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NORMAL FAMILIES OF HOLOMORPHIC FUNCTIONS WITH MULTIPLE ZEROS

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Abstract. Let $\mathcal{F}$ be a family of functions holomorphic on a domain D in $\mathbb{C}$, all of whose zeros are multiple. Let h be a function meromorphic on D, $h \not\equiv 0, \infty$. Suppose that for each $f \in \mathcal{F}$, $f'(z) \neq h(z)$ for $z \in D$. Then $\mathcal{F}$ is a normal family on D.

1. Introduction

This paper is a complement to [5], where the following result was established.

Theorem A ([5, Theorem 3])**.** *Let $\mathcal{F}$ be a family of functions meromorphic on a domain D in $\mathbb{C}$, all of whose zeros have multiplicity at least 4. Let h be a function holomorphic on D, $h \not\equiv 0$. Suppose that for each $f \in \mathcal{F}$, $f'(z) \neq h(z)$ for $z \in D$. Then $\mathcal{F}$ is a normal family on D.*

We show by the following example that the constant 4 in Theorem A is sharp.

Example. Let $D = \{z : |z| < 1\}$ and $\mathcal{F} = \{f_n\}$, where

$$f_n(z) = \frac{(z - \frac{1}{n})^3}{z - \frac{3}{n}} = z^2 + \frac{3}{n^2} + \frac{8}{n^3(z - \frac{3}{n})}.$$

Clearly, $\mathcal{F}$ fails to be equicontinuous at 0 and hence is not normal in any neighborhood of 0. However, all zeros of functions in $\mathcal{F}$ have multiplicity 3, and $f_n'(z) \neq 2z$ on $\mathbb{C}$.

Here we prove the following theorem.

Theorem. *Let $\mathcal{F}$ be a family of functions holomorphic on a domain D in $\mathbb{C}$, all of whose zeros are multiple. Let h be a function meromorphic on D, $h \not\equiv 0, \infty$. Suppose that for each $f \in \mathcal{F}$, $f'(z) \neq h(z)$ for $z \in D$. Then $\mathcal{F}$ is a normal family on D.*

Easy examples show that no such result holds if the assumption of multiple zeros is dropped.

The proof of the Theorem follows the proof of Theorem 2 in [5] rather closely. However, certain new difficulties already arise in the case $h(z) = z$. The reader is

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invited to try to adapt the proof of [5] to this special case before reading the proof given below.

2. Auxiliary results

First let us set some notation. We denote the unit disc by Δ. More generally, $\Delta(w,r) = \{z : |z-w| < r\}$ and $\Delta'(w,r) = \{z : 0 < |z-w| < r\}$.

Our theorem is a generalization of the following result, which is a special case of [6, Theorem 1.1] (cf. [2, Lemma 6] and [5, Theorem 1]).

Lemma 1. *Let $\mathcal{F}$ be a family of functions holomorphic on a domain D in $\mathbb{C}$, all of whose zeros are multiple. Let h be a function meromorphic on D, $h \not\equiv \infty$, such that $h(z) \neq 0$ for $z \in D$. Suppose that for each $f \in \mathcal{F}$, $f'(z) \neq h(z)$ for $z \in D$. Then $\mathcal{F}$ is a normal family on D.*

Let g be meromorphic on the domain D in $\mathbb{C}$. Then

$$g^{\#}(z) = \frac{|g'(z)|}{1+|g(z)|^2}$$

is the spherical derivative of g. It is easy to see that a meromorphic function with bounded spherical derivative can have order at most 2. For entire functions, more can be said.

Lemma 2 ([1, Theorem 3], cf. [3, Theorem 5])**.** *An entire function which has bounded spherical derivative is of exponential type.*

We also require the following renormalization result, which has become a standard tool in the study of normal families.

Lemma 3 ([4, Lemma 2], cf. [7, pp. 216–217])**.** *Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity of at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal, there exist, for each $0 \leq \alpha \leq k$,*

(a) *a number $0 < r < 1$,*
(b) *points z_n, $|z_n| < r$,*
(c) *functions $f_n \in \mathcal{F}$, and*
(d) *positive numbers $\rho_n \to 0$*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) = g_n(\zeta) \to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$ such that $g^{\#}(\zeta) \leq g^{\#}(0) = kA + 1$.

Naturally, when the functions in Lemma 3 are holomorphic, convergence is locally uniform (with respect to the Euclidean metric) and the limit function is entire.

Finally, a simple application of Lemma 3 yields the following result.

Lemma 4. *Let $\{f_n\}$ be a sequence of functions holomorphic on a domain D in $\mathbb{C}$, all of whose zeros are multiple. Let $\{h_n\}$ be a sequence of functions holomorphic on D such that $h_n \to h$ locally uniformly on D, where $h(z) \neq 0$ for $z \in D$. Suppose that for each n, $f_n'(z) \neq h_n(z)$ for $z \in D$. Then $\{f_n\}$ is a normal family on D.*

Proof. Otherwise, there is a disc (which we may assume to be Δ) contained in D on which $\{f_n\}$ is not normal. Then, taking an appropriate subsequence of $\{f_n\}$ and renumbering, we have, by Lemma 3 (with $\alpha = k = A = 1$), points z_n ($|z_n| < r < 1$) and numbers $\rho_n \to 0^+$ such that

$$\frac{f_n(z_n + \rho_n\zeta)}{\rho_n} = g_n(\zeta) \to g(\zeta)$$

locally uniformly on $\mathbb{C}$, where g is a nonconstant entire function, all of whose zeros are clearly multiple, satisfying $g^{\#}(\zeta) \le g^{\#}(0) = 2$. By Lemma 2, g is of exponential type. Taking an additional subsequence if necessary and renumbering, we may assume that $z_n \to z_0 \in \Delta$, so that $h_n(z_n + \rho_n\zeta) \to h(z_0) \ne 0$ locally uniformly on $\mathbb{C}$. Now since $g_n'(\zeta) = f_n'(z_n + \rho_n\zeta) \ne h_n(z_n + \rho_n\zeta) \to h(z_0)$ and $g_n' \to g'$ locally uniformly on $\mathbb{C}$, it follows from Hurwitz' Theorem that either $g'(\zeta) \equiv h(z_0)$ or $g'(\zeta) \ne h(z_0)$ for all $\zeta \in \mathbb{C}$. In the first instance, $g(\zeta) = h(z_0)\zeta + C$; in the second, either $g(\zeta) = Ae^{B\zeta} + h(z_0)\zeta + C$ or $g(\zeta) = (A + h(z_0))\zeta + C$, where $A \ne 0$, $B \ne 0$, and C are complex constants. In either case, we obtain a contradiction to the fact that all zeros of the nonconstant function g are multiple. □

3. Proof of the Theorem

By Lemma 1, it suffices to prove that $\mathcal{F}$ is normal at points for which $h(z) = 0$. So we may assume, making standard normalizations, that $\mathcal{F}$ satisfies the conditions of the Theorem and that h is holomorphic on Δ, with

$$h(z) = z^k + a_{k+1}z^{k+1} + \cdots = z^k b(z), \qquad z \in \Delta,$$

where $k \ge 1$, $b(0) = 1$, and $h(z) \ne 0$ for $0 < |z| < 1$. The claim is that $\mathcal{F}$ is normal at 0. Arguing by contradiction, suppose that $\{f_n\} \subset \mathcal{F}$ but no subsequence of $\{f_n\}$ is normal at 0. Let $\mathcal{F}_1 = \{F_n\}$, where $F_n = f_n/h$. If $f \in \mathcal{F}$, $f'(0) \ne h(0) = 0$; hence, since all zeros of f are multiple, $f(0) \ne 0$. Thus, for any $F \in \mathcal{F}_1$, $F(0) = f(0)/h(0) = \infty$.

We shall first prove that $\mathcal{F}_1$ is normal on Δ. Suppose not. Then, renumbering, we have by Lemma 3 (again with $\alpha = k = A = 1$) that there exist $F_n \in \mathcal{F}_1$, $z_n \in \Delta$ ($|z_n| \le r < 1$), and $\rho_n \to 0^+$ such that

$$\frac{F_n(z_n + \rho_n\zeta)}{\rho_n} = g_n(\zeta) \to g(\zeta)$$

spherically uniformly on compact subsets of $\mathbb{C}$, where g is a nonconstant meromorphic function on the plane, all of whose zeros are multiple, such that $g^{\#}(\zeta) \le g^{\#}(0) = 2$.

We consider the following two cases.

(a) Suppose $z_n/\rho_n \to \infty$. Then, since $g_n(-z_n/\rho_n) = F_n(0)/\rho_n$, the pole of g_n corresponding to that of F_n at 0 drifts off to infinity; g_n converges to g locally uniformly on $\mathbb{C}$; and hence, by Lemma 2, g is entire of exponential type. We have

$$F_n'(z) = \frac{f_n'(z)h(z) - f_n(z)h'(z)}{h(z)^2} = \frac{f_n'(z)}{h(z)} - \frac{h'(z)}{h(z)}F_n(z).$$

Thus,

$$\begin{aligned} g_n'(\zeta) = F_n'(z_n + \rho_n\zeta) &= \frac{f_n'(z_n + \rho_n\zeta)}{h(z_n + \rho_n\zeta)} - \frac{h'(z_n + \rho_n\zeta)}{h(z_n + \rho_n\zeta)}F_n(z_n + \rho_n\zeta) \\ &= \frac{f_n'(z_n + \rho_n\zeta)}{h(z_n + \rho_n\zeta)} - \left(\frac{k}{z_n/\rho_n + \zeta} + \rho_n \, \frac{b'(z_n + \rho_n\zeta)}{b(z_n + \rho_n\zeta)}\right) \, \frac{F_n(z_n + \rho_n\zeta)}{\rho_n}. \end{aligned}$$

Clearly,

$$\lim_{n\to\infty} \frac{k}{z_n/\rho_n+\zeta} = 0 \quad \text{and} \quad \lim_{n\to\infty} \rho_n \frac{b'(z_n+\rho_n\zeta)}{b(z_n+\rho_n\zeta)} = 0$$

uniformly on compact sets of $\mathbb{C}$. Thus,

$$\frac{f_n'(z_n+\rho_n\zeta)}{h(z_n+\rho_n\zeta)} = g_n'(\zeta) + \left(\frac{k}{z_n/\rho_n+\zeta} + \rho_n \frac{b'(z_n+\rho_n\zeta)}{b(z_n+\rho_n\zeta)}\right) g_n(\zeta)$$

converges locally uniformly to $g'(\zeta)$ on $\mathbb{C}$. Since $f_n'(z)/h(z) \neq 1$, by Hurwitz' Theorem either $g' \equiv 1$ or $g'(\zeta) \neq 1$ for all $\zeta \in \mathbb{C}$. Just as in the proof of Lemma 4, neither of these alternatives is consistent with the fact that all zeros of g are multiple.

(b) So we may assume that $z_n/\rho_n \to \alpha$, a finite complex number. We have

$$\frac{F_n(\rho_n\zeta)}{\rho_n} = \frac{F_n\left(z_n+\rho_n\left(\zeta - z_n/\rho_n\right)\right)}{\rho_n} \to g(\zeta-\alpha) = \tilde{g}(\zeta),$$

the convergence being spherically uniform on compact sets of $\mathbb{C}$ and hence uniform on compacta disjoint from the poles of $\tilde{g}$. Clearly, all zeros of $\tilde{g}$ are multiple and $\tilde{g}$ has a single pole at 0, of order k.

Now

$$\lim_{n\to\infty} \frac{h(\rho_n\zeta)}{\rho_n^k} = \zeta^k$$

uniformly on compact subsets of $\mathbb{C}$. Thus, writing

$$G_n(\zeta) = \frac{f_n(\rho_n\zeta)}{\rho_n^{k+1}} = \frac{h(\rho_n\zeta)}{\rho_n^k}\ \frac{f_n(\rho_n\zeta)}{\rho_n h(\rho_n\zeta)} = \frac{h(\rho_n\zeta)}{\rho_n^k}\ \frac{F_n(\rho_n\zeta)}{\rho_n},$$

we have

$$G_n(\zeta) \to \zeta^k \tilde{g}(\zeta) = G(\zeta)$$

locally uniformly on $\mathbb{C}\backslash\{0\}$ and hence (by the maximum principle) locally uniformly on $\mathbb{C}$. Clearly, G is an entire function, all of whose zeros are multiple, which has bounded spherical derivative and hence is of exponential type. Since the pole of $\tilde{g}$ at 0 has order k, $G(0) \neq 0$.

We claim that $G'(\zeta) \neq \zeta^k$ for all $\zeta \in \mathbb{C}$. Indeed, suppose that $G'(\zeta_0) = \zeta_0^k$. Since G is holomorphic and

$$G_n'(\zeta) - \frac{h(\rho_n\zeta)}{\rho_n^k} = \frac{f_n'(\rho_n\zeta) - h(\rho_n\zeta)}{\rho_n^k} \neq 0,$$

we have by Hurwitz' Theorem $G'(\zeta) \equiv \zeta^k$; hence, $G(\zeta) = \zeta^{k+1}/(k+1) + C$. Since all zeros of G are multiple, $C = 0$; but this contradicts $G(0) \neq 0$.

Thus $G'(\zeta) \neq \zeta^k$ for all $\zeta \in \mathbb{C}$. It follows that either $G(\zeta) = Ae^{B\zeta} + \zeta^{k+1}/(k+1) + C$ or $G(\zeta) = \zeta^{k+1}/(k+1) + A\zeta + C$ for complex constants $A \neq 0$, $B \neq 0$, and C. This is inconsistent with the fact that all zeros of G are multiple unless $k = 1$ and $G(\zeta) = (\zeta + A)^2/2$. It remains to rule out this last possibility.

Suppose, therefore, that $G(\zeta) = (\zeta + A)^2/2$. Then since

$$\frac{f_n(\rho_n\zeta)}{\rho_n^2} \to \frac{(\zeta + A)^2}{2} \tag{1}$$

locally uniformly on $\mathbb{C}$, there exist points $\zeta_n \to -A$ such that $f_n(\rho_n\zeta_n) = 0$. The sequence $\{f_n\}$ fails to be normal at 0; on the other hand, by Lemma 1, $\{f_n\}$ is normal on $\Delta' = \{z : 0 < |z| < 1\}$. Since the functions f_n are holomorphic on Δ, it follows that they tend to ∞ locally uniformly on Δ'. Suppose now that there exists

$\delta > 0$ such that, for each n, f_n has only the single zero $\xi_n = \rho_n \zeta_n$ in $\Delta(0, \delta)$ for all n. Put

$$H_n(z) = \frac{f_n(z)}{(z - \xi_n)^2}. \tag{2}$$

Then $\{H_n\}$ is a sequence of nonvanishing holomorphic functions on $\Delta(0, \delta)$ tending to ∞ locally uniformly on $\Delta'(0, \delta)$. It follows that the sequence $\{1/H_n\}$ of holomorphic functions tends to 0 locally uniformly on $\Delta'(0, \delta)$ and hence, by the maximum principle, on $\Delta(0, \delta)$. Thus $H_n \to \infty$ locally uniformly on $\Delta(0, \delta)$. In particular, $H_n(2\rho_n\zeta_n) \to \infty$. But by (1) and (2),

$$H_n(2\rho_n\zeta_n) = \frac{f_n(2\rho_n\zeta_n)}{(\rho_n\zeta_n)^2} \to \frac{1}{2},$$

a contradiction.

Thus, taking a subsequence if necessary, we may assume that for any $\delta > 0$, f_n has at least two distinct zeros in $\Delta(0, \delta)$ for n sufficiently large. Choose η_n such that $f_n(\eta_n) = 0$ and f_n has no zeros on $\Delta'(\xi_n, |\eta_n - \xi_n|)$; then $\eta_n \to 0$. By (1), $\eta_n/\rho_n \to \infty$, so that $\xi_n/\eta_n = \rho_n\zeta_n/\eta_n \to 0$. Put

$$K_n(z) = \frac{f_n((\eta_n - \xi_n)z)}{(\eta_n - \xi_n)^2}, \quad h_n(z) = \frac{h((\eta_n - \xi_n)z)}{\eta_n - \xi_n}.$$

Then $\{K_n\}$ is a sequence of holomorphic functions, all of whose zeros are multiple, which are defined for each $z \in \mathbb{C}$ for n sufficiently large. Similarly, the sequence of holomorphic functions $\{h_n\}$ is defined for each $z \in \mathbb{C}$ for n sufficiently large; and $h_n(z) \to z$ locally uniformly on $\mathbb{C}$. Clearly, $K_n'(z) \neq h_n(z)$. Hence, by Lemma 4, $\{K_n\}$ is normal on $\mathbb{C} \setminus \{0\}$. We claim that $\{K_n\}$ is also normal at 0. Indeed, otherwise $K_n \to \infty$ locally uniformly on $\mathbb{C} \setminus \{0\}$. But this is impossible, as $K_n(\eta_n/(\eta_n - \xi_n)) = 0$ and $\eta_n/(\eta_n - \xi_n) \to 1$. Thus $\{K_n\}$ is normal on $\mathbb{C}$. Taking a subsequence and renumbering, we have $K_n \to K$ locally uniformly on $\mathbb{C}$ for an entire function K, all of whose zeros are multiple. Since $K_n'(z) \neq h_n(z)$ and $h_n(z) \to z$, either $K'(z) \neq z$ for all $z \in \mathbb{C}$ or $K'(z) \equiv z$. But $K_n(\xi_n/(\eta_n - \xi_n)) = 0$ and $\xi_n/(\eta_n - \xi_n) \to 0$, so that $K(0) = 0$ and hence $K'(0) = 0$. Thus $K'(z) \equiv z$. It follows that $K(z) \equiv z^2/2$. But $K_n(\eta_n/(\eta_n - \xi_n)) = 0$ and $\eta_n/(\eta_n - \xi_n) \to 1$, so that $K(1) = 0$. This contradiction completes the proof that $\mathcal{F}_1$ is normal on Δ.

Finally, since $\mathcal{F}_1$ is normal on Δ and $F_n(0) = \infty$ for all n, there exists $\delta > 0$ such that $|F_n(z)| \geq 1$ on $\Delta(0, \delta)$ for all n. Thus $f_n(z) \neq 0$ on $\Delta(0, \delta)$, and hence $1/f_n$ is holomorphic on $\Delta(0, \delta)$ for all n. Choosing δ small enough that $|h(z)| \geq |z|^k/2$ for $|z| \leq \delta$, we have

$$\left|\frac{1}{f_n(z)}\right| = \left|\frac{1}{F_n(z)} \cdot \frac{1}{h(z)}\right| \leq \frac{2^{k+1}}{\delta^k}, \quad |z| = \frac{\delta}{2}.$$

By the maximum principle, this bound holds throughout $\Delta(0, \delta/2)$. It follows that $\{f_n\}$ is normal at 0, which contradicts the original choice of $\{f_n\}$. Thus $\mathcal{F}$ is normal at 0, and the Theorem is proved.

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