. A thick edge is subdivided into three thick edges, each with the same direction as the original edge. This determines the orientation of every edge in the subdivision of every tile type of $\mathcal{R}$ except for the two thick edges in $\mathcal{R}(t_3)$ which are contained in tiles of type t_4. We orient these two edges from left to right. Note that $\mathcal{R}$ does not preserve orientation; for example, the structure map of the tile of type 2 reverses orientation on every subtile of type 3. The complex $S_{\mathcal{R}}$ is obtained from the union of the tile types by identifying all of the thin edges and identifying all of the thick edges. The subdivision map of $S_{\mathcal{R}}$ can be defined so that $\mathcal{R}$ has bounded valence and mesh approaching 0. Given tile types t_i and t_j, there exists a positive integer n such that $\mathcal{R}^n(t_i)$ contains a copy of t_j. In other words, $\mathcal{R}$ is irreducible.

The expansion complex X is constructed as in Lemma 2.4 of [9]. Figure 23 shows the complexes X_0, X_1, X_2, and X_3. The tiles in X_0 all have type t_1. The orientations of the boundary edges of X_0 indicate that two tiles of X_0 are identified with t_1 in an orientation-preserving way and two tiles of X_0 are identified with t_1 in an orientation-reversing way. The complex X_{n-1} naturally embeds in X_n for every positive integer n and we construct X as the direct limit of this directed system. For each nonnegative integer n, let $\operatorname{im}(X_n)$ denote the image of X_n in X, and for each positive integer n, let $R_n = \operatorname{im}(X_n) \setminus \operatorname{int}(\operatorname{im}(X_0))$. The R_n's are rings whose union is $X \setminus \operatorname{int}(\operatorname{im}(X_0))$. We show that for each n, the fat cut modulus $m(R_n, \mathcal{S}(X))$ is bounded above by $3/4$. Since there is a constant K such that $M(R_n, \mathcal{S}(X)) \le K \cdot m(R_n, \mathcal{S}(X))$ by the bounded valence theorem, [5, Theorem 6.2.4] or [6, Theorem 1.6], it follows by Theorem 5.5 of [9] that X is not parabolic. Hence X is hyperbolic.

Lemma 5.1. *Let n be a positive integer. Then $m(R_n, \mathcal{S}(X)) \le 3/4$.*

Proof. For each edge e of $\partial(\operatorname{im}(X_n))$, let p_e be the fat path in R_n with underlying curve a topological path which joins the ends of R_n, has one boundary point in the interior of e, and intersects only thick edges of R_n (that is, edges whose images in $S_{\mathcal{R}}$ are the images of thick edges from the tile types). We also require that p_e meets $\partial(\operatorname{im}(X_m))$ in exactly one point for every $m \le n$. Define a weight function w on R_n in the manner of [5] by $w = \sum_{e \in \partial(\operatorname{im}(X_n))} p_e$. Since there are $4 \cdot 3^{n-1}$ edges in

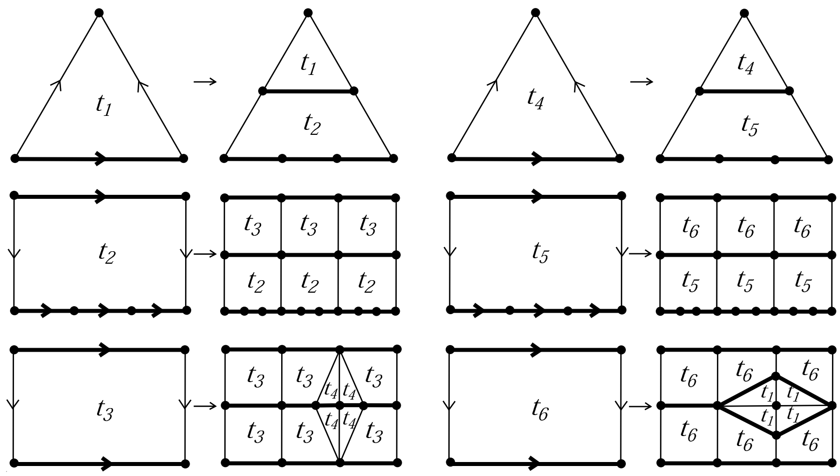

Figure 22. The subdivisions of the six tile types

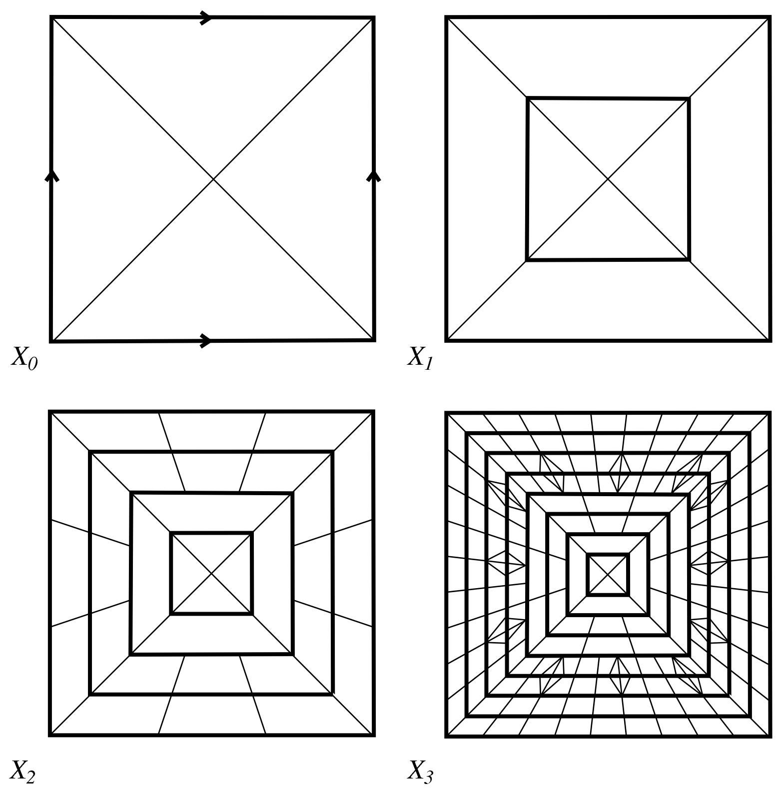

Figure 23. X_i, $i \in \{0, 1, 2, 3\}$

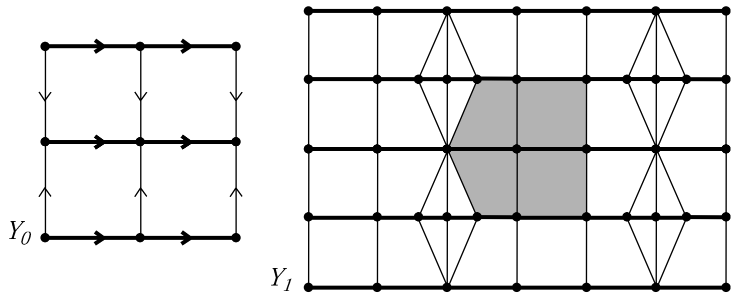

FIGURE 24. Y_0 and Y_1

$\partial(\mathrm{im}(X_n))$ and each closed curve in R_n separating the ends of R_n must intersect each fat path p_e, the circumference $C(R_n, w) \geq 4 \cdot 3^{n-1}$. But each boundary component of R_n is an underlying curve for a fat path with weight $4 \cdot 3^{n-1}$, so $C(R_n, w) = 4 \cdot 3^{n-1}$.

For each $k \in \{1, \ldots, n\}$, $\mathrm{im}(X_k) \backslash \mathrm{int}(\mathrm{im}(X_{k-1}))$ contains $4 \cdot 6^{k-1}$ tiles with weight 3^{n-k}; all of the other tiles have weight 0. The area of R_n with respect to w is

$$\begin{aligned} A(R_n, w) &= \sum_{k=1}^{n} 4 \cdot 6^{k-1} \cdot (3^{n-k})^2 = 4 \cdot 3^{2n-2} \cdot \sum_{k=1}^{n} (2/3)^{k-1} \\ &= 4 \cdot 3^{2n-1} \cdot (1 - (2/3)^n). \end{aligned}$$

Since $m(R_n, \mathcal{S}(X)) = \inf_\omega \frac{A(R_n,\omega)}{C(R_n,\omega)^2}$,

$$m(R_n, \mathcal{S}(X)) \leq \frac{A(R_n, w)}{C(R_n, w)^2} = \frac{3}{4}\left(1 - \left(\frac{2}{3}\right)^n\right) \leq \frac{3}{4}.$$

□

While the expansion complex X is constructed as a direct limit starting from the complex X_0, which is a union of four tiles of type t_1, the expansion complex Y is constructed as a direct limit starting from the complex Y_0, which is a union of four tiles of type t_3. Figure 24 shows Y_0 and Y_1. The orientations of the edges of Y_0 indicate that the top two tiles of Y_0 are identified with t_3 in an orientation-preserving way, and the bottom two tiles of Y_0 are identified with t_3 in an orientation-reversing way. The complex Y_0 naturally embeds in Y_1; its image is shown in gray.

Figure 25 shows part of the expansion complex Y with the fixed point p of the expansion map at the center. (The rings that are drawn in gray will be described later.) Note that Y has vertical strips composed of tiles of type t_3. If t is a tile of type t_3 containing an interior point q such that $\varphi^{-k}(q)$ is in a tile of type t_3 for each positive integer k, then t is part of a bi-infinite "vertical" strip of tiles of type t_3 in which adjacent tiles intersect in a thick edge. If t is in one of these vertical strips, then t is also in a bi-infinite "horizontal" strip in which adjacent tiles intersect in a thin edge. The horizontal strips may contain tiles of types t_3, t_4, t_5, and t_6.

We define an "island" to be the closure of a connected component of the complement of the union of the closed vertical strips of Y. If U is an island of Y and $q \in U$, then the depth of U is the maximum positive integer k such that $\varphi^{1-k}(q)$ is in a tile of type t_4. We define a "chain of islands" to be a connected component of the union of the islands. Any island is in a chain of islands, and all of the islands

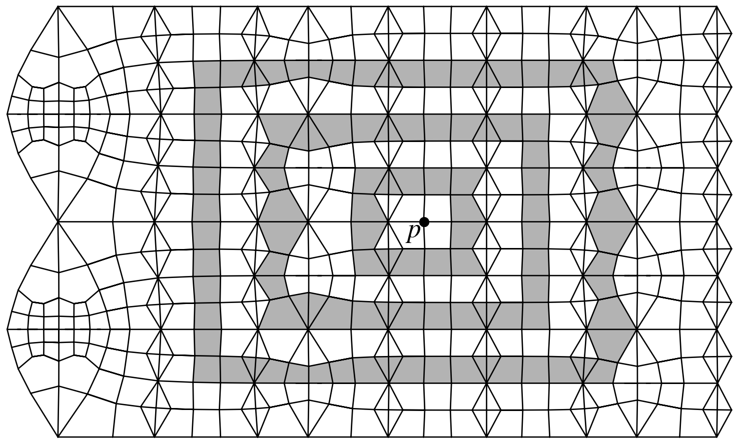

FIGURE 25. Part of the expansion complex Y and the rings C_1, C_3, and C_5

in a chain of islands have the same depth. For example, Figure 26 shows parts of three chains of islands of depth 1, one chain of islands of depth 2, and one chain of islands of depth 4. Note that for any positive integer n, there are at least 3^n vertical strips between any two chains of islands of depth n and there are 3^{n-1} vertical strips separating p and the nearest chain of islands of depth n.

There is a cellular map $\pi\colon Y \to \mathbb{R}^2$ from Y to the square tiling of the plane which maps p to $(0,0)$, maps each of the vertical strips injectively to a vertical strip in $\mathbb{R}^2$, maps each of the horizontal strips to a horizontal strip in $\mathbb{R}^2$, and maps each tile which is not in a vertical strip either to a vertex or to an edge.

We will use Theorem 5.6 of [9] and the layer theorem [7, Theorem 3.1] to show that Y is parabolic. Let A_0 be the star of $(0,0)$ in the square tiling of $\mathbb{R}^2$, let $A_1 = \mathrm{star}(A_0) \setminus \mathrm{int}(A_0)$, and define A_n recursively for $n \geq 2$ by $A_n = \mathrm{star}(A_{n-1}) \setminus (\mathrm{int}(A_{n-1}) \cup A_{n-2})$. Then $\bigcup_{n>0} A_n = \mathbb{R}^2 \setminus \mathrm{int}(A_0)$ and if $n > 0$, then A_n is a ring made up out of $8n+4$ squares. For each positive integer n, let C_n be the closure of $\pi^{-1}(\mathrm{int}(A_n))$. Each C_n is a ring; Figure 25 shows C_1, C_3, and C_5 drawn in gray. We show in Lemma 5.2 that for each n the fat flow modulus $M(C_n, \mathcal{S}(Y)) \geq \frac{1}{27(n+1)\ln(n+1)}$. It follows from the layer theorem that for each positive integer n the modulus of the ring which contains C_1 and C_n, and whose boundary is contained in $C_1 \cup C_n$, is at least $(1/27)\sum_{k=1}^{n} \frac{1}{(k+1)\ln(k+1)}$. Since $\sum_{k=1}^{\infty} \frac{1}{(k+1)\ln(k+1)}$ is divergent, the moduli of these rings are not bounded and so by Theorem 5.6 of [9] Y is not hyperbolic. Hence Y is parabolic.

Lemma 5.2. *Let n be a positive integer. Then*

$$M(C_n, \mathcal{S}(Y)) \geq \frac{1}{27(n+1)\ln(n+1)}.$$

Proof. Let w be the weight function which assigns weight 1 to each tile in C_n and assigns weight 0 to every other tile. Then the height $H(C_n, w) = 1$ and so $M(C_n, w) = \frac{H(C_n,w)^2}{A(C_n,w)} = \frac{1}{A(C_n,w)}$. Since every tile in C_n has weight 1, $A(C_n, w)$ is just the number of tiles in C_n. We view C_n as being decomposed into a top (which is contained in a horizontal strip), a bottom (which is also in a horizontal strip),

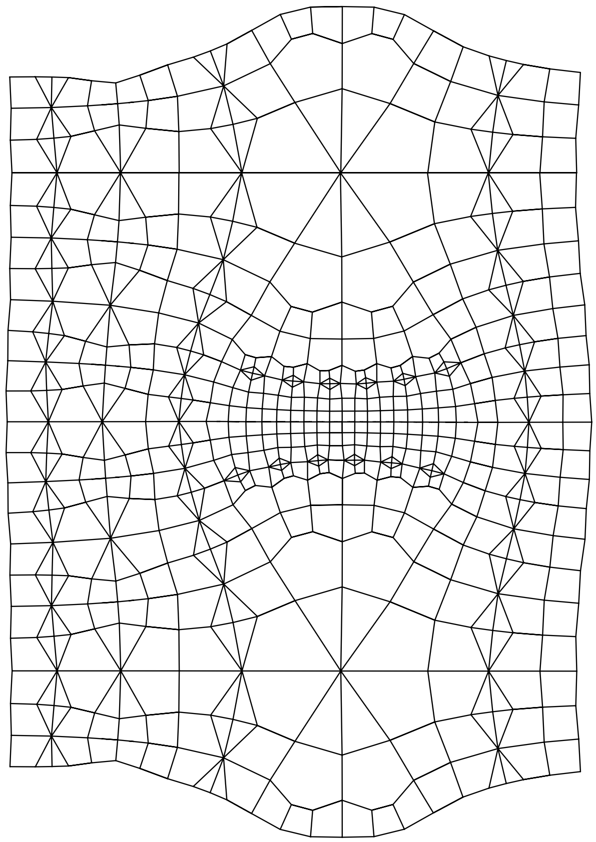

FIGURE 26. Chains of islands in Y

and two sides (each of which is in a vertical strip). We see that C_n contains $8n+4$ tiles of type t_3. In addition it can contain tiles of type t_4, t_5, and t_6 from the top and from the bottom. Let $i = \lfloor \frac{\ln(n+1)}{\ln(3)} \rfloor$. Since for each k there are 3^{k-1} vertical strips separating p from the nearest chain of islands of depth k, the top and bottom can contain tiles in islands of depth k only if $n+1 > 3^{k-1}$, which occurs only if $k \leq i+1$. Since the top and bottom each intersect only $2n+2$ vertical strips and chains of islands of depth k are separated by at least 3^k vertical strips, the top and bottom can each intersect at most $1 + \frac{2n+2}{3^k}$ islands of depth k. Furthermore, the number of tiles that the top or bottom can have in a given island of depth k is 2 if $k=1$ and $2 \cdot 3^{k-2}$ if $k > 1$. Hence

$$
\begin{aligned}
A(C_n, w) &\leq (8n+4) + 2 \cdot 2 \cdot \left(1 + \frac{2n+2}{3}\right) + 2\sum_{k=2}^{i+1} 2 \cdot 3^{k-2}\left(1 + \frac{2n+2}{3^k}\right) \\
&= \frac{32}{3}(n+1) + \frac{8}{9}(n+1)(i) + 2 \cdot (3^i - 1) \\
&\leq 11(n+1) + (n+1)\ln(n+1) + 2(n+1) \\
&\leq 13(n+1) + (n+1)\ln(n+1).
\end{aligned}
$$

Since $1 \leq \ln(4) = 2\ln(2)$, we have $13 \leq 26\ln(2) \leq 26\ln(n+1)$. So $A(C_n, w) \leq 27(n+1)\ln(n+1)$. Hence

$$M(C_n, \mathcal{S}(Y)) \geq \frac{1}{27(n+1)\ln(n+1)}. \qquad \square$$

Finally, note that X_0 is $\mathcal{R}$-isomorphic to a subcomplex of $\mathcal{R}(t_6)$, and so every compact $\mathcal{R}$-subcomplex of X is $\mathcal{R}$-isomorphic to a subcomplex of Y. Similarly, Y_0 is $\mathcal{R}$-isomorphic to a subcomplex of $\mathcal{R}^2(t_3)$, and so every compact $\mathcal{R}$-subcomplex of Y is $\mathcal{R}$-isomorphic to a subcomplex of X.

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Department of Mathematics, Brigham Young University, Provo, Utah 84602
E-mail address: cannon@math.byu.edu

Department of Mathematics, Virginia Tech, Blacksburg, Virginia 24061
E-mail address: floyd@math.vt.edu
URL: http://www.math.vt.edu/people/floyd

Department of Mathematics, Eastern Michigan University, Ypsilanti, Michigan 48197
E-mail address: walter.parry@emich.edu