

FREE SUBGROUPS OF SURFACE MAPPING CLASS GROUPS

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ABSTRACT. We quantify the generation of free subgroups of surface mapping class groups by pseudo-Anosov mapping classes in terms of their translation distance and the distance between their axes in Teichmüller's metric. The method makes reference to Teichmüller space only.

1. INTRODUCTION

Free subgroups of mapping class groups have attracted considerable attention from a number of authors over a number of years. It is a classical result, proven independently and simultaneously by Ivanov [11] and by McCarthy [16], that any collection \mathcal{A} of pseudo-Anosov mapping classes with pairwise distinct axes freely generates a free group of rank $|\mathcal{A}|$, so long as each element is first raised to a sufficiently high power. The supporting argument is based on the so-called Ping-Pong Lemma, quoted here in Section 4, and makes reference to Thurston's boundary of Teichmüller space.

The purpose of this work is to establish a ping-pong argument entirely inside Teichmüller space. This allows us to not only recover the result of Ivanov and of McCarthy by somewhat different means, but also to use the Teichmüller metric to quantify sufficiently high powers in a natural way. Our work relies on an application of Minsky's Bounded Projection Theorem [19]. One may also use Thurston's train tracks to quantify sufficiently high powers; see Hamidi-Tehrani [8].

The plan of this paper is as follows. In Section 2 we introduce all the background and notation we shall need. In Section 3 we recover the known result that axes of independent pseudo-Anosov mapping classes cannot be asymptotic. In Section 4 we exhibit ping-pong sets inside Teichmüller space for independent pseudo-Anosov mapping classes and use these to show that sufficiently high powers of independent pseudo-Anosov mapping classes generate a free group. In Section 5 we give a quantitative version of this result, with Theorem 5.7 our aim. Sections 4 and 5 are logically independent, but both rest firmly on Sections 2 and 3.

2. BACKGROUND

We refer the reader to [2], [9] and [12] for detailed studies of geodesic and measured geodesic laminations, Teichmüller spaces and mapping class groups, respectively, and recall only what we need here. Throughout this paper, a *surface* Σ will

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mean an orientable connected surface of negative Euler characteristic, with genus g and p punctures, and with empty boundary. A *curve* on Σ is the free homotopy class of a simple closed loop that is neither homotopic to a point nor to a puncture, and we denote by $\mathcal{S}(\Sigma)$ the set of all curves on Σ . We say that two curves are *disjoint* if they can be realised disjointly, and if they are not disjoint we say they *intersect essentially* or are *transverse*.

2.1. Teichmüller space. The *Teichmüller space* $T(\Sigma)$ is the space of all marked finite area hyperbolic structures on Σ , up to homotopy. More specifically, a point in $T(\Sigma)$ is an equivalence class $[(\sigma, f)]$, where σ is a finite area hyperbolic structure on Σ and $f : \Sigma \rightarrow \sigma$ is a homeomorphism, called a *marking* of σ . One declares two pairs (σ, f) and (σ', g) to be equivalent if and only if $g \circ f^{-1}$ is homotopic to an isometry between σ and σ' . To simplify our notation, we will use the same symbol to denote a point in $T(\Sigma)$ as to denote a particular marked finite area hyperbolic structure on Σ .

The space $T(\Sigma)$ is homeomorphic to an open ball of dimension $6g - 6 + 2p$, and can be compactified by attaching the space $\text{PML}(\Sigma)$ of all *projective measured laminations* on Σ , topologically a sphere of dimension $6g - 7 + 2p$. The closed ball $T(\Sigma) \cup \text{PML}(\Sigma)$ is sometimes known as the *Thurston compactification of Teichmüller space*, and we shall denote it by $\overline{T(\Sigma)}$.

Any curve $\alpha \in \mathcal{S}(\Sigma)$ induces a length function ℓ_α on $T(\Sigma)$, where $\ell_\alpha(x)$ denotes the length of the unique geodesic representative of the class α in the hyperbolic structure x on Σ . Given $\epsilon > 0$, the ϵ -*thick part* of Teichmüller space is defined by

$$T_{\geq \epsilon}(\Sigma) = \{x \in T(\Sigma) : \ell_\alpha(x) \geq \epsilon \text{ for all } \alpha \in \mathcal{S}(\Sigma)\},$$

and the ϵ -*thin part* of Teichmüller space is defined by

$$T_{\leq \epsilon}(\Sigma) = \{x \in T(\Sigma) : \ell_\alpha(x) \leq \epsilon \text{ for some } \alpha \in \mathcal{S}(\Sigma)\}.$$

The space $T(\Sigma)$ admits two natural metrics, the Teichmüller metric and the Weil-Petersson metric. In this paper we will only consider the Teichmüller metric, and refer the reader to [25] for a thorough study of the Weil-Petersson geometry. Given points $x, y \in T(\Sigma)$, the *Teichmüller distance* between $x = [(\sigma, f)]$ and $y = [(\sigma', g)]$ is defined as

$$d_T(x, y) = \frac{1}{2} \inf \{\log(K(h))\},$$

where the infimum ranges over all the quasiconformal homeomorphisms h in the homotopy class of $f \circ g^{-1}$ and $K(h)$ is the dilatation of h . By a celebrated result of Teichmüller, for any two points in Teichmüller space there is a unique quasiconformal homeomorphism h (in the appropriate homotopy class) realising their distance. Endowed with the Teichmüller metric, $T(\Sigma)$ is a uniquely geodesic, proper metric space. It is worth noting, however, the Teichmüller metric is not non-positively curved in any standard sense (see [14], [15], [18], and [3]).

For subsets X and Y of $T(\Sigma)$, we define the *nearest point distance* $d_T(X, Y)$ to be

$$d_T(X, Y) = \inf \{d_T(x, y) : x \in X, y \in Y\}.$$

2.2. Mapping class groups. The *mapping class group* $\text{MCG}(\Sigma)$ of Σ is the group of all homotopy classes of orientation preserving self-homeomorphisms of Σ . It is a finitely presented group, generated by a finite collection of Dehn twists about simple closed curves of Σ . There is a natural action of $\text{MCG}(\Sigma)$ on $T(\Sigma)$ by changing the

marking; the quotient $T(\Sigma)/\text{MCG}(\Sigma)$ is the *moduli space* of Σ . Except for a few low-dimensional cases, the mapping class group $\text{MCG}(\Sigma)$ is isomorphic to an index 2 subgroup of the full isometry group of both the Teichmüller metric and the Weil-Petersson metric, by results of Royden [22] and Masur-Wolf [15], respectively.

There are some similarities between the action of $\text{MCG}(\Sigma)$ on $T(\Sigma)$, equipped with the Teichmüller metric, and that of a geometrically finite group acting isometrically on a simply-connected Hadamard manifold of pinched negative curvature. For example, the elements of $\text{MCG}(\Sigma)$ can be classified in a manner that mimics the classification of the isometries of the pinched Hadamard manifold according to their dynamics on its ideal boundary. An infinite order mapping class is either *reducible*, so it fixes some non-empty and finite collection of disjoint curves, or is otherwise pseudo-Anosov, and fixes exactly two ideal points.

A *pseudo-Anosov* mapping class ϕ is represented by a pseudo-Anosov diffeomorphism Σ . That is, there exists a real number $r = r(\phi) > 1$, the dilatation of ϕ , such that for any hyperbolic metric (Σ, σ) there exists a unique diffeomorphism f representing ϕ and two measured laminations, λ^- and λ^+ , geodesic in σ such that $f(\lambda^+) = r\lambda^+$ and $f(\lambda^-) = \frac{1}{r}\lambda^-$. We shall write λ^\pm to denote either element of the set $\{\lambda^+, \lambda^-\}$. The measured lamination λ^\pm satisfies the following three fundamental properties [23]:

- (1) λ^\pm is *uniquely ergodic*: If μ is a measured lamination whose support $\text{supp}(\mu)$ is equal to $\text{supp}(\lambda^\pm)$, then μ and λ^\pm are proportional.
- (2) λ^\pm is *minimal*: If μ is a measured lamination satisfying $\text{supp}(\mu) \subseteq \text{supp}(\lambda^\pm)$, then either $\text{supp}(\mu) = \emptyset$ or $\text{supp}(\mu) = \text{supp}(\lambda^\pm)$.
- (3) λ^\pm is *maximal*: If μ is a measured lamination satisfying $\text{supp}(\lambda^\pm) \subseteq \text{supp}(\mu)$, then $\text{supp}(\mu) = \text{supp}(\lambda^\pm)$.

We will say a projective measured lamination is *uniquely ergodic* if one (and hence any) representative of its projective class is a uniquely ergodic measured lamination, and whenever λ is uniquely ergodic we shall also use, where there can be no ambiguity, λ to denote both a measured lamination and its projective class.

The fixed point set $\text{Fix}(\phi)$ of ϕ in $\overline{T(\Sigma)}$ is precisely $\{\lambda^+, \lambda^-\}$. These fixed points behave like attracting and repelling fixed points for ϕ . More specifically, with $s \in \{1, -1\}$, for any neighbourhood U of λ^s in $\overline{T(\Sigma)}$ and any compact set K in $T(\Sigma) \setminus \{\lambda^{-s}\}$ we have $\phi^{sn}(K) \subseteq U$ for sufficiently large n (see [12]). It is known that a pseudo-Anosov mapping class ϕ fixes a bi-infinite Teichmüller geodesic, the *axis* of ϕ , on which it acts by translation. By the above discussion, the set of accumulation points of this axis on $\text{PML}(\Sigma)$ is $\text{Fix}(\phi) = \{\lambda^+, \lambda^-\}$.

The *translation distance* $\text{Tr}(\phi) = \inf\{d_T(x, \phi(x)) : x \in T(\Sigma)\}$ of a pseudo-Anosov mapping class is always realised, and is always realised on the axis of ϕ . Furthermore, both Tr and the property of being pseudo-Anosov are invariant under conjugation. The following result is due to Ivanov [10].

Theorem 2.1 ([10]). *For a surface Σ and $L > 0$, there are only finitely many conjugacy classes of pseudo-Anosov mapping classes of translation distance at most L .*

It follows that there exists a constant $\ell_{\min} = \ell_{\min}(\Sigma) > 0$ such that all pseudo-Anosov mapping classes in $\text{MCG}(\Sigma)$ have translation distance at least ℓ_{\min} . Lower bounds for ℓ_{\min} , in terms of the topological type of Σ , have been found by Penner [20].

The following terminology is due to Minsky [19].

Definition 2.2. For a surface Σ and $\epsilon > 0$, a Teichmüller geodesic c is said to be ϵ -precompact if c is entirely contained in the ϵ -thick part $T_{\geq\epsilon}(\Sigma)$ of $T(\Sigma)$.

For any pseudo-Anosov mapping class ϕ , the projection of its axis into moduli space is compact. Moreover, by the continuity of the length functions $x \rightarrow \ell_\alpha(x)$, there is a uniform lower bound on all $\ell_\alpha(x)$, where $\alpha \in \mathcal{S}(\Sigma)$ and x lies on the axis of ϕ . Applying Theorem 2.1 to remove all dependence on ϕ yields the following.

Corollary 2.3. *For a surface Σ and $L > 0$, there exists a positive real number $\epsilon = \epsilon(L, \Sigma)$ such that, if ϕ is any pseudo-Anosov mapping class of translation distance at most L , the axis of ϕ is ϵ -precompact.*

Let $\phi, \psi \in \text{MCG}(\Sigma)$ be two pseudo-Anosov mapping classes and let $\text{Fix}(\phi)$ and $\text{Fix}(\psi)$ be their respective fixed point sets in $\overline{T(\Sigma)}$. It is known (see [17]) that either $\text{Fix}(\phi) = \text{Fix}(\psi)$, in which case ϕ and ψ have non-zero powers which are powers of the same pseudo-Anosov mapping class, or $\text{Fix}(\phi) \cap \text{Fix}(\psi) = \emptyset$. This prompted the following definition.

Definition 2.4 ([17]). For a surface Σ , two pseudo-Anosov mapping classes $\phi, \psi \in \text{MCG}(\Sigma)$ are said to be *independent* if $\text{Fix}(\phi) \cap \text{Fix}(\psi) = \emptyset$.

3. DIVERGENCE OF PSEUDO-ANOSOV AXES

It is one consequence of Minsky's Bounded Projection Theorem [19], found by Farb-Mosher [7], that the axes of two independent pseudo-Anosov mapping classes cannot be asymptotic in $T(\Sigma)$; that is, their Hausdorff distance is not finite. We offer a proof of a very much related result, namely that the distance function between the axes of independent pseudo-Anosov mapping classes is a proper function. (The corresponding result for the Weil-Petersson metric is due to Daskalopoulos-Wentworth [6].) We shall not need the Bounded Projection Theorem here, but instead a theorem of Wolpert [24] and a special case of Lemma 2.1 from [5].

Theorem 3.1 ([24]). *For a surface Σ and $D > 0$, let $x, y \in T(\Sigma)$ with $d_T(x, y) \leq D$. Then, for any curve $\alpha \in \mathcal{S}(\Sigma)$, we have*

$$e^{-2D}\ell_\alpha(x) \leq \ell_\alpha(y) \leq e^{2D}\ell_\alpha(x).$$

Although the intersection number of two arbitrary projective measured laminations is not well defined, we can still decide whether their intersection number should be zero or non-zero.

Lemma 3.2 ([5]). *For a surface Σ , let $\lambda, \lambda' \in \text{PML}(\Sigma)$. Let (x_n) be a sequence of points in $T(\Sigma)$ converging to λ and let (α_n) be a sequence of curves converging to λ' as measured laminations. Suppose there exists $R > 0$ such that $\ell_{\alpha_n}(x_n) \leq R$. Then, $i(\lambda, \lambda') = 0$.*

We are now ready to prove the distance function restricted to the axes of a pair of pseudo-Anosov mapping classes is a proper map. Recall, a map is said to be *proper* if the preimage of any compact subset of the range is compact.

Proposition 3.3. *For a surface Σ , let $\phi, \psi \in \text{MCG}(\Sigma)$ be two independent pseudo-Anosov mapping classes and let $c, c' : \mathbb{R} \rightarrow T(\Sigma)$ be arc-length parametrizations of their respective axes. Then, the map $(t, s) \rightarrow d_T(c(t), c'(s))$ is a proper map.*

Proof. Suppose the result is not true. Then, there are unbounded sequences (t_n) and (s_n) in \mathbb{R} and a real number $M > 0$ such that $d_T(c(t_n), c'(s_n)) \leq M$ for all $n \in \mathbb{N}$. By passing to subsequences if need be, we may assume that (t_n) and (s_n) are each either monotonically increasing or monotonically decreasing. Let us suppose they are both monotonically increasing, as the remaining cases can be treated analogously.

Let $\lambda, \mu \in \text{PML}(\Sigma)$ be such that $x_n = c(t_n) \rightarrow \lambda$ and $y_n = c'(s_n) \rightarrow \mu$ in $\overline{T(\Sigma)}$, as $n \rightarrow \infty$. Then, for all $n \in \mathbb{N}$, there exists $x'_n \in c(\mathbb{R})$ such that $d_T(x_n, x'_n) \leq \text{Tr}(\phi)$ and $x'_n = \phi^{k_n}(x_0)$, for some $x_0 \in c$ and $k_n = k(n)$. In particular, we have $k_n \rightarrow \infty$ as $n \rightarrow \infty$. Since $d_T(x_n, y_n) \leq M$, we see that $d_T(x'_n, y_n) \leq M + \text{Tr}(\phi)$. Denote by M' the upper bound $M + \text{Tr}(\phi)$.

Choose any curve $\alpha \in \mathcal{S}(\Sigma)$ and let $\alpha_n = \phi^{k_n}(\alpha)$, noting (α_n) converges to λ as measured laminations. It follows from the definition of the action of $\text{MCG}(\Sigma)$ on $T(\Sigma)$ that $\ell_{\alpha_n}(x'_n) = \ell_{\alpha}(x_0)$. This fact, together with the upper bound in Theorem 3.1, implies $\ell_{\alpha_n}(y_n) \leq e^{2M'} \ell_{\alpha}(x_0)$. Therefore, the sequences (y_n) and (α_n) satisfy the hypotheses of Lemma 3.2, with R equal to $e^{2M'} \ell_{\alpha}(x_0)$, and we conclude $i(\lambda, \mu) = 0$. As λ and μ are both maximal and both minimal, we find $\text{supp}(\lambda) = \text{supp}(\mu)$ and thus $\lambda = \mu$, since λ and μ are uniquely ergodic. However, this is contrary to the assumption that ϕ and ψ be independent as pseudo-Anosov mapping classes. \square

An immediate consequence of Proposition 3.3 is the following.

Corollary 3.4 ([7]). *For a surface Σ , let ϕ and ψ be two independent pseudo-Anosov mapping classes and let $c, c' : \mathbb{R} \rightarrow T(\Sigma)$ be arc-length parametrizations of their respective axes. Then, the Hausdorff distance between $c(\mathbb{R})$ and $c'(\mathbb{R})$ is infinite.*

4. PING-PONGING IN TEICHMÜLLER SPACE

The purpose of this section is to exhibit ping-pong sets inside Teichmüller space for any given finite family of independent pseudo-Anosov mapping classes. This recovers the theorem of Ivanov [11] and McCarthy [16]. In Section 5 we shall give a quantitative version of this result.

For a closed subset C of $T(\Sigma)$ and a point $x \in T(\Sigma)$, one defines the closest-point projection of x into C as

$$\pi_C(x) = \{y \in C : d_T(x, y) \leq d_T(x, z) \text{ for all } z \in C\}.$$

Note $\pi_C(x)$ is non-empty since $T(\Sigma)$ is a proper metric space. For another subset $C' \subset T(\Sigma)$, we define

$$\pi_C(C') = \bigcup_{x \in C'} \pi_C(x).$$

Given any path $c : \mathbb{R} \rightarrow T(\Sigma)$ we shall also use π_c to denote $\pi_{c(\mathbb{R})}$, the projection to the image of c . The next result is Minsky's Bounded Projection Theorem, a union of Contraction Theorem (1), Corollary 4.1 and Theorem 4.2 from [19], and highlights some of the hyperbolic behaviour of any thick part of Teichmüller space. For its statement, we shall first need the following standard definition.

Definition 4.1. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $K \geq 1, \kappa \geq 0$. A (K, κ) -quasi-isometric embedding of X into Y is a map $f : X \rightarrow Y$ such that

$$\frac{1}{K}d_X(x, x') - \kappa \leq d_Y(f(x), f(x')) \leq Kd_X(x, x') + \kappa,$$

for all $x, x' \in X$. A (K, κ) -quasi-geodesic in Y is a (K, κ) -quasi-isometric embedding of a closed subinterval of \mathbb{R} into Y .

Note, a quasi-geodesic need not be continuous. In what follows, $B_r(x)$ denotes the compact ball in $T(\Sigma)$ of radius r and centre x .

Theorem 4.2 ([19]). *For a surface Σ and $\epsilon > 0$, there exists a constant $b = b(\epsilon, \Sigma)$ such that the following hold.*

- (1) *Given any ϵ -precompact geodesic c and any $x \in T(\Sigma)$, we have*

$$\text{diam}(\pi_c(B_{d_T(x, c(\mathbb{R}))}(x))) \leq b.$$

- (2) *Given any ϵ -precompact geodesic c and any $x, y \in T(\Sigma)$, we have*

$$\text{diam}(\pi_c(x) \cup \pi_c(y)) \leq d_T(x, y) + 4b.$$

- (3) *Given $K \geq 1$ and $\kappa \geq 0$, there exists a non-negative real number $M = M(K, \kappa, \epsilon, \Sigma)$ such that the following holds: If q is a (K, κ) -quasi-geodesic path in $T(\Sigma)$ whose endpoints are connected by an ϵ -precompact Teichmüller geodesic c , then the image of q is contained in the closed M -neighbourhood of the image of c .*

The next result is a key ingredient for the ping-pong argument. Roughly speaking, it says precompact Teichmüller geodesics diverge “sufficiently fast”. We shall make use of the following notation: Given an embedding $c : \mathbb{R} \rightarrow T(\Sigma)$ and two of its points $x = c(t), x' = c(t')$, we will write $x < x'$ if $t < t'$.

Proposition 4.3. *For a surface Σ and $\epsilon > 0$, let $c, c' : \mathbb{R} \rightarrow T(\Sigma)$ be arc-length parametrizations of two ϵ -precompact Teichmüller geodesics so that $O = c(0)$ and $O' = c'(0)$ realise the nearest point distance D between $c(\mathbb{R})$ and $c'(\mathbb{R})$. Then, there exist two points $P^+, P^- \in c(\mathbb{R})$ and two points $Q^+, Q^- \in c'(\mathbb{R})$, with $P^- < O < P^+$ and $Q^- < O' < Q^+$, such that the following hold:*

- (1) *For all $x \in c(\mathbb{R})$ with $x > P^+$ and all $y \in c'(\mathbb{R})$ with $y > Q^+$,*

$$d_T(x, y) > \max\{d_T(O, x), d_T(O', y)\}.$$

- (2) *For all $x \in c(\mathbb{R})$ with $x < P^-$ and all $y \in c'(\mathbb{R})$ with $y < Q^-$,*

$$d_T(x, y) > \max\{d_T(O, x), d_T(O', y)\}.$$

Proof. We show only the first part of the proposition, as the second part follows by an analogous argument. Suppose, for contradiction, the statement is not true. Then, we can find two unbounded sequences (x_n) and (y_n) of points in $c(\mathbb{R})$ and $c'(\mathbb{R})$, respectively, such that $O < x_n < x_{n+1}$, $O' < y_n < y_{n+1}$ and $d_T(x_n, y_n) \leq \max\{d_T(O, x_n), d_T(O', y_n)\}$ for all $n \in \mathbb{N}$. Passing to a further subsequence if need be, we have $d_T(O, x_n) \geq d_T(O', y_n)$ for all n , or we have $d_T(O, x_n) \leq d_T(O', y_n)$ for all n . Without loss of generality, let us suppose the former holds.

Let g_n be the $c(\mathbb{R})$ -segment between O and x_n , noting the length of g_n tends to infinity as $n \rightarrow \infty$. Let q_n be the concatenation of the unique Teichmüller

geodesic segment from O to O' , the $c'(\mathbb{R})$ -segment from O' to y_n , and the unique Teichmüller geodesic segment from y_n to x_n . Then,

$$\begin{aligned} \text{length}(q_n) &\leq D + d_T(O', y_n) + d_T(x_n, y_n) \\ &\leq D + d_T(O, x_n) + d_T(O, x_n) = D + 2\text{length}(g_n). \end{aligned}$$

Moreover,

$$\text{length}(g_n) = d_T(O, x_n) \leq d_T(O, O') + d_T(O', y_n) + d_T(y_n, x_n) = \text{length}(q_n).$$

We deduce

$$\text{length}(g_n) \leq \text{length}(q_n) \leq D + 2\text{length}(g_n),$$

and therefore q_n is a $(2, D)$ -quasi-geodesic. In particular, notice that the constants of quasi-geodesicity are independent of n . By Theorem 4.2(3), there is a constant $M = M(2, D, \epsilon, \Sigma)$ such that the image of q_n is contained in the closed M -neighbourhood of the image of g_n , for all $n \in \mathbb{N}$. Since $d_T(O', y_n)$ tends to infinity as $n \rightarrow \infty$, this implies that the Hausdorff distance between $c([0, \infty))$ and $c'([0, \infty))$ is at most M . According to Proposition 3.3, this is a contradiction. \square

Corollary 4.4. *For a surface Σ and $\epsilon > 0$, let $c, c' : \mathbb{R} \rightarrow T(\Sigma)$ be arc-length parametrizations of two ϵ -precompact Teichmüller geodesics. Then, the set $\pi_{c'}(c(\mathbb{R}))$ is bounded.*

Proof. Let D be the nearest-point distance between $c(\mathbb{R})$ and $c'(\mathbb{R})$, and choose $O \in c(\mathbb{R})$ and $O' \in c'(\mathbb{R})$ such that $d_T(O, O') = D$. Note $O' \in \pi_{c'}(O)$ and $O \in \pi_c(O')$. Suppose, for contradiction, that $\pi_{c'}(c(\mathbb{R}))$ is not bounded. Then, there exists a sequence (x_n) of points from $c(\mathbb{R})$ with $d_T(O', y_n) \rightarrow \infty$, for any $y_n \in \pi_{c'}(x_n)$. Furthermore, the sequence (x_n) satisfies $d_T(O, x_n) \rightarrow \infty$, by Theorem 4.2(2). Passing to further subsequences if need be, we may assume $x_n > P^+$ and $y_n > Q^+$, say, where $P^+ \in c(\mathbb{R})$ and $Q^+ \in c'(\mathbb{R})$ are the respective points given by Proposition 4.3.

We have $d_T(x_n, y_n) > \max\{d_T(O, x_n), d_T(O', y_n)\}$ and, according to Theorem 4.2(1), we also have $\text{diam}(\pi_{c'}(B_{d_T(x_n, c'(\mathbb{R}))}(x_n))) \leq b$, for all $n \in \mathbb{N}$. However, $O \in B_{d_T(x_n, c'(\mathbb{R}))}(x_n)$ and so $O' \in \pi_{c'}(B_{d_T(x_n, c'(\mathbb{R}))}(x_n))$ for all n . From this we deduce $d_T(O', y_n) \leq b$ for all n , and this is a contradiction. \square

Given an arc-length parametrization $c : \mathbb{R} \rightarrow T(\Sigma)$ of a Teichmüller geodesic and a real number $R > 0$, we introduce the subsets

$$\Pi(c, R) = \{x \in T(\Sigma) : \pi_c(x) \subset c([R, \infty))\}$$

and

$$\Pi(c, -R) = \{x \in T(\Sigma) : \pi_c(x) \subset c((-\infty, -R])\}$$

of $T(\Sigma)$. We note that, if $R < R'$, then $\Pi(c, R') \subset \Pi(c, R)$ and $\Pi(c, -R') \subset \Pi(c, -R)$.

Corollary 4.5. *For a surface Σ and $\epsilon > 0$, let $c, c' : \mathbb{R} \rightarrow T(\Sigma)$ be arc-length parametrizations of two ϵ -precompact Teichmüller geodesics. Then, there exists $R > 0$ such that the sets $\Pi(c, R)$, $\Pi(c, -R)$, $\Pi(c', R)$ and $\Pi(c', -R)$ are pairwise disjoint.*

Proof. That $\Pi(c, R) \cap \Pi(c, -R)$ and $\Pi(c', R) \cap \Pi(c', -R)$ are both the empty set for sufficiently large R is a trivial consequence of Minsky's Bounded Projection Theorem. Let us just show $\Pi(c, R) \cap \Pi(c', R) = \emptyset$, since the remaining cases can

be proven analogously. We again argue by contradiction, by supposing that for all $n \in \mathbb{N}$ there exists $x_n \in \Pi(c, n) \cap \Pi(c', n)$. Let $y_n \in \pi_c(x_n)$ and let $z_n \in \pi_{c'}(x_n)$, for all n . Note that, in particular, $(y_n)_{n \in \mathbb{N}}$ and $(z_n)_{n \in \mathbb{N}}$ are unbounded sequences on the geodesics $c(\mathbb{R})$ and $c'(\mathbb{R})$, respectively. On passing to subsequences if need be, we have $d_T(x_n, y_n) \geq d_T(x_n, z_n)$, for all n , or $d_T(x_n, y_n) \leq d_T(x_n, z_n)$, for all n . Without loss of generality, we assume the former holds.

The diameter of the projection of $B_{d_T(x_n, y_n)}(x_n)$ into $c(\mathbb{R})$ is at most b , by Theorem 4.2(1). In particular, $d_T(\pi_c(z_n), \pi_c(x_n)) \leq b$ and it follows $d_T(\pi_c(c'), y_n) \leq b$ for all $n \in \mathbb{N}$. This is a contradiction, since $\pi_c(c')$ has bounded diameter, by Corollary 4.4, and (y_n) is an unbounded sequence on the geodesic $c(\mathbb{R})$. \square

Corollary 4.5 gives us enough information to apply the following lemma, the statement of which is recorded from [4], and deduce sufficiently high powers of n independent pseudo-Anosov mapping classes freely generate a free group of rank n .

Lemma 4.6 (Ping-Pong Lemma). *Let X be a set and let f_1, \dots, f_n be bijections from X to itself. Suppose, for every $i = 1, \dots, n$, there exist pairwise disjoint subsets $A_1^+, A_1^-, \dots, A_n^+, A_n^-$ of X such that $f_i(X \setminus A_i^-) \subseteq A_i^+$ and $f_i^{-1}(X \setminus A_i^+) \subseteq A_i^-$, for each i . Then, under composition, f_1, \dots, f_n freely generate a free group of rank n .*

Corollary 4.7 ([11], [16]). *For a surface Σ , let ϕ_1, \dots, ϕ_n be pairwise independent pseudo-Anosov mapping classes in $\text{MCG}(\Sigma)$. Then, there exists a natural number N such that $\phi_1^N, \dots, \phi_n^N$ freely generate a free group of rank n .*

Proof. Let $c_i : \mathbb{R} \rightarrow T(\Sigma)$ be an arc-length parametrization of the axis of ϕ_i for each $i = 1, \dots, n$. By Corollary 2.3 there exists a real number $\epsilon > 0$ such that c_i is ϵ -precompact for each $i = 1, \dots, n$. We note ϵ depends only on the maximal translation distance among the ϕ_i .

Now Corollary 4.5 applied to all pairs c_j and c_k of parametrizations for $1 \leq j < k \leq n$ implies there exists $R > 0$ such that the sets $\Pi(c_1, \pm R), \dots, \Pi(c_n, \pm R)$ are all pairwise disjoint. Let $\ell_{\min} > 0$ be the minimal translation distance among all pseudo-Anosov mapping classes in $\text{MCG}(\Sigma)$. Let N be the least integer such that $N > 2R/\ell_{\min}$. The mapping classes $\phi_1^N, \dots, \phi_n^N$ and the sets $A_1^\pm = \Pi(c_1, \pm R), \dots, A_n^\pm = \Pi(c_n, \pm R)$ satisfy the hypotheses of Lemma 4.6, and we conclude $\phi_1^N, \dots, \phi_n^N$ freely generate a free group of rank n . \square

5. A QUANTITATIVE PING-PONG ARGUMENT FOR PSEUDO-ANOSOVs

The purpose of this section is to give the promised quantitative version of Corollary 4.7. On the way, we will show quantitative versions of Proposition 4.3 and Corollary 4.4 for the axes of pseudo-Anosov mapping classes. Let us begin with the following definition.

Definition 5.1. For a surface Σ , $R > 0$, and $x \in T(\Sigma)$, we say a curve $\alpha \in \mathcal{S}(\Sigma)$ is R -short on x if $\ell_\alpha(x) \leq R$, and we let $S_R(x)$ be the set of R -short curves on x .

The next result is a special case of the main result in [1], where Birman-Series show the number of simple closed geodesics on a given surface grows at most polynomially in the length bound. The degree of this polynomial depends only on the topological type of the surface. We note that an improved version of this result has been given by Rivin [21].

Theorem 5.2 ([1], [21]). *For a surface Σ and $R > 0$, there exists an integer $B = B(R, \Sigma)$ such that the cardinality of the set $S_R(x)$ is at most B for all $x \in T(\Sigma)$.*

Recall that a *pants decomposition* for Σ is a maximal collection of pairwise distinct and pairwise disjoint curves on Σ . It is a theorem of Bers that there exists a universal constant $R_* = R_*(\Sigma)$ such that every point $x \in T(\Sigma)$ has a pants decomposition whose curves each have length at most R_* in x .

Let $\epsilon > 0$ and consider the thick part $T_{\geq \epsilon}(\Sigma)$ of the Teichmüller space of Σ . A simple area argument shows that, given $\epsilon > 0$, there is a constant $R = R(\epsilon)$ such that, for every point $x \in T_{\geq \epsilon}(\Sigma)$, the set $S_R(x)$ contains a pants decomposition of Σ and a curve transverse to each curve in the pants decomposition. From this discussion, and from Theorem 2.1 and Corollary 2.3, we obtain the following lemma.

Lemma 5.3. *For a surface Σ and $L > 0$, there exists a real number $F = F(L, \Sigma) > 0$ such that, for any pseudo-Anosov of translation distance at most L and any point x on its axis, the set $S_F(x)$ contains a pants decomposition of Σ and a curve intersecting each curve in the pants decomposition essentially.*

Suppose that ϕ, ψ are independent pseudo-Anosov mapping classes of translation distance at most $L > 0$, with $c, c' : \mathbb{R} \rightarrow T(\Sigma)$ arc-length parametrizations of their respective axes. By Corollary 2.3, there exists $\epsilon = \epsilon(L, \Sigma) > 0$ such that $c(\mathbb{R})$ and $c'(\mathbb{R})$ are ϵ -precompact Teichmüller geodesics. By Theorem 4.2(3), for any $D > 0$ there exists a non-negative real number $M = M(2, D, \epsilon, \Sigma)$ such that any $(2, D)$ -quasi-geodesic connecting the ends of an ϵ -precompact geodesic lies in the closed M -neighbourhood of the geodesic. Let $F = F(L, \Sigma)$ be the constant given by Lemma 5.3, and recall $B = B(e^{2(M+L)}F, \Sigma) \geq 0$ is a uniform upper bound on the number of $e^{2(M+L)}F$ -short curves over all points in $T(\Sigma)$. Note, B depends only on D, L and Σ . We have the following result, a quantitative analogue of Proposition 4.3.

Proposition 5.4. *For a surface Σ , let ϕ, ψ be independent pseudo-Anosov mapping classes of translation distance at most $L > 0$. Let $c, c' : \mathbb{R} \rightarrow T(\Sigma)$ be arc-length parametrizations of their respective axes so that $O = c(0)$ and $O' = c'(0)$ together realise the nearest-point distance D of $c(\mathbb{R})$ and $c'(\mathbb{R})$. For*

$$R = \max\{B! + 2, (B! + 2)L\},$$

the following holds: For all $x \in c(\mathbb{R}) \setminus B_R(O)$ and all $y \in c'(\mathbb{R}) \setminus B_R(O')$,

$$d_T(x, y) > \max\{d_T(O, x), d_T(O', y)\}.$$

Proof. Suppose the result were not true. Then, there are points $x \in c(\mathbb{R})$ and $y \in c'(\mathbb{R})$ with $d_T(x, O) > R$, $d_T(y, O') > R$ and $d_T(x, y) \leq \max\{d_T(O, x), d_T(O', y)\}$. Assume without loss of generality that $d_T(O, x) \geq d_T(O', y)$. Let q be the concatenation of the unique geodesic segment between O and O' , the $c'(\mathbb{R})$ -segment between O' and y and the unique geodesic segment between y and x . Since $d_T(x, y) \leq \max\{d_T(O, x), d_T(O', y)\}$ we have, as in the proof of Proposition 4.3, that q is a $(2, D)$ -quasi-geodesic connecting O to x . Therefore the image of q is entirely contained in the closed M -neighbourhood of the unique geodesic connecting O to x , where $M = M(2, D, \epsilon, \Sigma)$ as per Theorem 4.2(3).

Let $I = B! + 2$. Since the translation distance of ψ is at most L and $R \geq (B! + 2)L = IL$, the points $y_1 = O', y_2 = \psi(O') \dots, y_I = \psi^{I-1}(O')$ all lie on the geodesic from O' to y , which is a geodesic subpath of the quasi-geodesic q . Since the

image of q is contained in the closed M -neighbourhood of the geodesic connecting O to x , there are points x_1, \dots, x_I in $c(\mathbb{R})$ such that $d(x_i, y_i) \leq M$ for all $i = 1, \dots, I$. Since the translation distance of ϕ is also at most L , it follows that there are points z_1, \dots, z_I (not necessarily distinct) such that $d_T(x_i, z_i) \leq L$ and $z_i = \phi^{j(i)}(O)$ for $i = 1, \dots, I$. Therefore $d_T(y_i, z_i) \leq M + L$ for $i = 1, \dots, I$.

By Lemma 5.3 there exists $F = F(L, \Sigma) > 0$ such that the set $S_F(O')$ of short curves in O' contains a set S consisting of a pants decomposition and a single curve intersecting each curve in the pants decomposition essentially. Then, every element of the set S is $e^{2(M+L)}F$ -short in O by Theorem 3.1. Now, if a curve α is F -short in O' , then $\psi^{i-1}(\alpha)$ is F -short in y_i for $i = 1, \dots, I$. Therefore $\phi^{-j(i)}\psi^{i-1}(\alpha)$ is $e^{2(M+L)}F$ -short in O . In particular, all elements of the set $\phi^{-j(i)}\psi^{i-1}(S)$ are $e^{2(M+L)}F$ -short in O . Moreover, note that the set $\phi^{-j(i)}\psi^{i-1}(S)$ also consists of a pants decomposition and a curve intersecting each curve in the pants decomposition transversally, since $\phi^{-j(i)}\psi^{i-1}$ is a mapping class.

Since the set of $e^{2(M+L)}F$ -short curves in O has cardinality at most B and $I > B! + 1$, there must be some $k \in \{0, \dots, I - 1\}$ such that $\phi^{-j(k)}\psi^{k-1}(\beta) = \beta$ for all $\beta \in S$. It follows $\phi^{j(k)}$ and ψ^{k-1} share the same action on the set $S(\Sigma)$ of all curves on Σ . Therefore $\phi^{j(k)}$ and ψ^{k-1} are either equal or, for only a few exceptional surfaces, perhaps differ by a hyperelliptic involution. Regardless, both share common fixed points in $T(\Sigma)$ and this is contrary to their independence. \square

Corollary 5.5. *For a surface Σ and $L > 0$, let ϕ, ψ be independent pseudo-Anosov mapping classes of translation distance at most $L > 0$. Let $c, c' : \mathbb{R} \rightarrow T(\Sigma)$ be arc-length parametrizations of their respective axes so that $O = c(0)$ and $O' = c'(0)$ together realise the nearest-point distance of $c(\mathbb{R})$ and $c'(\mathbb{R})$. Let R and b be as per Proposition 5.4 and Theorem 4.2, respectively. Then,*

$$\pi_c(c') \subseteq B(O, R + 4b) \cap c(\mathbb{R}) \quad \text{and} \quad \pi_{c'}(c) \subseteq B(O', R + 4b) \cap c'(\mathbb{R}).$$

Proof. We need only prove one of the inclusions. Suppose, for contradiction, that $\pi_{c'}(c)$ is not entirely contained in $B(O', R + 4b) \cap c'(\mathbb{R})$. Then, there are points $x \in c(\mathbb{R})$ and $y \in \pi_{c'}(x) \subset c'(\mathbb{R})$ with $d_T(O', y) > R + 4b$. By Theorem 4.2(2), we have

$$d_T(O, x) + 4b \geq \text{diam}(\pi_c(O) \cup \pi_c(x)) \geq d_T(O', y) > R + 4b$$

and hence $d_T(O, x) > R$. According to Proposition 5.4, we also have $d_T(x, y) > d_T(O, x)$. In particular, $O \in B_{d_T(x, y)}(x)$. By Theorem 4.2(1), $\text{diam}(\pi_{c'}(B_{d_T(x, y)}(x))) \leq b$ and so $d_T(O', y) \leq b$. This is a contradiction, and we deduce the corollary. \square

Corollary 5.6. *For a surface Σ and $L > 0$, let ϕ, ψ be independent pseudo-Anosov mapping classes of translation distance at most $L > 0$. Let $c, c' : \mathbb{R} \rightarrow T(\Sigma)$ be arc-length parametrizations of their respective axes so that $O = c(0)$ and $O' = c'(0)$ together realise the nearest-point distance of $c(\mathbb{R})$ and $c'(\mathbb{R})$. Let R and b be as per Proposition 5.4 and Theorem 4.2, respectively. Then, the sets $\Pi(c, R + 6b)$, $\Pi(c, -R - 6b)$, $\Pi(c', R + 6b)$ and $\Pi(c', -R - 6b)$ are pairwise disjoint.*

Proof. We prove $\Pi(c, R + 6b) \cap \Pi(c', R + 6b) = \emptyset$. Again, suppose that there exists $x \in T(\Sigma)$ such that $x \in \Pi(c, R + 6b) \cap \Pi(c', R + 6b)$. Let $y \in \pi_c(x), w \in \pi_{c'}(y)$ and $z \in \pi_{c'}(x)$. Without loss of generality, we assume that $d_T(x, y) \geq d_T(x, z)$. By Theorem 4.2(1) we have $\text{diam}(\pi_{c'}(B_{d_T(x, y)}(x))) \leq b$, and in particular $d_T(w, z) \leq b$.

On the other hand, appealing to Corollary 5.5, we have

$$d_T(w, z) \geq d_T(O', z) - d_T(O', w) \geq R + 6b - R - 4b = 2b > b$$

and this is a contradiction. \square

We are ready to give the promised quantitative version of Corollary 4.7. Recall, ℓ_{\min} is defined as the minimal translation distance among all pseudo-Anosov mapping classes from $\text{MCG}(\Sigma)$.

Theorem 5.7. *For a surface Σ and $L > 0$, let ϕ_1, \dots, ϕ_n be pseudo-Anosov mapping classes of translation distance at most L . Let R and b be as per Proposition 5.4 and Theorem 4.2, respectively. If $N > (2R + 12b)/\ell_{\min}$, then $\phi_1^N, \dots, \phi_n^N$ freely generate a free group of rank n .*

Proof. Corollary 5.6 implies the sets $\Pi(c_1, \pm(R+6b)), \dots, \Pi(c_n, \pm(R+6b))$ are ping-pong sets for $\phi_1^N, \dots, \phi_n^N$. The result now follows from the Ping-Pong Lemma. \square

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