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QUASI-METRIC AND METRIC SPACES

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ABSTRACT. We give a short review of a construction of Frink to obtain a metric space from a quasi-metric space. By an example we illustrate the limits of the construction.

1. Introduction

A *quasi-metric space* is a set Z with a function $\rho : Z \times Z \to [0, \infty)$ which satisfies the conditions

(1) $\rho(z, z') \geq 0$ for every z, $z' \in Z$ and $\rho(z, z') = 0$ if and only if $z = z'$;
(2) $\rho(z, z') = \rho(z', z)$ for every z, $z' \in Z$;
(3) $\rho(z, z'') \leq K \max\{\rho(z, z'), \rho(z', z'')\}$ for every z, z', $z'' \in Z$ and some fixed $K \geq 1$.

The function ρ is called in that case a *quasi-metric*, or more specifically, a K-quasi-metric. The property (3) is a generalized version of the *ultra-metric* triangle inequality (the case $K = 1$).

Remark 1.1. If (Z, d) is a metric space, then d is a K-quasi-metric for $K = 2$. In general d^p is not a metric on Z for $p > 1$. But d^p is still a 2^p-quasi-metric.

We are interested in the question of how to obtain a metric on Z. Our personal motivation comes from the study of the boundary at infinity of a Gromov hyperbolic space, where this question arises naturally, see e.g., [BS, chapter 2], [BoF]. The problem was studied by Frink in the interesting paper [Fr]. The motivation of Frink was to obtain suitable conditions for a topological space to be metrizable. Frink used a natural approach, which we call the *chain approach* to obtain a metric on Z. He showed that this approach works and gives a metric if the space (Z, ρ) satisfies the axioms (1) and (2) above instead of (3) the *weak triangle inequality*

($3'$) If $\rho(z, z') \leq \varepsilon$ and $\rho(z', z'') \leq \varepsilon$, then $\rho(z, z'') \leq 2\varepsilon$.

Observe that ($3'$) is equivalent to (3) with constant $K = 2$, but the formulation as a weak triangle inequality points out that the constant $K = 2$ plays a special role.

In this short note we give a review of Frink's approach and show that there exists a "natural" counterexample to the chain approach in case the weak triangle inequality ($3'$) is not satisfied.

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1.1. **Quasi-metrics and metrics.** Let (Z,ρ) be a quasi-metric space. We want to obtain a metric on Z. Since ρ satisfies all axioms of a metric space except the triangle inequality, the following approach is very natural. Define a map $d: Z\times Z \to [0,\infty)$, $d(z,z') = \inf \sum_i \rho(z_i, z_{i+1})$, where the infimum is taken over all sequences $z = z_0, \dots, z_{n+1} = z'$ in Z. By definition, d satisfies the triangle inequality. We call this approach to the triangle inequality the *chain approach*. The problem with the chain approach is that $d(z,z')$ could be 0 for different points z, z' and axiom (1) is no longer satisfied for (Z,d).

This chain approach is due to Frink, who realized that the approach works if the space (Z,ρ) satisfies the axioms (1), (2) and (3′) above. For the convenience of the reader we give a proof of Frink's result.

Theorem 1.2. *Let ρ be a K-quasi-metric on a set Z with $K \le 2$. Then, the chain construction applied to ρ yields a metric d with $\frac{1}{2K}\rho \le d \le \rho$.*

Proof. Clearly, d is nonnegative, symmetric, satisfies the triangle inequality and $d \le \rho$. We prove by induction over the length of sequences $\sigma = \{z = z_0, \dots, z_{k+1} = z'\}$, $|\sigma| = k+2$, that

$$(1) \qquad \rho(z,z') \le \sum(\sigma) := K\left(\rho(z_0,z_1) + 2\sum_{1}^{k-1}\rho(z_i,z_{i+1}) + \rho(z_k,z_{k+1})\right).$$

For $|\sigma| = 3$, this follows from the triangle inequality (3) for ρ. Assume that (1) holds true for all sequences of length $|\sigma| \le k+1$, and suppose that $|\sigma| = k+2$.

Given $p \in \{1,\dots,k-1\}$, we let $\sigma'_p = \{z_0,\dots,z_{p+1}\}$, $\sigma''_p = \{z_p,\dots,z_{k+1}\}$, and note that $\sum(\sigma) = \sum(\sigma'_p) + \sum(\sigma''_p)$.

Because $\rho(z,z') \le K\max\{\rho(z,z_p),\rho(z_p,z')\}$, there is a maximal $p \in \{0,\dots,k\}$ with $\rho(z,z') \le K\rho(z_p,z')$. Then $\rho(z,z') \le K\rho(z,z_{p+1})$.

Assume now that $\rho(z,z') > \sum(\sigma)$. Then, in particular, $\rho(z,z') > K\rho(z,z_1)$ and $\rho(z,z') > K\rho(z_k,z')$. It follows that $p \in \{1,\dots,k-1\}$ and thus by the inductive assumption

$$\rho(z,z_{p+1}) + \rho(z_p,z') \le \sum(\sigma'_p) + \sum(\sigma''_p) = \sum(\sigma) < \rho(z,z').$$

On the other hand,

$$\rho(z,z') \le K\min\{\rho(z,z_{p+1}),\rho(z_p,z')\} \le \rho(z,z_{p+1}) + \rho(z_p,z')$$

because $K \le 2$; a contradiction. Now, it follows from (1) that $\rho \le 2Kd$; hence, d is a metric as required. ☐

2. Example

In this section we construct for any given $\varepsilon > 0$ a quasi-metric space (Z,ρ) such that the chain approach does not lead to a metric space and the following holds: For every triple of points z_0, z_1, z_2 we have

$$\rho(z_1,z_2) \le (1+\varepsilon)[\rho(z_1,z_0) + \rho(z_0,z_2)].$$

Let, therefore, $a \in (0,\frac{1}{2})$ be a given constant. Let Z be the set of dyadic rationals of the interval $[0,1]$. Then Z is the disjoint union of Z_n , $n \in \mathbb{N}$, where $Z_0 = \{0,1\}$, and $Z_n = \{\frac{k}{2^n} : 0 < k < 2^n, k \text{ odd}\}$ for $n \ge 1$. If $z \in Z_n$, we say that the level of z is n and write $\ell(z) = n$. For the following construction it is useful to see Z embedded by $z \mapsto (z,\ell(z))$ as a discrete subset of the plane. Let $z = \frac{k}{2^n} \in Z_n$ with $n \ge 1$, then we define the right and the left neighbors $l(z) = \frac{k-1}{2^n}$ and $r(z) = \frac{k+1}{2^n}$. We see that $\ell(l(z)), \ell(r(z)) < n$ and clearly $l(z) < z < r(z)$, where we take the usual ordering induced by the reals. Given $z \in Z$ with $\ell(z) \geq 1$ we consider the *right path* $z, r(z), r^2(z), \ldots$ and the left path $z, l(z), l^2(z), \ldots$. Note that after a finite number of steps, the right path always ends at 1 and the left path always ends at 0.

We use the following facts:

Fact 1: Consider for an arbitrary $z \in Z$ the levels of the vertices on the right and on the left path, i.e., $\ell(l(z)), \ell(l^2(z)), \ldots$ and $\ell(r(z)), \ell(r^2(z)), \ldots$. Then all intermediate levels n with $0 < n < \ell(z)$ occur exactly once (either on the right or on the left path). E.g., consider $11/64$ which is of level 6. The left path is $11/64, 5/32, 1/8, 0$ (containing the intermediate levels 5 and 3), the right path is $11/64, 3/16, 1/4, 1/2, 1$ (containing the remaining intermediate levels $4, 2$ and 1). This fact can be verified by looking at the dyadic expansion of z, e.g., $11/64 = 0.001011$. Note that the dyadic expression of $l(z)$ is obtained from the one of z by removing the last 1 in this expression, i.e., $l(0.001011) = 0.00101$. The dyadic expression of $r(z)$ is obtained by removing the last consecutive sequence of 1's and putting a 1 instead of the 0 in the last entry before the sequence, e.g., $r(0.001011) = 0.0011$. Therefore the levels of the left path (resp. of the right path) correspond to the places with a 1 (resp. with a 0) in the dyadic expansion.

Fact 2: Let $l^k(z)$ be a point on the left path and $\ell(l^k(z)) \geq 1$. Let m be the integer, such that $r^m(z)$ is the first point on the right path with $\ell(r^m(z)) < \ell(l^k(z))$, then $r(l^k(z)) = r^m(z)$. A corresponding statement holds for points on the right path. This fact can also be verified by looking at the dyadic expansion.

We consider the graph whose vertex set is Z, and the edges are given by the pairs $\{0, 1\}, \{z, r(z)\}, \{z, l(z)\}$, where the $z \in Z$ are points with level ≥ 1. One can visualize this graph nicely, if we use the realization of Z in the plane described above. In this picture we can see the edges as line intervals and the graph is planar. In this picture the left path from a point z with $\ell(z) \geq 1$ can be viewed as the graph of a piecewise linear function defined on the interval $[0, z]$ (here $z \in [0, 1]$) and the right path as the graph of a piecewise linear function on $[z, 1]$. The union of these two paths form a "tent" in this picture (see Figure 1).

Fact 3: Below this tent there lies no point of Z.

To every edge in this graph we associate a length. To the edge $\{0, 1\}$ we associate the length 1, and to an edge of the type $\{z, l(z)\}$ and $\{z, r(z)\}$ we associate the length $a^{\ell(z)}$. Now we define the quasi-metric ρ. First, set $\rho(0, 1) = 1$. Let $z, z' \in Z$ be points such that z, z' is not the pair $0, 1$. Let us assume $z < z'$. Then we consider the right path $z, r(z), r^2(z), \ldots, 1$ starting from z, and the left path $z', l(z'), l^2(z'), \ldots, 0$ starting from z'. Then the properties from above imply that these two paths intersect at a unique point $r^k(z) = l^s(z')$. Then we obtain a V-shaped path $z, r(z), \ldots, r^k(z) = l^s(z'), \ldots, l(z'), z'$ formed by edges from our graph from z to z'. We define $\rho(z, z')$ to be the sum of the lengths of the edges of this path.

The main point is now to show that ρ is a quasi-metric space. Before we prove this, we show that the chain approach does not give a metric. Therefore, consider for any integer n the chain $0, 1/2^n, 2/2^n, 3/2^n, \ldots, 2^n/2^n = 1$. By our definition $\rho(i/2^n, (i+1)/2^n) = a^n$. Thus the length of the chain is $2^n a^n$ which converges to 0 since $a < 1/2$.

It remains to show that ρ is a quasi-metric.

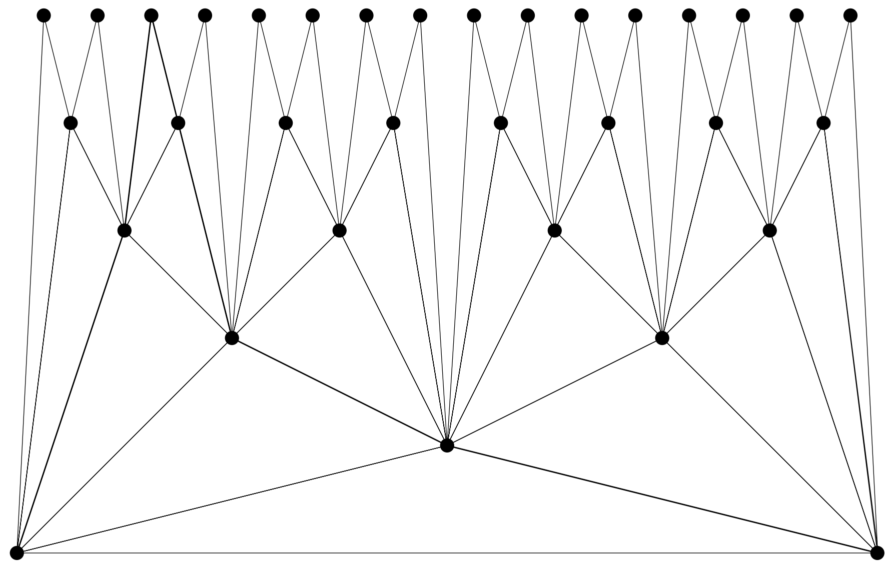

FIGURE 1. Graph with a tent

Fact 4: If $z \in Z_n$, then $\rho(z,0)+\rho(z,1)=\tau_n$ where
$$\tau_n = a + a^2 + \cdots + a^{n-1} + 2a^n.$$

To obtain this fact consider the tent formed by the left path from 0 to z and the right path from z to 1 and consider the levels of the points on this path. By Fact 1, all intermediate levels occur exactly once. Thus the formula comes immediately from the definition of ρ. Note that the $2a^n$ comes from the two edges starting at the top point z of the tent.

Note that in the "limit case" $a = 1/2$ we have $\tau_n = 1$ for all n. For $a < 1/2$ we easily compute
$$2a = \tau_1 > \tau_2 > \cdots > \tau_\infty = \lim \tau_n = \frac{a}{1-a}.$$

Consider now the following special triangle z_0, z_1, z_2, with the properties:
z_1 lies on the left path starting from z_0,
z_2 lies on the right path starting from z_0, and
z_2 lies on the right path starting from z_1.
These conditions imply that $z_1 \leq z_0 \leq z_2$ and $\ell(z_0) \geq \ell(z_1) \geq \ell(z_2)$.

Let $n = \ell(z_0)$ and $m = \ell(z_1)$. Then Fact 4 applied to the tents $0, z_0, 1$ and $0, z_1, 1$ implies that
$$\rho(z_1, z_0) + \rho(z_0, z_2) - \rho(z_1, z_2) = \tau_n - \tau_m < 0.$$
Hence, $\{z_1, z_2\}$ is the longest side of that triangle.

We obtain from the above inequalities in particular that
$$\rho(z_1, z_2) - (\rho(z_1, z_0) + \rho(z_0, z_2)) \leq \tau_m - \tau_\infty = a^m \frac{1-2a}{1-a},$$
where the last equality is an easy computation. Since $\rho(z_1, z_2) \geq a^m$, we obtain
$$\frac{\rho(z_1, z_o) + \rho(z_0, z_2)}{\rho(z_1, z_2)} \geq 1 - \frac{1-2a}{1-a},$$

and hence
$$\rho(z_1, z_2) \leq (1 + \varepsilon_a)(\rho(z_1, z_0) + \rho(z_0, z_2)),$$
where $\varepsilon_a \to 0$ for $a \to 1/2$. Actually $1 + \varepsilon_a = (1 - a)/a$.

We consider now an arbitrary (nondegenerate) triangle. We number the vertices such that $z_1 < z_0 < z_2$.

Consider the V-shaped path from z_1 to z_2, let $\tilde{z} = r^k(z_1) = l^s(z_2)$ be the "lowest" point on this path. By symmetry of the whole argument we assume without loss of generality that $z_0 \leq \tilde{z}$. Now (using Fact 3) we see that the left path staring at z_0 will intersect the right path starting in z_1. Let z_1' be the intersection point. Let z_2' be the first point, where the right path starting at z_0 coincides with the right path starting at z_1. Fact 3 implies that $z_2' \leq \tilde{z}$. Note that now the triangle z_1', z_0, z_2' is a special triangle as discussed above. Further note that
$$\begin{aligned}\rho(z_1, z_0) &= \rho(z_1, z_1') + \rho(z_1', z_0),\\ \rho(z_0, z_2) &= \rho(z_0, z_2') + \rho(z_2', z_2),\\ \rho(z_1, z_2) &= \rho(z_1, z_1') + \rho(z_1', z_2') + \rho(z_2', z_2).\end{aligned}$$
Therefore we see as above
$$\rho(z_1, z_2) \geq \rho(z_1, z_o) + \rho(z_0, z_2).$$

We compute
$$\frac{\rho(z_1, z_o) + \rho(z_0, z_2)}{\rho(z_1, z_2)} \geq \frac{\rho(z_1', z_o) + \rho(z_0, z_2')}{\rho(z_1', z_2')} \geq 1 - \frac{1 - 2a}{1 - a},$$
where the last inequality is from the special case. Thus also in this case we obtain
$$\rho(z_1, z_2) \leq (1 + \varepsilon_a)(\rho(z_1, z_o) + \rho(z_0, z_2)).$$

3. Final remarks

In this remark we discuss some related results of [BoF]. As already mentioned, quasi-metrics play an important role in the study of Gromov hyperbolic metric spaces. Indeed, the boundary $Z = \partial_\infty X$ of a Gromov hyperbolic space X carries a natural quasi-metric $\rho(\xi, \eta) = e^{-(\xi|\eta)_o}$, where $(.|.)_o$ is the Gromov product with respect to some basepoint $o \in X$. For a quasi-metric space (Z, ρ) denote by $d = \mathrm{ca}(\rho)$ the pseudometric which is obtained from ρ by the chain approach. We call (Z, ρ) an *LM-space* (Lipschitz metrizable), if $\mathrm{ca}(\rho)$ is bi-Lipschitz to ρ. Hence, a quasi-metric space is LM if and only if the following two conditions hold:

(1) The chain approach gives a metric.

(2) The metric from the chain approach is bi-Lipschitz to ρ.

Consider now the whole family of quasi-metrics ρ^s, $s \in (0, \infty)$. If (Z, ρ) is a quasi-metric space, then ρ^s is a 2-quasi-metric for $s > 0$ sufficiently small and thus the chain approach works and gives a metric bi-Lipschitz to ρ^s. This allows us to define, for a quasi-metric space, a critical exponent $s_0 \in (0, \infty]$ with the following property: ρ^s is LM for all $s < s_0$ and ρ^s is not LM for all $s > s_0$. In the case that $Z = \partial_\infty X$ is the boundary of a Gromov hyperbolic space X, the number $K_u(X) = -s_o^2$ is called the *asymptotic upper curvature bound* of X. This invariant is defined and studied in [BoF]. Using modifications of our example above, one can construct Gromov hyperbolic spaces with interesting properties with respect to the asymptotic upper curvature bound. One can, in particular, give examples of quasi-metric spaces (Z, ρ) arising as boundaries of Gromov hyperbolic spaces such that the critical quasi-metric ρ^{s_0} is (resp. is not) bi-Lipschitz to a metric. These and related questions will be studied elsewhere.

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